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SUPERLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

In this paper we consider nonlinear elliptic boundary value problems of the type

$$(1.1)\qquad \begin{aligned} \Delta u + \lambda_1 u + g(u) &= f(x)\,, && x \text{ in } \Omega\,,\\ Bu &= 0\,, && x \text{ on } \partial\Omega\,, \end{aligned}$$

where Ω is a smooth bounded domain in R^N and λ_1 is the smallest eigenvalue of $-\Delta$ together with the homogeneous boundary conditions $Bu = 0$. The nonlinear function $g: R \to R$ considered here is superlinear and is one of the following two types:

(i) $\lim_{|u|\to\infty} g(u) = \infty$ (or $-\infty$), referred here as "two-side unbounded" and g is superlinear;

(ii) $g(u)$ approaches ∞ (or $-\infty$) in one side superlinearly and g is bounded in the other, referred as "one-side unbounded".

Some typical examples of nonlinear problems to which the results of this paper may be applied are as follows:

$$(a)\qquad \begin{aligned} \Delta u + \lambda u - e^u &= f(x)\,, && x \in \Omega\,,\\ u &= 0\,, && x \in \partial\Omega \end{aligned}$$

for $\lambda > \lambda_1$ (the case when $\lambda \leqq \lambda_1$ has been well-studied in the literature).

$$(b)\qquad \begin{aligned} \Delta u + \lambda_1 u + \alpha|u|^p u^+ + \beta|u|^q u^- + \psi(u) &= f(x)\,, && x \text{ in } \Omega\,,\\ u &= 0\,, && x \text{ on } \partial\Omega\,, \end{aligned}$$

where the restrictions on α, β, p and q may be seen later.

The nonlinearities here are referred to as "jumping nonlinearities" in the literature.

$$(c)\qquad \begin{aligned} \Delta u + u^2 &= f(x)\,, && x \in \Omega \subset R^N \text{ and } N \leqq 2\,,\\ u &= 0\,, && x \in \partial\Omega\,. \end{aligned}$$

$$(d)\qquad \begin{aligned} \Delta u + g(u) &= f(x)\,, && x \in \Omega\,,\\ \partial u/\partial n &= 0\,, && x \in \partial\Omega\,, \end{aligned}$$

where $g(u) = \begin{cases} |u|\,, & u \leqq 0\\ a \sin u\,, & u > 0\,. \end{cases}$

It must be noted here that the results of this paper imply the existence of a solution to (d) if $\int_\Omega f(x)\,dx \geqq 0$ in contrast to what one would obtain by applying Landesman-Lazer type results [8, 12].

By appropriately rewriting (a), (b), (c) in the form (1.1) it can be seen that they are examples of "two-side unbounded" nonlinearities whereas (d) is an example of the "one-side unbounded" type. A discussion of how the results for the above examples compare with the existing literature may be seen in Section 4.

The qualitative type of results obtained in this paper may be briefly described as follows: rewriting (1.1) as

$$\Delta u + \lambda_1 u + g(u) = f_0\, \phi(x) + f_1(x)\,, \quad Bu = 0\,, \tag{1.2}$$

where $\phi(x)$ is the eigenfunction associated with λ_1, and f_1 is orthogonal to ϕ, we obtain estimates on f_0 in order that (1.2) has a solution for a given $f_1(x)$. We then show that the set of f_0 for which (1.2) is solvable for a given $f_1(x)$ is a semibounded interval. Thus an auxiliary functional can be defined mapping $f_1(x)$ to the finite extreme point of this interval. We use this functional to discuss the multiplicity of solutions of (1.2). Three general techniques are utilised in this paper: a priori bounds, upper and lower solutions, the Leray-Schauder degree.

2. THE "TWO-SIDE UNBOUNDED" CASE

Let Ω be a bounded domain in R^N, $N \geqq 1$, with a smooth boundary $\partial\Omega$ and such that the maximum principle holds, and let λ_1 be the first eigenvalue of $-\Delta$ together with Dirichlet boundary conditions and ϕ the corresponding normalized eigenfunction which is known to be nonnegative. With respect to the nonlinear function $g: \bar{\Omega} \times R \to R$ we assume that:

(2.1) g is locally Lipschitzian in u (uniformly in x) and α-Hölder continuous in x so that the corresponding Nemytskii operator generated by g is well-defined from $C^{0,\alpha}$ to $C^{0,\alpha}$.

$$\lim_{|u|\to\infty} g(x, u) = \infty \quad \text{uniformly in} \quad x \in \bar{\Omega}\,; \tag{2.2}$$

$$\lim_{u\to-\infty} [g(x, u) + \lambda_1 u] = \infty \quad \text{uniformly in} \quad x \in \bar{\Omega}\,; \tag{2.3}$$

(2.4) there exist $\gamma, \beta \in L^\infty(\Omega)$ with $\gamma, \beta \geqq 0$ such that

$$|g(x, u)| \leqq \gamma(x)\, u^\sigma + \beta(x)\,, \quad \text{for} \quad x \in \bar{\Omega} \quad \text{and} \quad u \geqq 0\,,$$

where $\sigma < (N + 1)/(N - 1)$ if $N \geqq 2$.

For any $f \in C^{0,\alpha}(\bar{\Omega})$ we consider, for the existence and multiplicity of classical solutions the nonlinear problem,

$$\Delta u + \lambda_1 u + g(x, u) = f(x)\,, \quad x \in \Omega\,, \qquad u = 0\,, \quad x \in \partial\Omega\,. \tag{2.5}$$

STEP I. A Priori Bounds

Lemma 2.1. *Any possible solution $u(x)$ of (2.5), for a given f, satisfies*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqq \varrho(\|f\|_{C^{0,\alpha}})$$

where ϱ is a non decreasing function (depending only on g).

Note. Throughout the rest of this section C will denote a constant indepedent of u and instead of using various symbols for constants in the inequalities, we will use C always.

Proof. Let $u(x)$ be a solution of (2.5). Then $u(x)$ can be decomposed as $u = u_0\phi + u_1$ where u_1 is orthogonal to ϕ.

By applying maximum principle arguments and utilizing (2.2), (2.3) we derive that for any solution $u(x)$ of (2.5)

$$u(x) \geqq C\,, \quad x \in \Omega$$

and

$$|g(x, u(x))| \leqq \gamma(x)\,|u(x)|^\sigma + \beta^*(x)\,, \quad x \in \Omega\,, \tag{2.6}$$

where $\beta^*(x) = \beta(x) + \max\{|g(x, \xi)|: C \leqq \xi \leqq 0,\ x \in \bar{\Omega}\}$.
Then, from (2.5) and Garding's inequality, there exists $C > 0$ such that

$$\|u_1\|^2_{W^{1,2}} \leqq C\left[\int_\Omega g(x, u)\, u_1\, dx - \int_\Omega f u_1\, dx\right]. \tag{2.7}$$

Also

$$\int_\Omega g(x, u)\, u_1\, dx = \int_\Omega g^\mu g^{1-\mu} u_1\, dx\,,$$

where $\mu \in (0, 1)$ is to be chosen later. Then

$$\int_\Omega g(x, u)\, u_1\, dx \leqq \left[\int_\Omega |g|\, \phi\, dx\right]^\mu \left[\int_\Omega \frac{|g|\, |u_1|^{1/1-\mu}}{\phi^{\mu/1-\mu}}\, dx\right]^{1-\mu}. \tag{2.8}$$

Further, as $\mathrm{g} = \inf\{g(x, u): x \in \bar{\Omega},\ u \in R\} > -\infty$, then

$$\int_\Omega |g|\, \phi\, dx \leqq \int_\Omega g\phi\, dx + 2|\mathrm{g}| \int_\Omega \phi\, dx\,.$$

Since u is a solution to (2.5), $\int_\Omega g\phi\, dx = \int_\Omega f\phi\, dx$. Thus

$$\int_\Omega |g|\, \phi\, dx \leqq \int_\Omega f\phi\, dx + 2|\mathrm{g}| \int_\Omega \phi\, dx \leqq C^{1/\mu}\,.$$

From (2.8),

$$\int_\Omega g(x, u)\, u_1\, dx \leqq C\left[\int_\Omega \frac{|g|\, |u_1|^{1/1-\mu}}{\phi^{\mu/1-\mu}}\, dx\right]^{1-\mu} \leqq$$

$$\leqq C\left[\int_\Omega \frac{(\gamma|u|^\sigma + \beta^*)\, |u_1|^{1/1-\mu}}{\phi^{\mu/1-\mu}}\, dx\right]^{1-\mu} \leqq$$

$$\leqq \left[\|\gamma\|_\infty \int_\Omega \frac{|u|^\sigma |u_1|^{1/1-\mu}}{\phi^{\mu/1-\mu}} \, dx + \|\beta^*\|_\infty \int_\Omega \frac{|u_1|^{1/1-\mu}}{\phi^{\mu/1-\mu}} \, dx \right]^{1-\mu}.$$

We now let $\mu = 2/(N+1)$ and denote by $K(\sigma)$ the jonstant such that

$$|u|^\sigma = |u_0\phi + u_1|^\sigma \leqq K(\sigma) \left[|u_0|^\sigma + |u_1|^\sigma\right].$$

Then by using the inequality ([4, 6])

$$\left\| \frac{v}{\phi^\tau} \right\|_q \leqq C\|v\|_{W^{1,2}}$$

for $v \in H_0^{1,2}(\Omega)$, $0 \leqq \tau \leqq 1$ and

$$\frac{1}{q} = \frac{1}{2} - \frac{1-\tau}{N}$$

we have

$$\int_\Omega g(x, u)\, u_1 \, dx \leqq C\left[|u_0|^{\sigma((N-1)/(N+1))} \|u_1\|_{W^{1,2}} + \|u_1\|_{W^{1,2}}^{\sigma((N-1)/(N+1))+1} + \|u_1\|_{W^{1,2}}\right].$$

From (2.7) we conclude

$$\|u_1\|_{1,2} \leqq C\left[|u_0|^{\sigma((N-1)/(N+1))} + \|u_1\|_{W^{1,2}}^{\sigma((N-1)/(N+1))} + 1\right].$$

Hence, there exist constants C, D such that

(2.9) $$\|u_1\|_{W^{1,2}} \leqq C|u_0|^\lambda + D, \quad \lambda < 1.$$

Since $u(x)$ is a solution to (2.5) we have

$$\int_\Omega g(x, u_0\phi + u_1(x))\, \phi(x) \, dx \leqq \|f\|_\infty \int_\Omega \phi \, dx$$

and this in conjunction with (2.9) we conclude that

$$\|u\|_{W^{1,2}} \leqq \tilde{\varrho}(\|f\|_\infty).$$

By using standard bootstrap and regularity arguments from the theory of elliptic p.d.e. it follows that

$$\|u\|_{C^{2,\alpha}} \leqq \varrho(\|f\|_{C^{0,\alpha}}).$$

Remark. The proof of Lemma 2.1 follows along the lines of [4].

Before we proceed to the next step we introduce the set-valued mapping R defined as follows: for any f_1 (fixed) such that $\int_\Omega f_1\phi \, dx = 0$ let

(2.10) $R(f_1) = \{\mu \in R$: there exists a solution of (2.5) with

$$f(x) = f_1(x) + \mu\, \phi(x)\}.$$

Let g be defined by

(2.11) $$g = \operatorname*{Inf}_{u \in R} g(u) < \infty.$$

Then, multiplying (2.5) by ϕ and integrating, it follows that

(2.12) $$R(f_1) \subset [g, \infty).$$

STEP II. Qualitative Properties of R

Lemma 2.2. *If* $\mu \in R(f_1)$, *then* $[\mu, \infty) \subset R(f_1)$.

Proof. We prove this by using the method of upper and lower solutions. For any $v \geqq \mu \geqq 0$ it follows from

$$\Delta u + \lambda_1 u = f_1(x) - g(x, u) + \mu\, \phi(x) \leqq f_1(x) - g(x, u) + v\, \phi(x),$$

that the solution $u(x)$ corresponding to the μ-problem (i.e., $\Delta u + \lambda_1 u + g(x, u) = f_1(x) + \mu\, \phi(x)$) is an upper solution for the v-problem. A lower solution to the v-problem can be easily constructed such that the upper and lower solutions are ordered. Hence there exists a solution to v-problem, i.e., $v \in R(f_1)$.

Lemma 2.3. $R(f_1)$ *is closed and nonempty.*

Proof. Let $\mu_n \to \mu$ and $\mu_n \in R(f_1)$. Then from the a priori estimates of Step 1 it follows that $\|u_n\|_{C^{2,\alpha}}$ is equibounded where u_n is a solution corresponding to the μ_n-problem. This implies that a limit a point u of a subsequence $\{u_n\}$ is a solution of

$$\Delta u + \lambda_1 u + g(x, u) = f_1 + \mu\phi,$$
$$u = 0,$$

i.e., $u \in R(f_1)$ implying $R(f_1)$ is closed.

Finally, let $\tilde{f}_1$ be the unique function satisfying

$$\Delta \tilde{f}_1 + \lambda_1 \tilde{f}_1 = f_1 - g_1(x, 0) \quad \text{in} \quad \Omega,$$

where $g(x, 0) = g_1(x, 0) + g_0\phi$, g_1 orthogonal to ϕ, $\int_\Omega \tilde{f}_1\phi\, dx = 0$ and $\tilde{f}_1 = 0$ on $\partial\Omega$. Further let

$$\mu_0 = \sup\left\{\frac{g(x, \tilde{f}_1(x)) - g(x, 0)}{\phi(x)} : x \in \Omega\right\}.$$

Then it follows from the Lipschitz property in (2.1) that $\mu_0 < \infty$ and $\tilde{f}_1$ is an upper solution for the $\mu_0 + g_0$-problem. Proceeding as in Lemma 2.2, we conclude that $\mu_0 + g_0 \in R(f_1)$ i.e., $R(f_1)$ is nonempty.

The above lemmas in Step 2 enable us to define the functional Φ as follows:

$$\Phi : f_1 \in C^{0,\alpha} \to R, \quad \int_\Omega f_1\phi = 0 \quad \text{and} \quad \Phi(f_1) = \min R(f_1). \tag{2.13}$$

STEP III. Existence of Solutions

It now easily follows from the above considerations that:

Theorem 2.1. *The nonlinear problem*

$$\Delta u + \lambda_1 u + g(x, u) = f_1 + \mu\phi, \quad \textit{in} \quad \Omega,$$
$$u = 0, \quad \textit{on} \quad \partial\Omega$$

has a solution if and only if $\mu \geqq \Phi(f_1)$.

STEP IV. Existence of Multiple Solutions

We first note that if $\mu > \Phi(f_1)$, then a solution of the $\Phi(f_1)$-problem is a strict upper solution for the μ-problem. As remarked before, a strict lower solution can be easily chosen. Then, following the lines of [6], it can be seen that there exists a constant $K > 0$ and a bounded, open subset θ_μ of the space made up of all functions u in $C^{0,\alpha}(\bar{\Omega})$ satisfying $u = 0$ on $\partial\Omega$ such that

$$\deg(I - T_{\mu,f_1}, \theta_\mu, 0) \neq 0, \tag{2.14}$$

where T_{μ,f_1} is defined as follows:

$$T_{\mu,f_1}v = 0 \quad \text{and} \quad (\Delta - KI)\,u + g(x, v) + (\lambda_1 + K)\,v = f_1(x) + \mu\,\phi(x)\,, \quad \text{in} \quad \Omega$$

together with $u = 0$ on $\partial\Omega$. In other words T_{μ,f_1} may also be denoted by

$$T_{\mu,f_1} = (\Delta - KI)^{-1}\left[-g - (\lambda_1 + K)I + f_1 + \mu\,\phi(x)\right].$$

By virtue of the a priori estimates of Step I, there exists a ball B containing θ_μ such that $I - T_{\lambda,f_1} \neq 0$ on ∂B where $\Phi(f_1) \leqq \lambda \leqq \mu$. This implies that $\deg(I - T_{\lambda,f_1}, B, 0)$ is a constant for $\lambda \in [\Phi(f_1), \mu]$. Also $\deg(I - T_{\nu,f_1}, B, 0) = 0$ for all $\nu < \Phi(f_1)$, because there are no solutions when $\nu < \Phi(f_1)$. Hence, by the continuity of the Leray-Schauder degree, we conclude that

$$\deg(I - T_{\Phi(f_1),f_1}, B, 0) = 0\,.$$

The excision property of the Leray-Schauder degree then enables us to obtain the existence of a second solution for the μ-problem where $\mu > \Phi(f_1)$.

We now discuss the continuity of the functional Φ. Let $f_{1n} \to f_1$ in $C^{0,\alpha}$ where f_{1n} is orthogonal to ϕ. That $\{\Phi(f_{1n})\}$ is bounded may be seen from the μ_0 introduced in the proof of Lemma 2. Let $\{f_{1k}\}$ be a subsequence such that $\Phi(f_{1k}) \to \mu$. Further let u_k be a solution of the $\Phi(f_{1k})$-problem, i.e.,

$$\Delta u_k + \lambda_1 u_k + g(x, u_k) = f_{1k} + \Phi(f_{1k})\,\phi\,, \quad \text{in} \quad \Omega\,,$$
$$u_k = 0\,, \quad \text{on} \quad \partial\Omega\,.$$

Passing to the limits (using subsequences) we derive that $\mu \geqq \Phi(f_1)$ and

$$\liminf \Phi(f_{1n}) \geqq \Phi(f_1)\,.$$

If $\mu > \Phi(f_1)$, let θ be such that $\deg(I - T_{\mu,f_1}, \theta, 0) \neq 0$. Hence for sufficiently large k, $\deg(I - T_{\mu,f_{1k}}, \theta, 0) \neq 0$ and this implies $\Phi(f_{1,k}) \leqq \mu$ for large k. Hence $\limsup \Phi(f_{1,n}) \leqq \Phi(f_1)$ and this establishes the continuity of Φ from the set of functions f_1 in $C^{0,\alpha}$ and orthogonal to ϕ into R. We can now state the following:

Theorem 2.2. *There exists a functional Φ: $C^{0,\alpha} \cap$ [functions orthogonal to ϕ] $\to R$ which is continuous and such that the nonlinear problem*

$$\Delta u + \lambda_1 u + g(x, u) = f_1(x) + f_0(x)\,, \quad \textit{in} \quad \Omega\,,$$
$$u = 0\,, \quad \textit{on} \quad \partial\Omega$$

has:

a) *no solution if* $f_0 < \Phi(f_1)$;

b) *at least one solution if* $f_0 = \Phi(f_1)$;

c) *at least two solutions if* $f_0 > \Phi(f_1)$.

Remark 1. All the results above held when (2.2) and (2.3) are replaced by

$$\lim_{|u| \to \infty} g(x, u) = -\infty \quad \text{uniformly in} \quad x \in \bar{\Omega},$$

$$\lim_{u \to \infty} [g(x, u) + \lambda_1 u] = -\infty \quad \text{uniformly in} \quad x \in \bar{\Omega}.$$

Growth hypotheses on g analogous to (2.4) must also be assumed.

3. THE "ONE-SIDE UNBOUNDED" CASE

In this section we consider the nonlinear problem

$$\Delta u = g(u) - f(x), \quad x \in \Omega, \qquad \partial u / \partial n = 0, \quad x \in \partial \Omega, \tag{3.1}$$

where $g: R \to R$ satisfies:

(3.2) g is locally Lipschitz, i.e., there exists for each bounded subset K of R a constant $c(K)$ such that

$$|g(u) - g(v)| \leqq c|u - v|, \quad u, v \in K;$$

$$\lim_{u \to -\infty} g(u) = \infty; \tag{3.3}$$

$$|g(u)| \leqq M \quad \text{for} \quad u \geqq 0; \tag{3.4}$$

(3.5) there exist α, β and p such that

$$|g(u)| \leqq \alpha |u|^p + \beta$$

for all $u \in R$ where $1 \leqq p < \infty$ if $N = 2$ and $1 \leqq p < N/(N - 2)$ if $N > 2$.

Let us assume that $f(x)$ in (3.1) is in $C^{0,\alpha}(\bar{\Omega})$ for $\alpha \in (0, 1)$. We now introduce the truncated function g_n defined by

$$g_n(u) = \begin{cases} g(-n) & \text{if} \quad u \leqq -n \\ g(u) & \text{if} \quad u \geqq -n, \end{cases}$$

for any positive integer n. We first note that g_n is also a Lipschitz function from R to R. Associated with (3.1) we thus have the nonlinear problem

$$\Delta u = g_n(u) - f(x), \quad x \in \Omega, \qquad \partial u / \partial n = 0, \quad x \in \partial \Omega. \tag{3.6}$$

Clearly, if u is a solution of (3.1), it must be a solution of (3.6) for some $n = n_0$ and hence a solution for all $n \geqq n_0$.

STEP I. A Priori Bounds

We now obtain a priori bounds on the solutions of (3.6). Thus let $u = u_0 + u_1$, $u_0 \in R$ and $\int_\Omega u_1 \, dx = 0$, be a solution (3.6) for an arbitrary but fixed n. Then we know that there exists $a_q > 0$ such that

$$\|u_1\|_{W^{2,q}} \leqq a_q \|\Delta u\|_q , \quad 1 < q < \infty .$$

But

$$\|\Delta u\|_q \leqq \|g_n(u)\|_q + \|f\|_q .$$

Further

$$\|g_n(u)\|_q = \left[\int_\Omega |g_n(u)|^q \, dx\right]^{1/q} \leqq$$

$$\leqq \left[(\alpha\|u\|_\infty^p + \beta)^{q-1} \int_\Omega |g_n(u)| \, dx\right]^{1/q},$$

by virtue of (3.5). Also noting that $\int_\Omega g_n(u)\, dx = \int_\Omega f \, dx$ implies $\int_\Omega |g(u)| \, dx \leqq \leqq \int_\Omega f \, dx + 2[\inf g(\xi)] \text{ meas } (\Omega) = \gamma_1$, we have

$$\|g_n(u)\|_q \leqq \gamma_1^{1/q}(\alpha\|u\|^p + \beta)^{(q-1)/q} .$$

As in Section 2 let $K_1 > 0$ be such that

$$(|a| + |b|)^p \leqq K_1(|a|^p + |b|^p)$$

for all $a, b \in R$. Thus

$$\|g_n(u)\|_q \leqq \gamma_1^{1/q}(\alpha K_1|u_0|^p + \alpha K_1\|u_1\|^p + \beta)^{(q-1)/q} \leqq$$

$$\leqq K\gamma_1^{1/q}[(\alpha K_1)^{(q-1)/q} |u_0|^{p((q-1)/q)} + (\alpha K_1)^{q-1/q} \|u_1\|_\infty^{p(q-1)/q} + \beta^{(q-1)/q}] .$$

Since

$$p < \frac{N}{N-2} = \frac{N/2}{(N/2) - 1},$$

we can choose $q > N/2$ such that $p < q/(q - 1)$ and then let K_2 be such that

$$\|u\|_{W^{2,q}} \geqq K_2\|u\|_\infty \quad \text{for} \quad u \in W^{2,q}(\Omega) .$$

Letting $\sigma = p(q - 1)\, q^{-1} < 1$ we have

$$K_2\|u_1\|_\infty \leqq \|u_1\|_{W^{2,q}} \leqq a_q\|\Delta u\|_q \leqq \alpha_1|u_0|^\sigma + \alpha_1\|u_1\|_\infty^\sigma + \beta_1 + \|f\|_q$$

and thus we conclude that

$$\|u_1\|_\infty \leqq a|u_0|^\sigma + b ,$$

where a and b are independent of the "n" in (3.6).

STEP II. Equivalence of Solvability of (3.1) and (3.6) (For Large N)

Let $\xi_0 > 0$ be such that $g(\xi) > (\text{meas } \Omega)^{-1} \int_\Omega f \, dx$ for all $\xi \leqq -\xi_0$ (using (3.3)). Then $g_n(\xi) > (\text{meas } \Omega)^{-1} \int_\Omega f \, dx$ for all $\xi \leqq -\xi_0$ and $n \geqq \xi_0$.

Now let $\xi_1 > 0$ be such that $\xi + a|\xi|^\sigma + b < -\xi_0$ for $\xi \leqq -\xi_1$.

If $u_0 \leqq -\xi_1$, $u_0 + u_1(x) \leqq u_0 + \|u_1\|_\infty \leqq u_0 + a|u_0|^\sigma + b < -\xi_0$. Then

$g_n(u(x)) > (\text{meas}\,\Omega)^{-1} \int_\Omega f\,dx$ for all $x \in \Omega$ and $n \geqq \xi_0$. However for any solution u of (3.6) we have $\int_\Omega f\,dx = \int_\Omega g_n(u)\,dx$. Thus all possible solutions of (3.6) for $n \geqq \xi_0$ satisfy

$$u_0 \geqq -\xi_1 \quad \text{and} \quad \|u_1\|_\infty \leqq a|u_0|^\sigma + b\,.$$

Denoting by $M = \inf\{\xi - a|\xi|^\sigma - b : \xi \geqq -\xi_1\}$ we have for any possible solution of (3.6) for $n \geqq \xi_0$

$$u(x) \geqq u_0 - a|u_0|^\sigma - b \geqq M \quad \text{for all} \quad x \in \bar{\Omega}\,.$$

Let $n_0 > \max(|M|, \xi_0)$. Then if u_0 is a solution of (3.6) for $n \geqq n_0 \geqq \xi_0$, we have $u(x) \geqq M$ for $x \in \bar{\Omega}$ and then by the definition of g_n,

$$g_n(u(x)) = g(u(x))\,, \quad x \in \bar{\Omega}\,.$$

This implies that $u(x)$ is a solution of (3.1).

It is easy to see that if $u(x)$ is a solution of (3.1) then $u(x) \geqq M$ and $u(x)$ is a solution of (3.6) for $n \geqq n_0$.

Summarizing we have:

There exists n_0 such that the solution set of (3.1) coincides with the solution set of (3.6) for $n \geqq n_0$.

STEP III. Definition and Qualitative Properties of $R(f_1)$

As in Section 2. we write $f(x) = f_0 + f_1(x)$ with $\int_\Omega f_1\,dx = 0$ and define $R(f_1)$ as follows:

$$R(f_1) = \{\lambda \in R : \Delta u = g(u) - (f_1 + \lambda) \text{ in } \Omega\,,$$
$$\partial u/\partial n = 0 \text{ on } \partial\Omega \text{ has at least one solution}\}\,.$$

We now recall two results which will be used in the following discussions.

Lemma 3.1. [7] *If g is locally Lipschitz and bounded with $g(\infty) = \limsup_{u\to\infty} g(u)$, $g(-\infty) = \liminf_{u\to-\infty} g(u)$ then the boundary value problem*

$$\text{(3.7)} \qquad \Delta u = g(u) - f(x)\,, \quad \text{in} \quad \Omega$$
$$\partial u/\partial n = 0\,, \quad \text{on} \quad \partial\Omega$$

has at least one classical solution for $f(x) \in C^{0,\alpha}(\bar{\Omega})$ if

$$g(-\infty) > (\text{meas}\,\Omega)^{-1}\int_\Omega f\,dx > g(\infty)\,.$$

Lemma 3.2. [1] *If g is locally Lipschitz nad bounded, the existence of an upper and lower solution (not necessarily ordered) implies the existence of a solution of* (3.7).

For the truncated problem (3.6) we have $g_n(\infty) = g(\infty)$ and $g_n(-\infty) = g(-n)$. Thus, by Lemma 3.1, (3.6) has at least one solution if

$$\text{(3.8)} \qquad g(-n) > (\text{means}\,\Omega)^{-1}\int_\Omega f\,dx > g(\infty)\,.$$

Further by Step 2 it follows that (3.1) has a solution if

$$(3.9) \qquad (\text{meas } \Omega)^{-1} \int_\Omega f \, dx > g(\infty) .$$

This implies that $R(f_1) \supset (g(\infty), \infty)$. Also by integration of (3.1) it follows that $R(f_1) \subset [g, \infty)$, where g is as in (2.11).

Lemma 3.3. *$R(f_1)$ is connected.*

Let $\lambda_0 \in R(f_1)$ and $\lambda > \lambda_0$. We shall show that $\lambda \in R(f_1)$. Let $\tilde{\lambda} > \max \{\lambda, g(\infty)\}$ and choose n_0 (cf. Step II) such that the problems

$$(3.10) \qquad \Delta u = g(u) - (f_1(x) + \alpha) , \quad \partial u/\partial n = 0 , \quad \alpha = \lambda_0 , \quad \tilde{\lambda}_1 \text{ and } \lambda ,$$

is equivalent to (3.6) for $n = n_0$. Then solutions of (3.10) for $\alpha = \lambda_0$, $\tilde{\lambda}$ may be treated as lower and upper sooutions of

$$(3.11) \qquad \Delta u = g_{n_0}(u) - (f_1(x) + \lambda) , \quad \text{in} \quad \Omega$$
$$\partial u/\partial n = 0 , \quad \text{on} \quad \partial\Omega .$$

By Lemma 3.2, there exists a solution of (3.11) and thus $\lambda \in R(f_1)$.

STEP IV. Existence Result for (3.1)

As in Section 2, we now define $\Phi(f_1) = \inf R(f_1)$ and summarizing the above steps we have:

Theorem 3.2. *There exists a functional Φ: $C^{0,\alpha} \cap$ [functions orthogonal to the constant functions] $\to R$ such that the nonlinear problem*

$$\Delta u = g(u) - (f_1(x) + f_0) , \quad x \in \Omega ,$$
$$\partial u/\partial n = 0 , \quad x \in \partial\Omega$$

has:

a) *no solution if $f_0 < \Phi(f_1)$;*

b) *at least one solution if $f_0 > \Phi(f_1)$.*

Remark 3.1. The results of Sections 2 and 3 also hold in the case $N = 1$. In this case, however, the polynomial growth conditions (2.4) and (3.5) are not required.

Remark 3.2. Dirichlet or Neumann boundary conditions can be used in all of the above results in Sections 2 and 3.

4. DISCUSSIONS AND REMARKS

1. Introductory remarks via the bounded case. For the sake of clarity and continuity of the discussions that follow, we first review the case when g is a bounded, continuous function. Further let

$$(4.1) \qquad g(-\infty) = \lim_{u \to -\infty} g(u) < g(u) < g(\infty) = \lim_{u \to \infty} g(u) .$$

We consider the nonlinear problem

$$\text{(4.2)} \qquad \Delta u + \lambda_1 u + g(u) = f_0\,\phi(x) + f_1\,, \quad \text{in} \quad \Omega\,, \qquad Bu = 0\,, \quad \text{on} \quad \partial\Omega\,,$$

where λ_1 is the first eigenvalue of $-\Delta$ together with the boundary conditions $Bu = 0$ and ϕ is the corresponding L^2-orthonormalised eigenfunction. Then the well-known result due to Landesman and Lazer [9] states that the condition

$$\text{(4.3)} \qquad g(-\infty)\int_\Omega \phi\,\mathrm{d}x < f_0 < g(\infty)\int_\Omega \phi\,\mathrm{d}x$$

is a necessary and sufficient condition for the existence of a solution to (4.2). Corresponding to (4.1) there exist analogous results with the inequalities reversed.

In the case when (4.1) does not hold, (4.3) is not a necessary and sufficient condition for existence. However an existence result may be stated in terms of the functional Φ. Thus we have: given $g\colon R \to R$ bounded and locally Lipschitz there exist functionals Φ_+ and Φ_- such that $\Phi_- \leqq \Phi_+$ and the nonlinear problem (4.2) has at least one solution if $\Phi_-(f_1) < f_0 < \Phi_+(f_1)$, no solutions if $f_0 < \Phi_-(f_1)$ or $f_0 > \Phi_+(f_1)$. This result can be easily proved by following the ideas and results in [cf. Theorem 3.1, 1].

Returning to the case of the problem (4.2) under the hypothesis (4.1), it can be seen that $\Phi_+ = g(\infty)\int_\Omega \phi\,\mathrm{d}x$, $\Phi_- = g(-\infty)\int_\Omega \phi\,\mathrm{d}x$ and (4.2) has no solution if $f_0 = \phi_+$ or Φ_-. However in the case of the pendulum equation

$$u'' + a\sin u = f_0 + f_1(x)\,, \quad x \in (0, 2\pi)$$
$$u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,,$$

where $a > 0$, it can be proved that there exist solutions when $f_0 = \Phi_+(f_1)$ or $f_0 = \Phi_-(f_1)$. In this case, unlike the Landesman-Lazer situation, Φ_+ and Φ_- depend on f_1.

Finally we note that under the hypotheses of Theorem 2.1 (and Theorem 3.2) $g(\infty) = \infty$ and it is in this sense that the results of this paper may be seen as an extension of the g-bounded case to the cases of g being either one-side unbounded or two-side unbounded.

2. The approach of using the functionl Φ to discuss multiplicity and existence results for nonlinear elliptic problems is motivated by the ideas in [3].

3. Examples of the two-side unbounded case. In [10] the author studies the superlinear elliptic problem

$$\text{(4.4)} \qquad \Delta u + \lambda u - \mathrm{e}^u = f(x)\,, \quad \text{in} \quad \Omega\,, \qquad \partial u/\partial n = 0\,, \quad \text{on} \quad \partial\Omega\,,$$

for the case $\lambda = \lambda_2$, the second eigenvalue of $-\Delta$ together with Neumann boundary conditions. The author remarks that the ideas in [10] are not extendable when $\lambda > \lambda_2$ and that the monotonicity was essential to the methods of the paper.

We can write (4.4) as

$$\Delta u + \lambda_1 u + (\lambda - \lambda_1) u - e^u = f_0 \phi + f(x), \quad \text{in} \quad \Omega, \qquad \partial u/\partial n = 0, \quad \text{on} \quad \partial\Omega. \tag{4.5}$$

Note that with Neumann boundary condition $\lambda_1 = 0$ and $\phi = 1$. But we have written it in the above form to illustrate more general boundary conditions. Then $g(u) = (\lambda - \lambda_1) u - e^u$ and thus for $\lambda > \lambda_1$, $g(u) \to -\infty$ as $|u| \to \infty$, i.e., g is two-side unbounded. The case $\lambda \leqq \lambda_1$ is well-known in the literature of monotone operator theory. We can now apply Theorems 2.1 and 2.2 to conclude that, when $\lambda > \lambda_1$, there exists a $C^{0,\alpha}$-continuous functional Φ satisfying the following: for every $f_1(x) \in C^{0,\alpha}(\bar{\Omega})$ and $\int_\Omega f_1 \, dx = 0$, the nonlinear problem (4.5) has at least two solutions for $f_0 > \Phi(f_1)$, at least one solution for $f_0 = \Phi(f_1)$ and no solution for $f_0 < \Phi(f_1)$.

A second class of nonlinearities to which the results of Section 2 can be applied, is the class of jumping nonlinearities [2, 7]. Thus we consider the nonlinear problem

$$\Delta u + \alpha u^+ + \beta u^- + \psi(u) = f_0 \phi(x) + f_1(x), \quad x \in \Omega, \qquad Bu = 0, \quad x \in \partial\Omega, \tag{4.6}$$

where $u^+(x) = \max\{u(x), 0\}$ and $u^- = u - u^+$. Also $\psi: R \to R$ is assumed to be Lipschitz on bounded subsets of R and is itself bounded. Rewriting (4.6) as

$$\Delta u + \lambda_1 u + \alpha u^+ + \beta u^- - \lambda_1 u + \psi(u) = f_0 \phi(x) + f_1(x), \quad x \in \Omega, \qquad Bu = 0, \quad x \in \partial\Omega$$

it follows that $g(u) = \alpha u^+ + \beta u^- + \psi(u) - \lambda_1 u$ satisfies the hypotheses of Theorem 2.1 and 2.2 if $\beta > 0$ and $\alpha > \lambda_1$. Thus there exists a functional Φ such that (4.6) has at least two solutions if $f_0 > \Phi(f_1)$, at least one solution if $f_0 = \Phi(f_1)$ and no solution if $f_0 < \Phi(f_1)$.

It can be easily seen that the above results are also valid for the class of superlinear jumping nonlinearities

$$\Delta u + \lambda_1 u + \alpha|u|^p u^+ + \beta|u|^q u^- + \psi(u) = f_0 \phi(x) + f_1(x), \qquad Bu = 0$$

under appropriate hypotheses on p and q. Related results may also be seen in [14].

4. One-side unbounded case. The results of Section 3 can be applied, for example, to the nonlinear problem

$$\Delta u + \lambda_1 u + |u|^p u^+ = f_0 \phi(x) + f_1(x), \quad x \in \Omega, \qquad Bu = 0, \quad x \in \partial\Omega.$$

With appropriate restrictions on p related to the dimension N of the underlying domain Ω, the above problem was considered as an example of one-side unbounded nonlinearities in [12] for the case of Neumann boundary conditions. For related results concerning Landesman-Lazer type sufficient conditions for the above problem and a comparison with the results in [12] we refer to [8].

Another interesting consequence of the results in Section 3 is illustrated by the following example: consider the nonlinear problem:

$$\Delta u + g(u) = f_0 + f_1(x), \quad x \in \Omega, \qquad (4.7)$$
$$\partial u/\partial n = 0, \quad x \in \partial\Omega,$$

where

$$g(u) = \begin{cases} |u|^p, & u < 0 \quad \text{and } p \text{ as in (3.5)}, \\ a \sin u, & u \geqq 0. \end{cases}$$

where $a \leqq \lambda_2$, λ_2 being the first positive eigenvalue of the Neumann problem. By the results in [5],

$$\Delta u + a \sin u = f_1(x), \quad x \in \Omega, \qquad (4.8)$$
$$\partial u/\partial n = 0, \quad x \in \partial\Omega,$$

has a solution $u(x)$ for all $f_1(x)$ such that $\int_\Omega f_1(x)\,dx = 0$. It is obvious that if $u(x)$ is a solution then $u(x) + 2k\pi$ for any integer k is a solution. Choosing k large enough it follows that (4.8) has at least one positive solution. Clearly such a solution is also a solution of (4.7) for $f_0 = 0$. As in Section 3, $R(f_1)$ is nonempty and connected. Thus, for any f_1, $\Phi(f_1) \leqq 0$. And hence for any $f_0 \geqq \Phi(f_1)$, the nonlinear problem (4.7) has at least one solution. Note that in this case $\limsup_{u\to\infty} g(u) = a$ and thus application of Landesman-Lazer type results (cf. [12]) would have only allowed to infer the existence of solutions of (4.7) for $f_0 \geqq a$. We refer to [13] for results related to this section.

5. From the results of Section 2 we can also conclude that

$$\Delta u + u^2 = f_0 + f_1(x), \quad x \in \Omega,$$
$$\partial u/\partial n = 0, \quad x \in \partial\Omega,$$

can be studied for existence and multiplicity in $\Omega \subset R^N$, $N = 2, 3$. Related results for this class of problems may be seen in [11].

6. The proof in Sections 2 and 3 in conjunction with the results in [8] leads to a similar result as in [1]: under the hypotheses of either Section 3 (or Section 2), the existence of upper and lower solutions to (3.1) (or (2.5)), no necessarily ordered, implies the existence of a solution. This leads to the question: when g is one-side unbounded and does not have proper limits, does the existence of upper and lower solutions (which are not necessarily ordered) imply the existence of a solution?

7. The quantity μ_0 introduced in Lemma 2 of Section 2 may be used to estimate Φ.

8. The question of continuity and related properties of Φ in Theorem 3.2 remains open.

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