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SETS WITH NO UNCOUNTABLE BLACKWELL SUBSETS

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0. PRELIMINARIES

A measurable space $(S, \mathscr{B}(S))$ is *standard* if there is a complete, separable, metrisable topology on S generating the σ-field $\mathscr{B}(S)$ as its Borel structure. Equivalently, $(S, \mathscr{B}(S))$ is isomorphic with a Borel subset of the real line $\mathbb{R}$ under its relative Borel structure. A non-void collection $\mathscr{I}$ of sets in $\mathscr{B}(S)$ is a *σ-ideal* if it is closed under the taking of countable unions and if $B \in \mathscr{B}(S)$ and $N \in \mathscr{I}$ implies $B \cap N \in \mathscr{I}$. We insist that $S \notin \mathscr{I}$. If m is a probability measure on $\mathscr{B}(S)$, then $\mathscr{I}(m)$ is the σ-ideal of all m-null sets in $\mathscr{B}(S)$. A subset $X \subset S$ is *$\mathscr{I}$-dense* if $B \subset \mathscr{B}(S)$ and $B \subset S \setminus X$ implies $B \in \mathscr{I}$. Let f be a Borel-isomorphism between sets B_1 and B_2 in $\mathscr{B}(S)$. Then $T = \text{graph}(f)$ is an *$\mathscr{I}$-thread* if there is no set $N \in \mathscr{I}$ such that $T \subset \subset (N \times S) \cup (S \times N)$.

Let λ denote Lebesgue measure on the real line. An uncountable subset X of $\mathbb{R}$ is a *Sierpiński set* if $X \cap N$ is countable for each Borel set N with $\lambda N = 0$. An uncountable $X \subset \mathbb{R}$ is a *Lusin set* if $X \cap N$ is countable for each Borel set of first category in $\mathbb{R}$. For more information about such singular sets, consult the surveys [2] or [8].

A subset X of a standard space S has the *Blackwell property* if whenever $f: X \to \mathbb{R}$ is a one-one real function measurable with respect to the relative Borel structure $\mathscr{B}(X) = \{B \cap X : B \in \mathscr{B}(S)\}$, then f is a Borel-isomorphism of X onto its image $f(X)$. An exposition treating of this topic is [1]. In some ways, the Blackwell property functions as a measurable version of compactness. Every analytic set is Blackwell, but not every co-analytic set is. For these and other basic facts, see [1]. There have been a number of recent investigations along these lines: [3], [5], [6], [7], [10].

Using the continuum hypothesis (CH), Jasiński [7] has demonstrated the existence of Sierpiński and Lusin sets with and without the Blackwell property. In [10], his ideas were extended to a general class of singular sets, and it was shown that, roughly speaking, only relatively "large" Sierpiński and Lusin sets are Blackwell. To wit, we have the following

Lemma 1. *Let $\mathscr{I}$ be a σ-ideal in the standard structure $\mathscr{B}(S)$ and suppose that X is an $\mathscr{I}$-dense subset of S. If X has the Blackwell property, then $X \times X$ meets every $\mathscr{I}$-thread in $S \times S$.*

Indication. This follows from propositions 1 and 2 in [10].

We shall use this fact to construct (CH) Lusin and Sierpiński sets, each of whose uncountable subsets are not Blackwell. Assuming MA + (not-CH), such sets cannot exist.

1. MAIN RESULTS

For our construction of a highly non-Blackwell set, we shall employ a familiar result of Steinhaus slightly recast.

Let $r_1 \, r_2 \, r_3 \ldots$ be an enumeration of the non-zero rationals in the interval $(-1, 1)$. Define subsets R_n of the square $(0, 1) \times (0, 1)$ by

$$R_n = \{(x, y): y = x + r_n\}$$

and put $R = R_1 \cup R_2 \cup \ldots$.

Lemma 2. *Let A be a linear Borel set.*

1) *If A is of positive Lebesgue measure, then for some n, the set $(A \times A) \cap R_n$ has positive linear measure.*

2) *If A is of second category in $\mathbb{R}$, then for some n, the projection of $(A \times A) \cap R_n$ on either axis is of second category in $\mathbb{R}$.*

Proof. A classical theorem of Steinhaus [11] says that if A has positive Lebesgue measure, then the difference set $A - A = \{a - a': a, a' \in A\}$ contains an open interval about 0. So $(A \times A) \cap R$ is non-empty and must in fact have positive linear measure.

For the case where A is of second category, replace the theorem of Steinhaus by its category analogue, due to Pettis [4; p. 87]. Q.E.D.

Proposition 1 (CH): *There is an uncountable subset Y of $\mathbb{R}$, no uncountable subset of which is Blackwell. One may choose Y to be a Sierpiński set or a Lusin set.*

Construction. We build a Sierpiński set with the desired property. The method for Lusin sets is entirely analogous and is therefore omitted. List in transfinite series $N_0 \, N_1 \ldots N_\alpha \ldots \alpha < c$ all linear Borel sets of measure zero and put $M_\alpha = \bigcup \{N_\beta: \beta \leqq \alpha\}$ for each $\alpha < c$. Choose y_0 from the set $(0, 1) \setminus M_0$.

Suppose now that for $\alpha < c$, the set $Y_\alpha = \{y_\beta: \beta < \alpha\}$ has been defined in such a way that $y_\beta \in (0, 1) \setminus M_\beta$ for each β and so that no two elements of Y_α are equivalent (in the sense that their difference is rational). Choose y_α to be any member of $(0, 1) \setminus M_\alpha$ not equivalent to any point in Y_α.

Finally, define $Y = \{y_\alpha: \alpha < c\}$. Clearly, Y is a Sierpiński set. Suppose now that X is an uncountable subset of Y. Then X has positive outer measure. Let $S \supset X$ be a Borel set with $\lambda S = \lambda^* X$. (In the case of Lusin sets, use [4; p. 25] to find a Borel set $S \supset X$ such that all Borel sets contained in $S \setminus X$ are of first category in $\mathbb{R}$). Define $m = \lambda / \lambda S$ as a probability measure on the Borel subsets of S.

Now using lemma 2, we find n so that $T = (S \times S) \cap R_n$ has positive linear measure. Then X is $\mathscr{I}(m)$-dense in S, but $X \times X$ does not intersect the $\mathscr{I}(m)$-thread T. From lemma 1, we see that X cannot have the Blackwell property. Q.E.D.

Under Martin's Axiom, every linear set of cardinality less than c has outer measure zero and is of first category in $\mathbb{R}$ [9]. Using the same construction as above, one may demonstrate

Proposition 2 (MA). *There is a subset Y of $\mathbb{R}$ of cardinality c such that no subset of Y of cardinality c is Blackwell.*

However, there are limitations on the extent to which proposition 1 can ignore CH. If MA + (not-CH), then every uncountable set contains an uncountable Blackwell set:

Proposition 3 (MA): *If X is a linear set of power less than c, then X has the Blackwell property.*

Indication. See [1; p. 26].

We leave off with some unsettled business. Let X be a linear subset whose complement $\mathbb{R} \setminus X$ is totally imperfect (i.e. X is $\mathscr{I}$-dense, where $\mathscr{I}$ is the σ-ideal of countable sets). A characterisation of such sets with the Blackwell property is known [10], but

Question. *Does X necessarily contain a Blackwell set?*

The author can answer in the affirmative only if MA is assumed.

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