Czechoslovak Mathematical Journal

Kunwarjit S. Bagga; Lowell W. Beineke

# Uniquely realizable score lists in bipartite tournaments

*Czechoslovak Mathematical Journal*, Vol. 37 (1987), No. 2, 323–333

Persistent URL: http://dml.cz/dmlcz/102160

Czechoslovak Mathematical Journal, 37 (112) 1987, Praha

# UNIQUELY REALIZABLE SCORE LISTS IN BIPARTITE TOURNAMENTS

KUNWARJIT S. BAGGA*) and LOWELL W. BEINEKE, Fort Wayne

(Received November 27, 1985)

## 1. INTRODUCTION

An ordinary tournament is well-known as a finite directed graph in which each pair of vertices is joined by exactly one arc. Such a digraph can be obtained by assigning an orientation to each edge of a complete graph. A *bipartite tournament* is defined as an orientation of a complete bipartite graph. Thus an $m \times n$ bipartite tournament $T$ consists of two partite sets (say) $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of vertices, and $mn$ arcs between $X$ and $Y$. If the scores (or outdegrees) of $x_i$ and $y_j$ are denoted by $a_i$ and $b_j$, respectively, for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, then $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ are called the *score lists* of $T$.

If $A$ and $B$ are lists of nonnegative integers, then the pair $(A, B)$ will be called *realizable* if there exists a bipartite tournament with score lists $A$ and $B$. Several forms of necessary and sufficient conditions for realizability are known, and we shall state these in the following section. If $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ are integer lists, we define their *dual lists* $\bar{A} = [n - a_1, n - a_2, \ldots \ldots, n - a_m]$ and $\bar{B} = [m - b_1, m - b_2, \ldots, m - b_n]$. Clearly, if $(A, B)$ is realizable, so is $(\bar{A}, \bar{B})$ — one just reverses the orientation of all the arcs in a realization.

If a pair $(A, B)$ is realizable and all its realizations are isomorphic, then $(A, B)$ will be called *uniquely realizable*. As an illustration, it is easily seen that if one of the lists in a realizable pair has all its entries as 1's (we say that this list is constantly 1), then all realizations must be isomorphic. This is indicated in Figure 1, where

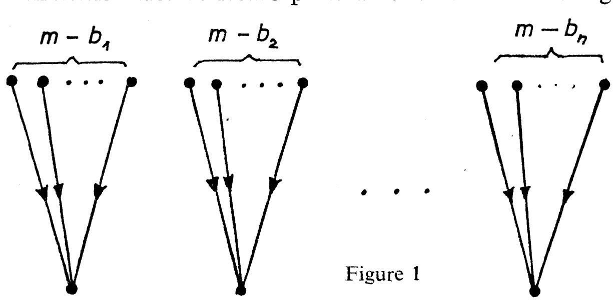


Figure 1

*) Some of the results proved in this paper form part of this author's doctoral thesis.

we assume that $A = [1, 1, \ldots, 1] = [1^m]$, $B = [b_1, b_2, \ldots, b_n]$, and that only $X$ to $Y$ arcs are shown.

We also observe that a pair $(A, B)$ is uniquely realizable if and only if $(\bar{A}, \bar{B})$ is uniquely realizable.

The purpose of this article is to determine which pairs $(A, B)$ are uniquely realizable. We shall answer this question for irreducible (i.e., strongly connected) bipartite tournaments. In this paper, we refer to irreducible components simply as components. From a known result stated in the next section (Theorem 2.4), it follows that the score lists determine a bipartite tournament up to its components. Thus a solution to the above problem for the irreducible case will lead to one for the class of all bipartite tournaments.

For recent surveys on bipartite tournaments and many of their properties, we refer the reader to [2], [3]. In the case of ordinary tournaments, Avery [1] proved that there are just four irreducible uniquely realizable score lists. We also note that our work on the uniqueness question was done independently of the work of Koren [5], who studied the uniqueness question in the context of bipartite graphs. Also, an incomplete form of this result was stated in [3].

## 2. CRITERIA FOR REALIZABILITY

There are several known criteria for determining whether or not a given pair of lists belongs to a bipartite tournament. The first that we give is a generalization of Gale's constructive criterion [4].

**Theorem 2.1.** *Let $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ be lists of integers, with $0 \leqq a_i \leqq n$ and $0 \leqq b_j \leqq m$. Suppose that $A'$ is obtained from $A$ by deleting one entry $a_i$, and $B'$ is obtained from $B$ by reducing the largest $n - a_i$ entries by 1 each. Then $(A, B)$ is realizable if and only if $(A', B')$ is realizable.*

The crucial idea in the proof of the necessary part of this result is to show that in a bipartite tournament any vertex in one set can dominate vertices of minimum scores in the other set (up to its score).

The generality of this result indicates that there is considerable choice in constructing a bipartite tournament from a realizable pair $(A, B)$. If, however, we begin with the lists $A$ and $B$ in nonincreasing order and then delete the first entry in $A$ and preserve the order in $B$ when reducing the appropriate number of its greatest entries by 1 each, then the result is prescribed and we call the bipartite tournament so obtained *canonical*, with lists $A$ and $B$.

We now turn to an existence criterion for realizability. It was found by Moon [6], who in fact established a more general result for multipartite tournaments.

**Theorem 2.2.** *A pair of lists $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ of nonnegative integers in nondecreasing order are the score lists of an $m \times n$*

*bipartite tournament if and only if, for* $k = 1, 2, \ldots, m$ *and* $l = 1, 2, \ldots, n$,

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \geqq kl\,, \tag{1}$$

*with equality when* $k = m$ *and* $l = n$.

*Furthermore, the realizations are irreducible if and only if* $a_1 > 0$, $b_1 > 0$ *and the inequalities* (1) *are all strict (except when* $k = m$ *and* $l = n$).

Another existence criterion is due to Ryser [7], and it was originally given in the context of row and column sums in matrices of zeros and ones. We state it here for the sake of completeness, although we shall not have occasion to use it.

**Theorem 2.3.** *Suppose that* $A = [a_1, a_2, \ldots, a_m]$ *and* $B = [b_1, b_2, \ldots, b_n]$ *are lists of nonnegative integers, with A in nonincreasing order. Then* $(A, B)$ *is realizable if and only if, for* $k = 1, 2, \ldots, m$,

$$\sum_{i=1}^{k} a_i \leqq \sum_{j=1}^{n} \min(k, m - b_j)\,, \tag{2}$$

*with equality when* $k = m$.

*Furthermore, the realizations are irreducible if and only if* $0 < b_j < m$ *for each* $j$, *and the inequalities* (2) *are strict for* $1 \leqq k < m$.

It has been observed by Beineke and Eggleton (independently, unpublished), that in criteria such as Moon's and Ryser's, one need only check the inequalities when there are jumps. A precise statement of this result and its proof will now be given. We first introduce some notation. For a list $L = [l_1, l_2, \ldots, l_p]$, let $L_q = \sum_{i=1}^{q} l_i$, for $1 \leqq q \leqq p$, and let $L_0 = 0$.

**Theorem 2.4.** *Suppose that* $A = [a_1, a_2, \ldots, a_m]$ *and* $B = [b_1, b_2, \ldots, b_n]$ *are nondecreasing integer lists, with* $0 \leqq a_i \leqq n$ *and* $0 \leqq b_j \leqq m$, *and such that*

(a) $A_m + B_n = mn$; *and,*

(b) $A_r + B_s \geqq rs$, *whenever* $a_r < a_{r+1}$ *and* $b_s < b_{s+1}$.

*Then* $(A, B)$ *is realizable.*

Proof. We wish to show that the inequality

$$A_k + B_l \geqq kl \tag{1}$$

holds for all $1 \leqq k \leqq m$ and $1 \leqq l \leqq n$. If this is not the case for some $k$ and $l$, let $q$ and $s$ be the smallest and let $r$ and $t$ the largest indices such that $a_{q+1} = a_k = a_r$ and $b_{s+1} = b_l = b_t$ $(q, s \geqq 0)$. Now $A_k + B_l < kl$. We claim that at least one of $A_k + B_s < ks$ and $A_k + B_t < kt$ must hold. For, otherwise $(l - s)\, b_l < k(l - s)$ and $(t - l)\, b_l > k(t - l)$, which is impossible. Hence, suppose that

(i) $A_k + B_s < ks$.

Now, by hypothesis,

(ii) $A_q + B_s \geqq qs$, and

(iii) $A_r + B_s \geqq rs$

(observe that if $r = m$, then $A_m + B_n = mn$ and $0 \leqq b_j \leqq m$ together imply (iii)). Then (i) and (ii) give $(k - q)\, a_k < (k - q)\, s$, while (i) and (iii) give $(r - k)\, a_k > > (r - k)\, s$. These again lead to a contradiction. The case $A_k + B_t < kt$ can similarly be treated. This completes the proof.

We now mention some known results about irreducible components. A component is called *dominating* if it has no incoming arcs. While an ordinary tournament has precisely one dominating component, the situation in the bipartite case is slightly different. It is described in the following result [2], and is a direct consequence of Moon's Theorem 2.2.

**Theorem 2.5.** *Let $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ be score lists (in nondecreasing order) of a reducible $m \times n$ bipartite tournament.*

(i) *If $a_m = n$ or $b_n = m$, then there is a corresponding trivial dominating component (consisting of one vertex which dominates all the vertices in the other partite set);*

(ii) *otherwise, if $k$ and $l$ are the largest indices with $k < m$ and $l < n$ such that*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j = kl\,,$$

*then the nontrivial dominating component consists of all the vertices in the two partite sets with scores exceeding $a_k$ and $b_l$, respectively.*

## 3. SOME SPECIAL SCORE LISTS

In this section we prove two results which will be used later. These concern the existence of bipartite tournaments and their properties under special conditions. The first involves pairs of lists in which one list is constant or near-constant.

**Theorem 3.1.** *Let $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ be a realizable pair of score lists, with $0 < a_i < n$ and $0 < b_j < m$. If $|a_i - a_k| \leqq 1$ for any $i, k = 1, 2, \ldots, m$, then any bipartite tournament with score lists $A$ and $B$ is irreducible.*

Proof. Suppose that $T$ is a reducible bipartite tournament on partite sets $X$ and $Y$, with score lists $A$ and $B$, respectively. Since $0 < a_i < n$ and $0 < b_j < m$, $T$ has at least two nontrivial components, say $C$ and $C'$, with $C$ being the dominating component. If $x_i \in C \cap X$ and $x_k \in C' \cap X$, then $x_i$ dominates all the vertices (in $Y$) dominated by $x_k$. Also, there exist $y_j \in C \cap Y$ and $y_l \in C' \cap Y$ such that $x_i \to y_j \to \to x_k$ and $x_i \to y_l \to x_k$. Thus $a_i = \text{score}\,(x_i) \geqq \text{score}\,(x_k) + 2 = a_k + 2$. This contradicts the hypothesis, and hence the theorem is proved.

**Theorem 3.2.** *Let* $A = [a_1, a_2, \ldots, a_m]$ *(in nondecreasing order) and* $B = [b^n]$ *be lists such that* $(A, B)$ *is realizable. Suppose that* $a_k$, $a_l$ *are two entries in* $A$ *with* $a_k > 0$ *and* $a_l < n$. *Define a new list* $A' = [a'_1, a'_2, \ldots, a'_m]$ *as follows:*

$$\begin{aligned} a'_k &= a_k - 1 \\ a'_l &= a_l + 1 \\ a'_i &= a_i \quad \text{for} \quad i \neq k, l\,. \end{aligned}$$

*Then* $(A', B)$ *is realizable.*

Proof. This follows at once from Theorem 2.4.

There are other results like the one in the above theorem. We state one more, although we shall not need it.

**Theorem 3.3.** *Suppose* $(A, B)$ *is irreducible (that is,* $(A, B)$ *is realizable and all its realizations are irreducible). If* $A'$ *is obtained from* $A$ *by adding* 1 *to some entry and* $B'$ *is obtained from* $B$ *by subtracting* 1 *from some entry, then* $(A', B')$ *is realizable.*

## 4. SOME LEMMAS

We are now ready to obtain necessary conditions on $A$ and $B$ so that $(A, B)$ is uniquely realizable. As observed earlier, it is enough to consider the irreducible case.

We find it convenient to introduce some notation. If $(A, B)$ is realizable, we shall let $T$ denote a realization on partite sets $X$ and $Y$. If $X_1$ and $Y_1$ are nonempty subsets of $X$ and $Y$, respectively, then $T(X_1, Y_1)$ is the subtournament of $T$ induced by the vertex sets $X_1$ and $Y_1$.

**Lemma 4.1.** *Suppose* $(A, B)$ *is uniquely realizable. For any entry* $a_i$ *in* $A$ *and* $b_j$ *in* $B$, *let* $X_i$ *and* $Y_j$ *be the subsets of* $X$ *and* $Y$ *consisting of vertices of scores* $a_i$ *and* $b_j$, *respectively. Then any cycle in* $T$ *contains the same number of arcs from* $X_i$ *to* $Y_j$ *as from* $Y_j$ *to* $X_i$.

Proof. If this were not the case for some cycle $Z$, then reversal of the arcs of $Z$ would produce an $(A, B)$ realization nonisomorphic to $T$.

**Lemma 4.2.** *Suppose that* $(A, B)$ *is irreducible and uniquely realizable. Then* $A$ *or* $B$ *is constant.*

Proof. If neither $A$ nor $B$ is constant, let $X_1$ be the set of vertices of minimum score in $X$ and let $X_2 = X - X_1$, and similarly define $Y_1$ and $Y_2$. Since $T$ is irreducible, every arc is contained in a cycle. Hence, by Lemma 4.1, none of the four subtournaments $T(X_i, Y_j)$ $(i, j = 1, 2)$ is unanimous (that is, has all its arcs directed from one partite set to the other).

Choose a vertex $x_1$ in $X_1$ of minimum score in $T(X_1, Y_1)$. Then $x_1$ must dominate some $y_2 \in Y_2$. We consider two cases depending on whether or not $y_2$ dominates some vertex in $X_2$.

Case (i). Every vertex in $X_2$ dominates $y_2$. Let $uv$ be an arc from $Y_1$ to $X_2$. Since score $(y_2) >$ score $(u)$ in $T$, there exists an $x \in X_1$ such that $y_2 \to x \to u$. (See Figure 2).

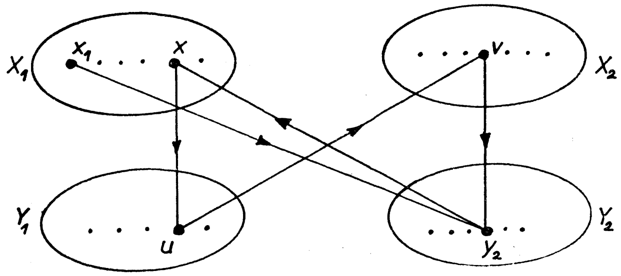


Figure 2

But then $x \to u \to v \to y_2 \to x$ is a cycle which violates Lemma 4.1.

Case (ii). Some vertex $x_2 \in X_2$ is dominated by $y_2$.

By our choice of $x_1$, there exists a $y_1 \in Y_1$ such that $y_1 \to x_1$. If $x_2 \to y_1$, we again get a 4-cycle which has one arc from $Y_1$ to $X_1$, but no arcs in the other direction. So, assume $y_1 \to x_2$. (See Figure 3). Since score $(y_2) >$ score $(y_1)$ in $T$, there exists an $x \in X$ with $y_2 \to x \to y_1$. If $x \in X_2$, we again get a forbidden cycle $x_1 \to y_2 \to x \to \to y_1 \to x_1$. Hence $x \in X_1$. Likewise, there exists a $y \in Y_1$ with $x_2 \to y \to x$. But then $x_1 \to y_2 \to x_2 \to y \to x \to y_1 \to x_1$ is a 6-cycle which violates Lemma 4.1.

Since all the possibilities have been exhausted, the lemma follows.

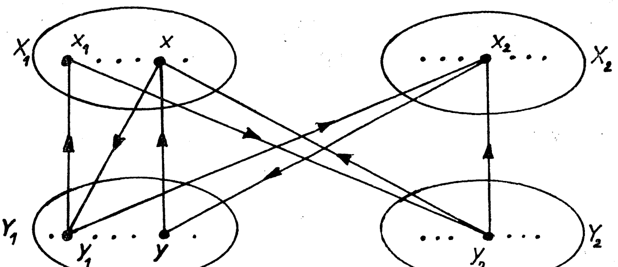


Figure 3

We noted in Section 1 that a realizable pair $(A, B)$ is uniquely realizable if one of the lists is constantly 1. We now assume, for the remainder of this section, that none of the lists $A$, $B$, $\bar{A}$ and $\bar{B}$ is constantly 1.

**Lemma 4.3.** *With the above assumptions on $A$ and $B$, if $(A, B)$ is irreducible and uniquely realizable, then $A$ or $B$ is nonconstant.*

Proof. Suppose, on the contrary, that $A = [a^m]$ and $B = [b^n]$, with $1 < a < n-1$ and $1 < b < m - 1$. We may also assume, without loss of generality, that $b \geqq \frac{1}{2}m$.

Consider the largest $r$ for which there is a unanimous $r \times a$ subtournament of an $(A, B)$ realization $T$, with all its arcs directed from $X$ to $Y$. It is easy to see that $1 \leqq r \leqq b \leqq m - r$. We claim that $r = m - b$. First suppose that $r = b$. Then $\frac{1}{2}m \leqq b \leqq m - r = m - b \leqq \frac{1}{2}m$ implies that $r = m - b$. Next, suppose that $r < b$. Now $([a^{m-r}], [b^a, (b - r)^{n-a}])$ is realizable. Hence, by Moon's Theorem 2.2, we get

$$\text{(i)} \qquad (m - r)\, a + (n - a)(b - r) \geqq (m - r)(n - a)\,.$$

Furthermore, if the lists $[a^{m-r-1}]$ and $[b^a, (b - r - 1)^{n-a}]$ form a realizable pair, we can get an $(A, B)$ realization with an $(r + 1) \times a$ unanimous subtournament. This contradicts the unique realizability of $(A, B)$. Hence, we have

$$\text{(ii)} \qquad (m - r - 1)\, a + (n - a)(b - r - 1) < (m - r - 1)(n - a)\,.$$

We note that the following also holds:

$$\text{(iii)} \qquad ma + nb = mn\,.$$

From (i), (ii), and (iii), it follows readily that $r = m - b$. Hence, our claim is proven.

We thus have the situation as shown in Figure 4, where $U$ is our unanimous $r \times a = (m - b) \times a$ subtournament.

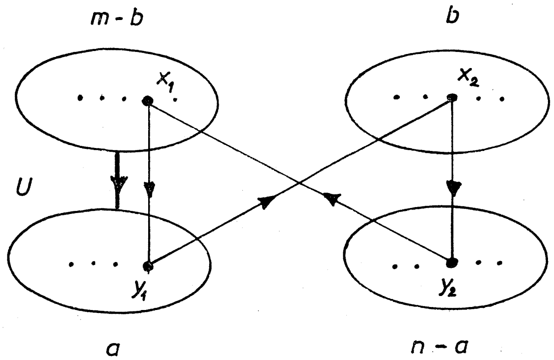


Figure 4

Choose an arc $x_1y_1$ in $U$ and another arc $x_2y_2$ in $T - U$. Then we have a 4-cycle $x_1 \to y_1 \to x_2 \to y_2 \to x_1$. On reversing the arcs of this 4-cycle, we get a realization $T'$ of $(A, B)$ in which $U$ is no longer unanimous. Also, neither $x_1$ nor $x_2$ can lie on an $(m - b) \times a$ unanimous subtournament of $T'$. Hence $T'$ has a different number of such subtournaments than $T$, and is consequently nonisomorphic to $T$, a contradiction. This completes the proof.

We may now assume that we have a pair $(A, B)$ which is uniquely realizable and irreducible; and moreover, without loss of generality, $A = [a^m]$, $1 < a < n - 1$, and $B = [b_1 \leqq b_2 \leqq \ldots \leqq b_r < b_{r+1} \leqq \ldots \leqq b_n]$, where $1 \leqq r < n$.

**Lemma 4.4.** *With A and B as above, the list B has precisely two distinct values.*

Proof. Suppose that $B = [b^r, c^s, d_1, \ldots, d_t]$ with $b < c < d_1 \leqq \ldots \leqq d_t$, and $r, s, t > 0$. Let $Y_1$, $Y_2$ and $Y_3$ be the subsets of $Y$ consisting of those vertices with scores in the sublists $[b^r]$, $[c^s]$ and $[d_1, \ldots, d_t]$, respectively.

We first show that in the subtournament $T(X, Y_1)$ of $T$, the scores of the vertices in $X$ take on precisely two distinct consecutive values. Since $(A, B)$ is uniquely realizable, the canonical realization of $T(X, Y_1)$ (as described in Section 2) shows that the scores, in $T(X, Y_1)$, of the vertices in $X$ assume at most two distinct consecutive values. Suppose that the score lists of $T(X, Y_1)$ are $[\alpha^m]$ and $[b^r]$. Since $T$ is irreducible, we clearly have $0 < \alpha < a$. Choose $y_1 \in Y_1$, $x_1 \in X$ and $y_2 \in Y - Y_1$ such that $y_1 \to x_1 \to y_2$. Since score $(y_2) >$ score $(y_1)$ in $T$, there exists an $x_2 \in X$ with $y_2 \to x_2 \to y_1$. On reversing the arcs of the 4-cycle $x_1 \to y_2 \to x_2 \to y_1 \to x_1$, we get a realization of $(A, B)$ which is not isomorphic to $T$.

We have thus proved that the score lists of $T(X, Y_1)$ are $[\alpha^p, (\alpha + 1)^q]$ and $[b^r]$ for some $p, q > 0$, $p + q = m$. Now let $\beta = a - \alpha$. Then the score lists of $T(X, Y - Y_1) = T(X, Y_2 \cup Y_3)$ are $[(\beta - 1)^q, \beta^p]$ and $[c^s, d_1, \ldots, d_t]$. Since $T$ is unique, so is $T(X, Y - Y_1)$. Hence, by Lemma 4.2, $T(X, Y - Y_1)$ is reducible. But one of its score lists is near-constant. Hence, Theorem 3.1 applies, and it follows that one of the following must hold:

$$\beta - 1 = 0\,, \quad c = 0\,, \quad \beta = s + t\,, \quad d_t = p + q = m\,.$$

Clearly, $c = 0$ and $d_t = m$ are impossible. Suppose that $\beta = 1$. Then the score lists of $T(X, Y_1 \cup Y_2)$ are $[(a - 1)^{p_1}, a^{q_1}]$ and $[b^r, c^s]$, where $p_1 + q_1 = m$. Moreover, a consideration of the score lists of $T(X, Y_1)$ gives $a \neq r + s$ and $q_1 > 0$. Hence, the score lists of $T(X, Y_1 \cup Y_2)$ are irreducible and uniquely realizable. An application of Lemma 4.2 shows that we must have $p_1 = 0$. But then the score lists of $T(X, Y_1 \cup Y_2)$ are $[a^m]$ and $[b^r, c^s]$, a contradiction to the irreducibility of $T$.

Finally, if $\beta = s + t$, then every vertex of $X$ must dominate all or all but one of the vertices in $Y_2 \cup Y_3$. From the canonical construction for $(A, B)$, it follows that $a \geqq n - 1$, again a contradiction.

This completes the proof of the lemma.

Let us now recapitulate the results proved thus far. If $(A, B)$ is an irreducible, uniquely realizable pair, then we have proved that one of the following must hold:

(i) one of $A$, $B$, $\bar{A}$ or $\bar{B}$ is constantly 1;

(ii) (without loss of generality) $A = [a^m]$, $2 \leqq a \leqq n - 2$, and $B = [b^r, c^s]$, with $1 \leqq b < c \leqq m - 1$, $r, s > 0$ and $r + s = n$.

**Lemma 4.5.** *Suppose that $(A, B)$ is irreducible, uniquely realizable, and* (ii) *above holds. Then $r = 1$ or $s = 1$.*

Proof. Suppose, to the contrary, that $r \geqq 2$ and $s \geqq 2$. If $Y_1$ is the subset of $Y$ consisting of those vertices with score $b$, then as in the proof of Lemma 4.4, the score lists of $T(X, Y_1)$ are $[\alpha^p, (\alpha + 1)^q]$ and $[b^r]$, while those of $T(X, Y - Y_1)$ are $[(\beta - 1)^q, \beta^p]$ and $[c^s]$, with $p, q > 0$, and $\alpha + \beta = a$. Since $(A, B)$ is irreducible and $B$ is nonconstant, we get $3 \leqq m = p + q$. Thus, $p \geqq 2$ or $q \geqq 2$.

We first consider the case when $0 < \alpha < r - 1$ and $1 < \beta < s$. If $p \geqq 2$, then the score lists of $T(X, Y_1)$ and $T(X, Y_2)$ imply the realizability of the pairs of lists $([\alpha - 1, \alpha^{p-2}, (\alpha + 1)^{q+1}], [b^r])$ and $([(\beta - 1)^{q+1}, \beta^{p-2}, \beta + 1], [c^s])$, respectively, by Theorem 3.2. If $T_1$ and $T_2$ are realizations of these two pairs on partite sets $X$, $Y_1$ and $X$, $Y_2$, respectively, then $T_1 \cup T_2$ is another realization of $(A, B)$, nonisomorphic to $T$, a contradiction. If, on the other hand, $q \geqq 2$, then the list pairs $([\alpha^{p+1}, (\alpha + 1)^{q-2}, \alpha + 2], [b^r])$ and $([\beta - 2, (\beta - 1)^{q-2}, \beta^{p+1}], [c^s])$ again lead to a contradiction.

We are now left to consider the case when one of $\alpha = 0$, $\alpha = r - 1$, $\beta = 1$ and $\beta = s$ holds. Suppose $\alpha = 0$. Then the score lists of $T(X, Y_1)$ are $[0^p, 1^q]$ and $[b^r]$, while those of $T(X, Y_2)$ are $[(a - 1)^q, a^p]$ and $[c^s]$. Hence, $q + rb = rm$, and since $r > 1$ and $m > b$, $q > 1$. It follows, again from Theorem 3.2, that the list pairs $([0^{p+1}, 1^{q-2}, 2], [b^r])$ and $([a - 2, a - 1)^{q-2}, a^{p+1}], [c^s])$ are realizable. This contradicts the unique realizability of $(A, B)$, as before. For the other possibilities, namely $\alpha = r - 1$, $\beta = 1$ or $\beta = s$, an analogous modification of the original pairs of lists can be made. This completes the proof.

## 5. THE MAIN THEOREM

We start by summarizing the results proved in Section 4. It has been shown that if $(A, B)$ is irreducible and uniquely realizable, then one of the lists (or its dual) consists entirely of 1's or one of the lists is constant and the other has exactly two distinct values, one of which appears precisely once. We are now ready to state and prove our main theorem, which gives necessary and sufficient conditions for unique realizability in the irreducible case.

**Theorem 5.1.** *An irreducible pair $(A, B)$ of score lists is uniquely realizable if and only if one of the following holds*:

(I) (*without loss of generality*) $A = [1^m]$ *and* $B$ *is arbitrary*;
($\bar{\mathrm{I}}$) *the dual of* (I), *that is*, $A = [(n - 1)^m]$ *and* $B$ *is arbitrary*;
(II) (*without loss of generality*) $A = [1^{m-1}, a]$ *and* $B = [b^n]$;
($\bar{\mathrm{II}}$) *the dual of* (II);
(III) (*without loss of generality*) $A = [1, a^{m-1}]$ *and* $B = [2^n]$;
($\bar{\mathrm{III}}$) *the dual of* (III).

Proof. The sufficiency of (I) has already been noted in Section 1, where Figure 1 shows the unique realization. Now suppose $T$ is a realization of $A = [1^{m-1}, a]$ and $B = [b^n]$, on partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, respectively.

If (say) $x_m$ has score a and it dominates (say) $y_1, y_2, \ldots, y_a$, then $T - x_m$ has score lists $A_1 = [1^{m-1}]$ and $B_1 = [(b-1)^{n-a}, b^a]$. Hence, by (I), $(A_1, B_1)$ is uniquely realizable. The unique realizability of $(A, B)$ follows. This proves the sufficiency of (II). The proof for (III) is similar, and the dual cases follow by a remark made in Section 1.

We now prove the necessity by induction on the total number $m + n$ of entries in $A$ and $B$. Since $(A, B)$ is irreducible, we have $m, n \geqq 2$. If (say) $m = 2$, then $B = [1^n]$, and the theorem follows. Now assume that the result holds for all irreducible and uniquely realizable pairs of score lists with combined length less than $m + n$, and consider such a pair $(A, B)$ with $|A| = m$ and $|B| = n$ $(m, n \geqq 3)$.

Suppose that $A$ and $B$ are not of the type (I) or $(\bar{\text{I}})$. Then, by the remarks at the beginning of this section, we have, without loss of generality, $A = [a^m]$ and $B = [b^{n-1}, c]$, with $1 < a < n - 1$, $1 \leqq b$, $c \leqq m - 1$ and $b \neq c$.

If $y$ is the vertex of score $c$ in a realization $T$ of $(A, B)$, then $T - y$ has score lists $A_1 = [(a-1)^{m-c}, a^c]$ and $B_1 = [b^{n-1}]$. Now the the unique realizability of $(A, B)$ implies that of $(A_1, B_1)$. Also, by Theorem 3.1, $(A_1, B_1)$ is irreducible. Hence, by the induction hypothesis, $A_1$ and $B_1$ must belong to one of the six given types. We consider these cases one by one.

(i) If $B_1 = [1^{n-1}]$, then $B = [1^{n-1}, c]$, so that $A$ and $B$ are of type (II).

(ii) If $B_1 = [(m-1)^{n-1}]$, then $A, B$ belong to $(\bar{\text{II}})$.

(iii) If $B_1 = [b^{n-1}]$ and $A_1 = [1^{m-1}, a]$, then $c = 1$ and $a = 2$, so that $A = [2^m]$ and $B = [1, b^{n-1}]$. This is of type (III).

(iv) If $B_1 = [b^{n-1}]$ and $A_1 = [d, (n-2)^{m-1}]$, we get $A$ and $B$ to be of type $(\bar{\text{III}})$.

(v) If $B_1 = [2^{n-1}]$ and $A_1 = [1, a^{m-1}]$, then $b = 2$, $a = 2$ and $c = m - 1$. Hence $A = [2^m]$ and $B = [2^{n-1}, m - 1]$. Using Moon's Theorem, we get

$$2m + 2(n-1) + (m-1) = mn\,;$$

or

$$m = \frac{2n-3}{n-3}.$$

It follows that $n = 6$ and $m = 3$. But then $A$ and $B$ are both constant, a contradiction. Thus, this case is not possible.

(vi) The impossibility of $B_1 = [(m-2)^{n-1}$ and $A_1 = [a^{m-1}, n-2]$ follows by duality.

This exhausts all the possibilities, and hence, by induction, the theorem is completely proved.

The results proved above can now be summed up as follows: An irreducible pair of score lists is uniquely realizable if and only if that pair or the dual pair is one of the following types (not mutually exclusive):

(1) one list is constantly 1 (and the other is arbitrary);

(2) one list is constantly 1 except for exactly one entry, and the other is constant;

(3) one list is constantly 2 and the other list is constant except for exactly one entry which is 1.

Now suppose that $(A, B)$ is a realizable pair and $Q_1, Q_2, \ldots, Q_p$ are the irreducible components of a realization $T$ of $(A, B)$. Also suppose that $Q_k$ has score lists $A_k$ and $B_k$, $1 \leqq k \leqq p$. Then it is a simple observation that $(A, B)$ is uniquely realizable if and only if $(A_k, B_k)$ is uniquely realizable for all $k$. Hence Theorems 2.4 and 5.1 together solve the uniqueness problem for the class of all bipartite tournaments.

*References*


[1] *P. Avery:* The condition for a tournament score sequence to be simple, J. Graph Theory *4* (1980), 157—164.

[2] *L. W. Beineke:* A tour through tournaments or bipartite and ordinary tournaments: A comparative survey, Combinatorics, London Mathematical Society Lecture Note Series 52, Cambridge University Press (1981), pp. 41—55.

[3] *L. W. Beineke* and *J. W. Moon:* On bipartite tournaments and scores, in The Theory and Applications of Graphs (ed. G. Chartrand et al), Wiley (1981), pp. 55—71.

[4] *D. Gale:* A theorem of flows in networks, Pacific J. Math. *7* (1957), 1073—1082.

[5] *M. Koren:* Pairs on sequences with a unique realization by bipartite graphs, J. Combin. Theory *21* (1976), 224—234.

[6] *J. W. Moon:* On the score sequence of an $n$-partite tournament, Canadian Math. Bulletin *5* (1962), 51—58.

[7] *H. J. Ryser:* Combinatorial Mathematics, Carus Mathematical Monographs No. 14, MAA, Washington, 1963.



*Authors' address:* Indiana University-Purdue University at Fort Wayne, 2101 Coliseum Boulevard East, Fort Wayne, Indiana 46805, U.S.A.