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A REMARK ON C^*-ALGEBRAS

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Besides the characterization of C^*-algebras due to Gelfand and Naimark [G-N] some others have been given [G1], [Vi 1]. From the point of view of physics the most satisfactory characterizations are those dealing with selfadjoint elements only. This may be found in the paper by Behncke [Be], unfortunately, not complete (see MR 39 #4685). Nevertheless, it can be shown that the condition of positivity of squares for selfadjoint commuting elements enables to prove the desirable result.

Theorem. *Let $\mathscr{A}$ be an algebra with involution and an identity element e. Denote by $\mathscr{S}$ the set of all selfadjoint elements of $\mathscr{A}$. Suppose that $\mathscr{S}$ is a real Banach space with a norm $|\cdot|$ and*

$1°$ $|u^2| = |u|^2$ *for* $u \in \mathscr{S}$;

$2°$ $|u^2 + v^2| \geqq \max(|u^2|, |v^2|)$ *for* $u, v \in \mathscr{S}$, $uv = vu$ (positivity of squares).

Then it is possible to equip $\mathscr{A}$ with a norm $| \ |_0$ which is an extension of the norm $|\cdot|$ and $(\mathscr{A}, |\cdot|_0)$ is a C^*-algebra.

Proof. Let $\mathscr{A}_1$ be a maximal commutative *-subalgebra of $\mathscr{A}$. Denote by $\mathscr{S}_1 = \mathscr{S} \cap \mathscr{A}_1$.

For $x, y \in \mathscr{S}_1$ and all real t,

$$4\,txy = (x + ty)^2 - (x - ty)^2\,.$$

Hence

$$4|t|\,|xy| \leqq |x + ty|^2 + |x - ty|^2 \leqq 2(|x|^2 + 2|t|\,|x|\,|y| + t^2|y|^2)\,.$$

It follows

$$2|xy| \leqq |t|^{-1}\,|x|^2 + 2|x|\,|y| + |t|\,|y|^2$$

for all real t. By minimizing the right-hand side we get $|xy| \leqq 2|x|\,|y|$ (see also [Vi 2]).

If we set $|x|_1 = 2|x|$ for $x \in \mathscr{S}$ we obtain $|xy|_1 \leqq |x|_1\,|y|_1$ for $x, y \in \mathscr{S}_1$ and

$$|u^2|_1 = 2|u^2| = 2|u|^2 = (\sqrt{2}\,|u|)^2 = (2^{-1/2}|u|_1)^2 = 2^{-1}|u|_1^2\,. \tag{1}$$

Further, for $z \in \mathscr{A}_1$, decompose z into selfadjoint parts, i.e. $z = x + iy$ with $x, y \in \mathscr{S}_1$ and set $|z|_1 = |x|_1 + |y|_1$.

For $\varphi = \xi + i\eta,\ \xi, \eta \in \mathscr{S}_1$

$$|z\varphi|_1 = |x\xi - y\eta|_1 + |x\eta + \xi y|_1 \leqq |z|_1\,|\varphi|_1\,,$$

$$|z + \varphi|_1 \leqq |z|_1 + |\varphi|_1\,,$$

$$|tz|_1 = |t|\,|z|_1 \quad \text{for } t \text{ real}$$

and

$$|\lambda z|_1 = |\lambda_1 x - \lambda_2 y|_1 + |\lambda_1 y + \lambda_2 x|_1 \leqq \sqrt{2}\,|\lambda|\,|z|_1$$

for complex $\lambda = \lambda_1 + i\lambda_2$.

Since $z^* = x - iy$ it follows from 1° and 2°

$$|z|_1^2 = |x|_1^2 + 2|x|_1\,|y|_1 + |y|_1^2 \leqq 4\max\left(|x|_1^2, |y|_1^2\right) =$$
$$= 16\max\left(|x|^2, |y|^2\right) = 16\max\left(|x^2|, |y^2|\right) \leqq 16|x^2 + y^2| =$$
$$= 8|x^2 + y^2|_1 = 8|zz^*|_1\,.$$

Finally, if we set $|z|_2 = \sup\limits_{0 \leqq \vartheta \leqq 2\pi} |e^{i\vartheta}z|_1$ we obtain a norm on $\mathscr{A}_1$ which is equivalent to $|\cdot|_1$.

$$|z\varphi|_2 \leqq |z|_2\,|\varphi|_2$$

and

(2) $$|z|_2^2 \leqq 8|zz^*|_2\,.$$

The completion $\tilde{\mathscr{A}}_1$ of the algebra $(\mathscr{A}_1, |\cdot|_2)$ is obviously a commutative algebra. Assume $\{z_n\}$ a Cauchy sequence in $\mathscr{A}_1$. Then $\{\operatorname{Re} z_n\}$ and $\{\operatorname{Im} z_n\}$ are Cauchy sequences in $\mathscr{S}_1$. Since $\mathscr{S}$ is complete the both sequences have a limit in $\mathscr{S}$ so that $\{z_n\}$ has a limit in $\mathscr{A}$ and $\tilde{\mathscr{A}}_1 \subseteq \mathscr{A}$. $\mathscr{A}_1$ is maximal commutative *-subalgebra of $\mathscr{A}$. This implies that $\mathscr{A}_1 = \tilde{\mathscr{A}}_1$ and $\mathscr{A}_1$ is complete. It follows from the maximality of $\mathscr{A}_1$ that $\sigma_{\mathscr{A}_1}(x) = \sigma_{\mathscr{A}}(x)$ so that $|x|_\sigma = \lim |x^n|_2^{1/n}$ for $x \in \mathscr{A}_1$.

Now take a $z \in \mathscr{A}_1$. Since $z^n \in \mathscr{A}_1$ as well, we get, according to (2), that

$$|z^n|_2^2 \leqq 8|z^n z^{n*}|_2 = 8|(zz^*)^n|_2\,,$$

and consequently,

(3) $$|z|_\sigma^2 \leqq |zz^*|_\sigma\,.$$

Similarly as in [Pt] (5,10) we shall show now that spectra of selfadjoint elements are real. Assume an $h = h^*$ in $\mathscr{A}_1$ such that $\alpha + i\beta \in \sigma(h)$ (with $\beta \neq 0$). Set $a = \beta^{-1}(h - \alpha)$, so that $a = a^*$ and $i \in \sigma(a)$. Then, for real τ, $i(\tau + 1) \in \sigma(a + \tau\, ie)$. Using subadditivity of the spectral radius on $\mathscr{S}$ and according to (3) we get

$$(\tau + 1)^2 \leqq |a + \tau\, ie|_\sigma^2 \leqq |(a - \tau\, ie)(a + \tau\, ie)|_\sigma =$$
$$= |a^2 + \tau^2 e|_\sigma \leqq |a^2|_\sigma + \tau^2|e|_\sigma = |a^2|_\sigma + \tau^2\,.$$

Hence $2\tau + 1 \leqq |a^2|_\sigma$ for all real τ, which is impossible. It follows that, for $u = u^*$, $\sigma(u^2)$ is nonnegative so that $e + u^2$ has an inverse in $\mathscr{A}_1 \subseteq \mathscr{A}$. According to [Vi 2] the algebra $\mathscr{A}$ equipped with the norm $|z|_0 = |zz^*|^{1/2}$ is a C^*-algebra. It follows from 1° that $|u| = |u|$ for $u = u^*$.

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