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# OSCILLATION PROPERTIES OF SOLUTIONS OF A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

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The present paper studies the oscillatory properties of the solutions of a class of integro-differential equations of the form

(1) $$[(Lx)(t)]^{(n)} + \int_{I_t} K(t, s, x(s))\, ds = 0 ,$$

where $n \geqq 1$; $I_t \subset J$, $J = [t_0, +\infty)$, $t_0 \in \mathbb{R}$; $K: J^2 \times \mathbb{R} \to \mathbb{R}$ $L: \tilde{C}^{n-1}(J, \mathbb{R}) \to \tilde{C}^{n-1}(J, \mathbb{R})$, $\tilde{C}^{n-1}(J, \mathbb{R})$ denoting the linear space of functions $x: J \to \mathbb{R}$, possessing locally absolutely continuous derivatives up to and including the order $n - 1$.

**Definition 1.** We will say that a proposition $Q$ is finally fulfilled if there exists a point $t_Q \in J$ such that the proposition $Q$ is true for every $t \geqq t_Q$.

The operator $L$ will be assumed to satisfy the conditions (A):

A1. If a function $\varphi \in \tilde{C}^{n-1}(J, \mathbb{R})$ is finally non-negative (non-positive), then the function $(L\varphi)(t)$ is also finally non-negative (non-positive).

A2. For every $\varepsilon > 0$ and every finally non-negative or non-positive function $\varphi \in \tilde{C}^{n-1}(J, \mathbb{R})$ for which such a point $\bar{t} = \bar{t}(\varphi, \varepsilon) \in J$ can be found that

(2) $$\inf_{t \geqq \bar{t}} |(L\varphi)(t)| \geqq \varepsilon ,$$

there are a set $E = E(\bar{t}, \varphi, \varepsilon) \subset J$, meas $E = +\infty$, and a number $\delta(\varphi, \varepsilon, \bar{t}, E) > 0$ such that the inequality $|\varphi(t)| \geqq \delta$ is fulfilled for every $t \in E$.

A3. If a function $\varphi \in \tilde{C}^{n-1}(J, \mathbb{R})$ is finally non-negative or non-positive and $\lim_{t \to +\infty} (L\varphi)(t) = 0$, then $\lim_{t \to +\infty} \varphi(t) = 0$.

Let a mapping $\mathscr{F}: t \mapsto I_t$ be given, where for every $t \in J$, $I_t$ is a bounded, non-empty and measurable subset of $J$, and let us introduce the notation

$$F_t = \{t\} \times I_t = \{(t, s) \mid s \in I_t\} \subset \mathbb{R}^2 ,$$

$$M_t = \bigcup_{s \in [t, +\infty)} F_s , \qquad s'_t = \inf_{s \in I_t} s , \qquad s''_t = \sup_{s \in I_t} s .$$

It will be assumed that the mapping $\mathscr{F}$ and the kernel $K$ satisfy the conditions (B):

B1. For every $\varepsilon > 0$ and $t' \in J$, there exists $\delta = \delta(\varepsilon, t') > 0$ such that if $|t' - t| <$

$< \delta$, the inequality

$$\text{meas}\left\{(I_t \setminus I_{t'}) \cup (I_{t'} \setminus I_t)\right\} < \varepsilon$$

holds.

B2. $\limsup_{t \to +\infty} s''_t = +\infty$.

B3. The function $K(t, s, u)$ is continuous at every point $(t, s, u) \in M_{t_0} \times \mathbb{R}$.

B4. For $(t, s, u) \in M_{t_0} \times \mathbb{R}$, the relation

$$u \,.\, K(t, s, u) \geqq 0$$

holds.

B5. For every $u_0 > 0$ the inequality

$$\liminf_{\substack{|u| \geqq u_0 \\ (t,s,u) \in M_{t_0} \times \mathbb{R} \\ t,s \to +\infty}} |K(t, s, u)| > 0$$

holds.

**Definition 2.** A function $x \in \tilde{C}^{n-1}(J, \mathbb{R})$ will be called a *regular solution* if it satisfies (1) almost everywhere for $t \in J$ and $\sup_{t \in [t', +\infty)} |x(t)| > 0$, $t' \in J$.

**Definition 3.** We will say that a regular solution is *oscillatory* if for every $t' \in J$ we have $\sup_{t \in [t', +\infty)} x(t) > 0$, $\inf_{t \in [t', +\infty)} x(t) < 0$.

**Theorem 1.** *Let the following conditions be fulfilled*:

1. Conditions (A) and (B) hold.
2. $\lim_{t \to +\infty} s'_t = +\infty$.
3. *For every measurable subset* $E \subset J$, $\text{meas}\, E = +\infty$, *the relation*

$$\int_E \text{meas}\left\{t \mid t \in J,\ s \in I_t\right\} \mathrm{d}s = +\infty \tag{4}$$

*holds.*

*Then for $n$ even every regular solution $x(t)$ of (1) oscillates, while for $n$ odd, it either oscillates or tends to zero for $t \to +\infty$.*

Proof. Assume that a non-oscillatory solution of (1) exists, and for definiteness suppose that $x(t) \geqq 0$ for $t \in \tilde{J} = [\tilde{t}, +\infty)$, $\tilde{t} \in J$. Then (1) implies that $[(Lx(t)]^{(n)} \leqq 0$ for $t \in \tilde{J}$ and hence there exists an integer $l$, $0 \leqq l \leqq n$, $l + n$ odd, such that for $t \geqq \tilde{t}$ the inequalities

$$\begin{aligned} &[(Lx)(t)]^{(i)} \geqq 0\,, \quad i = 0, \ldots, l\,, \\ &(-1)^{l+i}[(Lx)(t)]^{(i)} \geqq 0\,, \quad i = l + 1, \ldots, n \end{aligned} \tag{5}$$

hold. (See [1], Lemma 14.3, p. 289).

Let $n$ be an even number. (1) implies that

$$\int_{\tilde{t}}^{+\infty} \left(\int_{I_t} K(t, s, x(s))\, \mathrm{d}s\right) \mathrm{d}t < +\infty\,, \tag{6}$$

and taking into account (5), we conclude that $\liminf_{t \to +\infty} (Lx)(t) \geqq c > 0$ ($x(t)$ is a regular solution). Therefore, there exists a point $t \in J$ such that $(Lx)(t) \geqq \frac{1}{2}c$ for $t \geqq \bar{t}$. Condition A2 implies that there exist a set $E \subset [\bar{t}, +\infty)$, $\operatorname{meas} E = +\infty$, and a number $\delta > 0$ such that $x(t) \geqq \delta$ for $t \in E$. Condition B5 yields that there exist a constant $\gamma > 0$ and a point $t' \geqq \bar{t}$ such that the inequality $K(t, s, u) \geqq \gamma$ holds for $t, s \geqq t'$ and $u \geqq \delta$.

Employing the Fubini theorem and (4), we obtain

$$\int_{t'}^{+\infty} \left( \int_{I_t} K(t, s, x(s))\, ds \right) dt \geqq \int_{t'}^{+\infty} \left( \int_{I_t \cap [t', +\infty)} K(t, s, x(s))\, ds \right) dt =$$
$$= \int_{t'}^{+\infty} \left( \int_{\{t \mid t \in [t', +\infty), s \in I_t\}} K(t, s, x(s))\, dt \right) ds \geqq$$
$$\geqq \int_{E \cap [t', +\infty)} \left( \int_{\{t (t \in [t', +\infty), s \in I_t\}} K(t, s, x(s))\, dt \right) ds \geqq$$
$$\geqq \gamma \int_{E \cap [t', +\infty)} \operatorname{meas} \{t \mid t \in [t', +\infty),\ s \in I_t\}\, ds = +\infty\,,$$

which contradicts inequality (6).

Let $n$ be an odd number. Then (5) implies that either $\lim_{t \to +\infty} (Lx)(t) = 0$ and A3 yields $\lim_{t \to +\infty} x(t) = 0$, or $\lim_{t \to +\infty} (Lx)(t) > 0$, the latter case being treated as for $n$ — an even number.

Example 1. Put

$$(Lx)(t) := x(t) + \lambda x(t - \tau)\,, \quad \lambda, \tau > 0\,. \tag{7}$$

Then Lemma 2 of [2] immediately implies that the operator defined by equality (7) satisfies the conditions (A). Therefore, equation (1) involves integro-differential equations of neutral type as a particular case.

Remark 1. It is not difficult to see that if the operator $L$ is defined by equality (7), then condition 2 of Theorem 2 can be replaced by the following condition:

Let $\lim_{t \to +\infty} s'_t = +\infty$ and let for every sufficiently large $t^* \in J$ the relation

$$\sum_{i=0}^{+\infty} \left( \inf_{t^* + 2i\tau \leqq s \leqq t^* + 2(i+1)\tau} \operatorname{meas} \{t \mid t \in [t^*, +\infty),\ s \in I_t\} \right) = +\infty \tag{8}$$

hold. It is immediately verified that for (8) to hold, it is sufficient for sufficiently large $t^* \in J$ to fulfil the relation

$$\int_{t^*}^{+\infty} \left( \inf_{t^* \leqq \sigma \leqq s} \operatorname{meas} \{t \mid t \in [t^*, +\infty),\ \sigma \in I_t\} \right) ds = +\infty\,. \tag{9}$$

Remark 2. To supply an example when (9) holds, we have to put $I_t = [t - \omega, t]$, $\omega > 0$.

Condition 2 of Theorem 1 is quite essential for its proof, but it excludes the important special case $s'_t = \text{const}$. In order to cover this case as well we have to strengthen condition 3 of Theorem 1. The theorem that follows represents one of the possible variants of doing so.

**Theorem 2.** *Let the following conditions be fulfilled*:

1. *Conditions* (A) *and* (B) *hold.*

2. *For every constant* $c > 0$ *the inequality*

$$\sup_{\substack{(t,s,u)\in M_{t_0}\times \mathbb{R} \\ |s|\leqq c,\, |u|\leqq c}} |K(t, s, u)| < +\infty$$

*holds.*

3. *For every measurable subset* $E \subset J$, meas $E = +\infty$, *the relation*

$$\lim_{T\to+\infty} (T^{-1} \textstyle\int_E \operatorname{meas} \{t \mid t \in [t_0, T],\ s \in I_t\})\, \mathrm{d}s = +\infty \tag{10}$$

*holds.*

*Then every regular solution* $x(t)$ *of* (1) *either oscillates or* $\liminf_{t\to+\infty} |(Lx)(t)| = 0$.

Proof. Let $x(t)$ be a regular solution of (1) and for definiteness assume that $x(t) \geqq 0$ for $t \in \tilde{J} = [\tilde{t}, +\infty)$, $\tilde{t} \in \mathbb{R}$. For the assumption of the theorem to be fulfilled it is sufficient to show that if $\liminf_{t\to+\infty} (Lx)(t) > 0$ then

$$\textstyle\int_{\tilde{t}}^{+\infty} (\int_{I_t} K(t, s, x(s))\, \mathrm{d}s)\, \mathrm{d}t = +\infty\,. \tag{11}$$

Assume that $\liminf_{t\to+\infty} (Lx)(t) > 0$. Then for every $t \in \tilde{J}$ the equality

$$\textstyle\int_{\tilde{t}}^{T} (\int_{I_t} K(t,s,x(s))\, \mathrm{d}s)\, \mathrm{d}t = \int_{\tilde{t}}^{T} (\int_{I_t\cap J} K(t,s,x(s)\, \mathrm{d}s)\, \mathrm{d}t + \int_{\tilde{t}}^{T} (\int_{I_t\setminus J} K(t,s,x(s)\, \mathrm{d}s)\, \mathrm{d}t \tag{12}$$

holds.

The first integral on the right-hand side of equality (12) is positive for every $T > \tilde{t}$, it can be estimated as in the proof of Theorem 1 and for $T > \tilde{t}$ the following estimate holds:

$$\textstyle\int_{\tilde{t}}^{T} (\int_{I_t\cap J} K(t, s, x(s)\, \mathrm{d}s)\, \mathrm{d}t \geqq \gamma \int_{E\cap[t',+\infty)} \operatorname{meas} \{t \mid t \in [t', T],\ s \in I_t\}\, \mathrm{d}s\,.$$

The sets $I_t \setminus \tilde{J}$ are uniformly bounded for $t \geqq \tilde{t}$ and, taking into account condition 2 of Theorem 2, we conclude that the modulus of the second integral on the right-hand side of equality (12) tends to $+\infty$ as $O(T)$.

Hence from equality (12), taking into account (10) and passing to the limit for $T \to +\infty$, we conclude that relation (11) holds. This completes the proof of Theorem 2.

Remark 4. It is not difficult to see that if the relation

$$\lim_{T\to+\infty} (T^{-1} \textstyle\int_E \inf_{t_0\leqq\sigma\leqq s} \operatorname{meas} \{t \mid t_0 \leqq t \leqq T,\ \sigma \in I_t\}\, \mathrm{d}s = +\infty \tag{13}$$

holds, then condition (10) is also fulfilled.

Remark 5. Let $\sup_{t\in J} s_t' < +\infty$ and let $s_t''$ be a locally integrable function. Then condition (13) assumes the following form:

$$\lim_{T\to+\infty} (T^{-1} \textstyle\int_{[t_0,T]\cap\{t \mid s_t''\geqq t_0\}} s_\xi''\, \mathrm{d}\xi = +\infty\,.$$

Example 2. An example illustrating Theorem 2 can be obtained by putting $I_t = [0, t]$. That is, equation (1) contains the Volterra type integro-differential equations as a particular case.

*References*

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