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VECTOR FIELDS ON HYPERSPHERES

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0. Let $(M, ds^2 = g_{ij}\, dx^i\, dx^j)$ be a Riemannian manifold, ∇ the associated linear connection. For a q-times covariant tensor $T_{i\ldots j}$ on M, define the Laplace operator Δ_0 by

$$(0.1)\qquad (\Delta_0 T)_{i\ldots j} = g^{kl}\nabla_k \nabla_l T_{i\ldots j}\,.$$

For $q = 0$, Δ_0 coincides with the classical Laplacian Δ on functions; for $q > 0$ and T skew-symmetric, Δ_0 does not differ too much from the classical Laplacian $\Delta = -(d\delta + \delta d)$ on q-forms on M. Using Δ_0, we are able to define $\mathrm{Spec}_{[q]}(M)$ as the set of all λ's, $\lambda \in \mathbb{R}$, such that the equation

$$(0.2)\qquad (\Delta_0 + \lambda)\, T_{i\ldots j} = 0$$

admits a non-trivial solution. Because of the existence of the metric, we may study contravariant tensors as well.

The classical spectrum of the unit hypersphere $S^n \subset E^{n+1}$ is well known, see [1]; the eigen-functions are the restrictions of polynomials harmonic in E^{n+1}. In the following, I study the behavior of restrictions of arbitrary polynomials. Using the same method, I am going to study $\mathrm{Spec}_{[1]}(S^n)$ and, more generally, $\mathrm{Spec}_{[1]}(M)$. Instead of 1-forms, I am using, dually, vector fields; this enables me to consider the Lie brackets.

As mentioned above, $\mathrm{Spec}_{[1]}(S^n)$ does not differ substantially from the classical $\mathrm{Spec}^1(S^n)$. This comparison will be treated elsewhere, and it will show some discrepancies with the already published results, see, e.g., [4].

1. Let (M, ds^2) be a Riemannian manifold, $\dim M = n$. In a coordinate neighborhood $U \subset M$ we may write

$$(1.1)\qquad ds^2 = (\omega^1)^2 + \ldots + (\omega^n)^2\,,$$

$\omega^1, \ldots, \omega^n$ being linearly independent 1-forms on U. It is well known that there exists a unique set of 1-forms ω_i^j on U satisfying

$$(1.2)\qquad \omega_i^j + \omega_j^i = 0\,, \quad d\omega^i = \omega^j \wedge \omega_j^i\,;$$

here (and in the following)

$$(1.3)\qquad i, j, \ldots = 1, \ldots, n\,,$$

and we use the usual summation convention. The *curvature tensor* of (M, ds^2) is defined by

(1.4) $$d\omega_i^j = \omega_i^k \wedge \omega_k^j - \tfrac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l\,, \quad R^j_{ikl} + R^j_{ilk} = 0\,;$$

it satisfies

(1.5) $$R^j_{ikl} + R^i_{jkl} = 0\,, \quad R^j_{ikl} = R^l_{kij}\,, \quad R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0\,.$$

The *Ricci tensor* and the *scalar curvature* are defined by

(1.6) $$R_{ij} = R^k_{ijk}\,, \quad R = \delta^{ij}R_{ij}$$

respectively; δ^{ij}, δ_{ij} and δ_i^j are Kronecker's deltas ($= 1$ for $i = j$, $= 0$ otherwise).

Let $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ be a tensor on M. Its covariant derivatives with respect to the chosen coframes $\{\omega^i\}$ in U are defined by

(1.7) $$dT^{i_1 \ldots i_r}_{j_1 \ldots j_s} - \sum_{p=1}^{s} T^{i_1 \ldots i_r}_{j_1 \ldots j_{p-1} j j_{p+1} \ldots j_s}\,\omega^j_{j_p} + \sum_{q=1}^{r} T^{i_1 \ldots i_{q-1} i i_{q+1} \ldots i_r}_{j_1 \ldots j_s}\,\omega_i^{i_q} = T^{i_1 \ldots i_r}_{j_1 \ldots j_s;i}\,\omega^i\,;$$

its second covariant derivatives are, by definition, the covariant derivatives $T^{i_1 \ldots i_r}_{j_1 \ldots j_s;i;j} \equiv T^{i_1 \ldots i_r}_{j_1 \ldots j_r;ij}$ of the tensor $T^{i_1 \ldots i_r}_{j_1 \ldots j_s;i}$, etc.

Definition 1. The *Laplacian* of the tensor $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ is defined by

(1.8) $$(\Delta_0 T)^{i_1 \ldots i_r}_{j_1 \ldots j_s} \equiv \Delta_0 T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \delta^{ij} T^{i_1 \ldots i_r}_{j_1 \ldots j_r;ij}\,.$$

We say that $\lambda \in \mathbb{R}$ belongs to the (r, s)-*spectrum* of (M, ds^2), and we write $\lambda \in \operatorname{Spec}_{(r,s)}(M, ds^2)$, if there is a non-trivial tensor field $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ on M satisfying

(1.9) $$(\Delta_0 + \lambda)\, T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = 0\,.$$

For $\lambda \in \operatorname{Spec}_{(r,s)}(M, ds^2)$, the solutions of (1.9) are called *eigen-tensors*. They form an $\mathbb{R}$-module denoted by $\mathscr{E}^\lambda_{(r,s)}(M, ds^2)$.

The *main problem* is to determine, for a given (M, ds^2), the (r, s)-spectrum and, for each $\lambda \in \operatorname{Spec}_{(r,s)}(M, ds^2)$, the corresponding $\mathbb{R}$-module $\mathscr{E}^\lambda_{(r,s)}$. Of course, $\operatorname{Spec}(M, ds^2) = \operatorname{Spec}_{(0,0)}(M, ds^2)$ is the classical spectrum of (M, ds^2). In what follows, we will be mainly interested in $\operatorname{Spec}_{(1,0)}(M, ds^2)$, (M, ds^2) being the unit hypersphere $S^n(1)$ with its natural metric.

2. Let E^{n+1} be the $(n + 1)$-dimensional Euclidean space, V^{n+1} its vector space, $\langle\,,\rangle$ the scalar product and $S^n(1) \subset E^{n+1}$ a unit hypersphere. With each point $m \in S^n(1)$ of (a certain coordinate neighborhood of) $S^n(1)$, let us associate an orthonormal frame $\{v_\alpha\}$ with $v_i \in T_m(S^n(1))$, the unit normal vector v_{n+1} being oriented in such a way that $m + v_{n+1}$ is the center of $S^n(1)$; here, and in what follows,

(2.1) $$\alpha, \beta, \ldots = 1, \ldots, n + 1\,.$$

Then we may write

(2.2) $$dm = \omega^i v_i\,, \quad dv_i = \omega_i^j v_j + \omega_i^{n+1} v_{n+1}\,, \quad dv_{n+1} = \omega^i_{n+1} v_i$$

with

(2.3) $$\omega_i^j + \omega_j^i = 0\,, \quad \omega^i_{n+1} = -\omega^i\,, \quad \omega_i^{n+1} = \delta_{ij}\omega^j\,.$$

The curvature tensor, the Ricci tensor and the scalar curvature of $S^n(1)$ are

$$R^j_{ikl} = \delta_{ik}\delta^j_l - \delta_{il}\delta^j_k\,, \quad R_{ij} = (n-1)\,\delta_{ij}\,, \quad R = (n-1)\,n\,. \tag{2.4}$$

Let $F: V^{n+1} \times \ldots \times V^{n+1} \to V^{n+1}$ be a p-linear mapping. At each point $m \in S^n(1)$, F is given by

$$F(v_{\alpha_1}, \ldots, v_{\alpha_p}) = F^\beta_{\alpha_1\ldots\alpha_p} v_\beta\,. \tag{2.5}$$

Because of $(2.2_{2,3})$ and $(2.3_{2,3})$, we get

$$\mathrm{d}F^\beta_{\alpha_1\ldots\alpha_p} - \sum_{r=1}^{p} F^\beta_{\alpha_1\ldots\alpha_{r-1}\alpha\alpha_{r+1}\ldots\alpha_p}\omega^\alpha_{\alpha_r} + F^\gamma_{\alpha_1\ldots\alpha_p}\omega^\beta_\gamma = 0\,. \tag{2.6}$$

For a 0-linear mapping, i.e., for a fixed vector $A = A^\alpha v_\alpha$, (2.6) reduce to

$$\mathrm{d}A^i + A^j\omega^i_j = A^{n+1}\omega^i\,, \quad \mathrm{d}A^{n+1} = -\delta_{ij}A^j\omega^i\,. \tag{2.7}$$

For a linear mapping $B: V^{n+1} \to V^{n+1}$, we get

$$\begin{aligned} \mathrm{d}B^j_i + B^k_i\omega^j_k - B^j_k\omega^k_i &= (\delta^j_k B^{n+1}_i + \delta_{ik}B^j_{n+1})\,\omega^k\,,\\ \mathrm{d}B^{n+1}_i - B^{n+1}_j\omega^j_i &= (\delta_{ij}B^{n+1}_{n+1} - \delta_{jk}B^k_i)\,\omega^j\,,\\ \mathrm{d}B^i_{n+1} + B^j_{n+1}\omega^i_j &= (\delta^i_j B^{n+1}_{n+1} - B^i_j)\,\omega^j\,,\\ \mathrm{d}B^{n+1}_{n+1} &= -(\delta_{ij}B^j_{n+1} + B^{n+1}_i)\,\omega^i\,. \end{aligned} \tag{2.8}$$

Finally, for a bilinear mapping $C: V^{n+1} \times V^{n+1} \to V^{n+1}$, we have

$$\begin{aligned} &\mathrm{d}C^k_{ij} - C^k_{lj}\omega^l_i - C^k_{il}\omega^l_j + C^l_{ij}\omega^k_l = (\delta_{il}C^k_{n+1,j} + \delta_{jl}C^k_{i,n+1} + \delta^k_l C^{n+1}_{ij})\,\omega^l\,,\\ &\mathrm{d}C^j_{n+1,i} - C^j_{n+1,k}\omega^k_i + C^k_{n+1,i}\omega^j_k = (\delta_{ik}C^j_{n+1,n+1} - C^j_{ki} + \delta^j_k C^{n+1}_{n+1,i})\,\omega^k\,,\\ &\mathrm{d}C^j_{i,n+1} - C^j_{k,n+1}\omega^k_i + C^k_{i,n+1}\omega^j_k = (\delta_{ik}C^j_{n+1,n+1} - C^j_{ik} + \delta^j_k C^{n+1}_{i,n+1})\,\omega^k\,,\\ &\mathrm{d}C^{n+1}_{ij} - C^{n+1}_{kj}\omega^k_i - C^{n+1}_{ik}\omega^k_j = (\delta_{ik}C^{n+1}_{n+1,j} + \delta_{ik}C^{n+1}_{i,n+1} - \delta_{kl}C^l_{ij})\,\omega^k\,,\\ &\mathrm{d}C^{n+1}_{n+1,i} - C^{n+1}_{n+1,j}\omega^j_i = (\delta_{ij}C^{n+1}_{n+1,n+1} - C^{n+1}_{ji} - \delta_{jk}C^k_{n+1,i})\,\omega^j\,,\\ &\mathrm{d}C^{n+1}_{i,n+1} - C^{n+1}_{j,n+1}\omega^j_i = (\delta_{ij}C^{n+1}_{n+1,n+1} - C^{n+1}_{ij} - \delta_{jk}C^k_{i,n+1})\,\omega^j\,,\\ &\mathrm{d}C^i_{n+1,n+1} + C^j_{n+1,n+1}\omega^i_j = (\delta^i_j C^{n+1}_{n+1,n+1} - C^i_{j,n+1} - C^i_{n+1,j})\,\omega^j\,,\\ &\mathrm{d}C^{n+1}_{n+1,n+1} = -(C^{n+1}_{i,n+1} + C^{n+1}_{n+1,i} + \delta_{ij}C^j_{n+1,n+1})\,\omega^i\,. \end{aligned} \tag{2.9}$$

Definition 2. Let $B: V^{n+1} \to V^{n+1}$ be a linear mapping; let the *adjoint mapping* ${}^{(t)}B: V^{n+1} \to V^{n+1}$ be defined by

$$\langle B(u), v\rangle = \langle u, {}^{(t)}B(v)\rangle \quad \text{for each} \quad u, v \in V^{n+1}\,. \tag{2.10}$$

For a bilinear mapping $C: V^{n+1} \times V^{n+1} \to V^{n+1}$, we define the mappings ${}^{(1)}C$, ${}^{(2)}C$: $V^{n+1} \times V^{n+1} \to V^{n+1}$ by

$$\langle C(u, w), v\rangle = \langle {}^{(1)}C(v, w), u\rangle\,, \quad \langle C(w, u), v\rangle = \langle {}^{(2)}C(w, v), u\rangle \quad \text{for each} \quad u, v, w \in V^{n+1}\,, \tag{2.11}$$

respectively. Further, we define

$$\text{(2.12)} \qquad \operatorname{Tr} B = \sum_{\alpha=1}^{n+1} \langle B(e_\alpha), e_\alpha \rangle\,, \quad \operatorname{Tr} C = \sum_{\alpha=1}^{n+1} C(e_\alpha, e_\alpha)\,,$$

$\{e_\alpha\}$ being any orthonormal frame in V^{n+1}.

Definition 3. Let $F: V^{n+1} \times \ldots \times V^{n+1} \to V^{n+1}$ be a p-linear mapping, $S^n(1) \subset \subset E^{n+1}$ a unit hypersphere and v_{n+1} its field of unit normal vectors (with a chosen orientation). Then the function $vF: S^n(1) \to \mathbb{R}$ is defined by

$$\text{(2.13)} \qquad v\,F(m) = \langle F(v_{n+1}, \ldots, v_{n+1}), v_{n+1} \rangle$$

and the tangent vector field $\pi F: S^n(1) \to T S^n(1)$ by

$$\text{(2.14)} \qquad \pi\, F(m) = \text{pr.}\, F(v_{n+1}, \ldots, v_{n+1})\,,$$

pr.: $V^{n+1} \to T_m(S^n(1))$ being the orthogonal projection and $v_{n+1} = v_{n+1}(m)$.

Of course, in our notation,

$$\text{(2.15)} \qquad vF = F^{n+1}_{n+1,\ldots,n+1}\,, \quad \pi F = F^i_{n+1,\ldots,n+1} v_i\,.$$

3. In this section, let us reestablish some trivial results on Spec$(S^n(1))$.

Theorem 1. *Let $A \in V^{n+1}$ be a fixed vector. Then $vA \in \mathscr{E}^n_{(0,0)}$.*

Proof. Because of $vA = A^{n+1}$ and (2.7), we have $A^{n+1}_{;i} = -\delta_{ik}A^k$, $A^{n+1}_{;ij} = -\delta_{ij}A^{n+1}$, $\Delta A^{n+1} = -nA^{n+1}$. QED.

Theorem 2. *Let $B: V^{n+1} \to V^{n+1}$ be a linear mapping. Then we may write*

$$\text{(3.1)} \qquad vB = \frac{1}{n+1} \operatorname{Tr} B + f\,,$$

where

$$\text{(3.2)} \qquad \operatorname{Tr} B \in \mathscr{E}^0_{(0,0)}\,, \quad f = vB - \frac{1}{n+1} \operatorname{Tr} B \in \mathscr{E}^{2(n+1)}_{(0,0)}\,.$$

Proof. Let us write

$$\text{(3.3)} \qquad f^{(1)} := vB = B^{n+1}_{n+1}\,, \quad f^{(2)} := B^i_i\,.$$

Using (2.8) we find

$$\text{(3.4)} \qquad \Delta f^{(1)} = -2nf^{(1)} + 2f^{(2)}\,, \quad \Delta f^{(2)} = 2nf^{(1)} - 2f^{(2)}\,.$$

Hence $\Delta(f^{(1)} + f^{(2)}) = \Delta \operatorname{Tr} B = 0$ and

$$\{\Delta + 2(n+1)\} \left(vB - \frac{1}{n+1} \operatorname{Tr} B \right) = \frac{1}{n+1} \{\Delta + 2(n+1)\} (nf^{(1)} - f^{(2)}) = 0\,.$$

QED.

Theorem 3. *Let $C: V^{n+1} \times V^{n+1} \to V^{n+1}$ be a bilinear mapping. Then we may write*

$$\text{(3.5)} \qquad vC = \frac{1}{n+3} g + h$$

with

(3.6) $$g = \nu \operatorname{Tr}(C + {}^{(1)}C + {}^{(2)}C) \in \mathscr{E}^{n}_{(0,0)}\,,$$

$$h = \nu C - \frac{1}{n+3}\,\nu \operatorname{Tr}(C + {}^{(1)}C + {}^{(2)}C) \in \mathscr{E}^{3(n+2)}_{(0,0)}\,.$$

Proof. Consider the functions

(3.7) $$f^{(1)} := C^{n+1}_{n+1,n+1} = \nu C\,, \quad f^{(2)} := \delta^{kl} C^{n+1}_{kl}\,, \quad f^{(3)} := C^{k}_{k,n+1}\,,$$
$$f^{(4)} := C^{k}_{n+1,k}\,.$$

From (2.9) we get

(3.8) $$\begin{aligned} \Delta f^{(1)} &= -3nf^{(1)} + 2f^{(2)} + 2f^{(3)} + 2f^{(4)}\,,\\ \Delta f^{(2)} &= \ \ 2nf^{(1)} - (n+2)f^{(2)} - 2f^{(3)} - 2f^{(4)}\,,\\ \Delta f^{(3)} &= \ \ 2nf^{(1)} - 2f^{(2)} - (n+2)f^{(3)} - 2f^{(4)}\,,\\ \Delta f^{(4)} &= \ \ 2nf^{(1)} - 2f^{(2)} - 2f^{(3)} - (n+2)f^{(4)}\,, \end{aligned}$$

and, as a consequence,

(3.9) $$(\Delta + n)(f^{(1)} + f^{(2)}) = (\Delta + n)(f^{(1)} + f^{(3)}) = (\Delta + n)(f^{(1)} + f^{(4)}) = 0\,,$$
$$\{\Delta + 3(n+2)\}(nf^{(1)} - f^{(2)} - f^{(3)} - f^{(4)}) = 0\,.$$

We may write (3.5) with

(3.10) $$g = 3f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)}\,, \quad h = \frac{1}{n+3}(nf^{(1)} - f^{(2)} - f^{(3)} - f^{(4)})\,,$$

and we have

(3.11) $$g = \delta^{\alpha\beta} C^{n+1}_{\alpha\beta} + C^{\alpha}_{\alpha,n+1} + C^{\alpha}_{n+1,\alpha}\,.$$

From (2.11),

(3.12) $${}^{(1)}C^{\alpha}_{\beta\gamma} = \delta_{\varrho\beta}\delta^{\sigma\alpha}C^{\varrho}_{\sigma\gamma}\,, \quad \operatorname{Tr}{}^{(1)}C = \delta^{\beta\gamma}{}^{(1)}C^{\alpha}_{\beta\gamma} = \delta^{\alpha\gamma}C^{\beta}_{\gamma\beta}\,, \quad \nu \operatorname{Tr}{}^{(1)}C = C^{\beta}_{n+1,\beta}\,.$$

Similarly,

(3.13) $$\nu \operatorname{Tr}{}^{(2)}C = C^{\beta}_{\beta,n+1}\,,$$

and (3.6_1) is verified. Analogously, we verify (3.6_2). QED.

4. Let us consider the tangent vector fields of the type πF on $S^n(1)$, with $p = 0, 1, 2$.

Theorem 4. *Let $A \in V^{n+1}$ be a fixed vector. Then $\pi A \in \mathscr{E}^{1}_{(1,0)}$.*

Proof. By definition, $\pi A = A^i v_i$. From (2.7),

(4.1) $$A^i_{;j} = \delta^i_j A^{n+1}\,, \quad A^i_{;jk} = -\delta^i_j \delta_{kl} A^l\,, \quad \Delta_0 A^i = \delta^{jk} A^i_{;jk} = -A^i\,,$$

i.e., $(\Delta_0 + 1)(\pi A) = 0$. QED.

Theorem 5. *Let B: $V^{n+1} \to V^{n+1}$ be a linear mapping. Then we may write*

(4.2) $$\pi B = \tfrac{1}{2}V^{(1)} + \tfrac{1}{2}V^{(2)}$$

with

$$V^{(1)} = \pi(B + {}^{(t)}B) \in \mathscr{E}^{n+3}_{(1,0)}\,, \qquad V^{(2)} = \pi(B - {}^{(t)}B) \in \mathscr{E}^{n-1}_{(1,0)}\,. \tag{4.3}$$

Proof. Consider the vector fields

$$X^{(1)} := \pi B = B^i_{n+1} v_i\,, \quad X^{(2)} := \pi\, {}^{(t)}B = \delta^{ij} B^{n+1}_j v_i\,. \tag{4.4}$$

It is easy to check

$$\Delta_0 X^{(1)} = -(n+1)\, X^{(1)} - 2X^{(2)}\,, \quad \Delta_0 X^{(2)} = -2X^{(1)} - (n+1)\, X^{(2)}\,, \tag{4.5}$$

and the proof follows.

Theorem 6. *Let* $C\colon V^{n+1} \times V^{n+1} \to V^{n+1}$ *be a bilinear mapping and let* ${}^{(1)}C$, ${}^{(2)}C$ *be defined by* (2.11). *Then we may write*

$$\pi C = \frac{1}{n(n+3)}\, W^{(1)} + \frac{1}{3n}\, W^{(2)} + \frac{1}{3(n+3)}\, W^{(3)} \tag{4.6}$$

with

$$\begin{aligned} W^{(1)} &= \pi\, \mathrm{Tr}\, \{(n+2)\, C - {}^{(1)}C - {}^{(2)}C\} \in \mathscr{E}^{1}_{(1,0)}\,,\\ W^{(2)} &= n\pi(2C - {}^{(1)}C - {}^{(2)}C) - \pi\, \mathrm{Tr}\, (2C - {}^{(1)}C - {}^{(2)}C) \in \mathscr{E}^{2n+1}_{(1,0)}\,,\\ W^{(3)} &= (n+3)\, \pi(C + {}^{(1)}C + {}^{(2)}C) - \pi\, \mathrm{Tr}\, (C + {}^{(1)}C + {}^{(2)}C) \in \mathscr{E}^{2n+7}_{(1,0)}\,. \end{aligned} \tag{4.7}$$

In particular: The conditions

$$C = {}^{(1)}C = {}^{(2)}C\,, \quad \mathrm{Tr}\, C = 0 \tag{4.8}$$

imply $\pi C \in \mathscr{E}^{2n+7}_{(1,0)}$, *the conditions*

$$C + {}^{(1)}C + {}^{(2)}C = 0\,, \quad \mathrm{Tr}\, C = 0 \tag{4.9}$$

lead to $\pi C \in \mathscr{E}^{2n+1}_{(1,0)}$.

Proof. Consider the vector fields

$$\begin{aligned} X^{(1)} &:= \pi C = C^i_{n+1,n+1} v_i\,, & X^{(2)} &:= \pi\, \mathrm{Tr}\, C - \pi C = \delta^{jk} C^i_{jk} v_i\,,\\ X^{(3)} &:= \pi\, {}^{(1)}C = \delta^{ij} C^{n+1}_{j,n+1} v_i\,, & X^{(4)} &:= \pi\, \mathrm{Tr}\, {}^{(1)}C - \pi\, {}^{(1)}C = \delta^{ij} C^k_{jk} v_i\,,\\ X^{(5)} &:= \pi\, {}^{(2)}C = \delta^{ij} C^{n+1}_{n+1,j} v_i\,, & X^{(6)} &:= \pi\, \mathrm{Tr}\, {}^{(2)}C - \pi\, {}^{(2)}C = \delta^{ij} C^k_{kj} v_i\,. \end{aligned} \tag{4.10}$$

From (2.9) we find

$$\begin{aligned} \Delta_0 X^{(1)} &= -(2n+1)\, X^{(1)} + 2X^{(2)} - 2X^{(3)} - 2X^{(5)}\,,\\ \Delta_0 X^{(2)} &= 2nX^{(1)} - 3X^{(2)} + 2X^{(3)} + 2X^{(5)}\,,\\ \Delta_0 X^{(3)} &= -2X^{(1)} - (2n+1)\, X^{(3)} + 2X^{(4)} - 2X^{(5)}\,,\\ \Delta_0 X^{(4)} &= 2X^{(1)} + 2nX^{(3)} - 3X^{(4)} + 2X^{(5)}\,,\\ \Delta_0 X^{(5)} &= -2X^{(1)} - 2X^{(3)} - (2n+1)\, X^{(5)} + 2X^{(6)}\,,\\ \Delta_0 X^{(6)} &= 2X^{(1)} + 2X^{(3)} + 2nX^{(5)} - 3X^{(6)}\,, \end{aligned} \tag{4.11}$$

and the rest of the proof is easy. QED.

5. Let $B\colon V^{n+1} \to V^{n+1}$ be a linear mapping satisfying $B + {}^{(t)}B = 0$; for ${}^{(t)}B$, see Definition 2. Then $\pi B = B^i_{n+1} v_i$ and, because of (2.8), $B^i_{n+1;j} = -B^i_j$. Thus, see [2],

$$\delta_{ik} B^k_{n+1;j} + \delta_{jk} B^k_{n+1;i} = -\delta_{ik} B^k_j - \delta_{jk} B^k_i = 0\,,$$

and πB is a Killing vector field. Further, let $A \in V^{n+1}$ be a vector. Then $\pi A = A^i v_i$, and we have

$$\delta_{ik} A^k_{;j} + \delta_{jk} A^k_{;i} = 2A^{n+1} \delta_{ij}\,, \quad A^i_{;i} = nA^{n+1}\,,$$

i.e., πA is an infinitesimal conformal transformation.

Definition 4. Denote by $\mathscr{I}$ or $\mathscr{C}$ or $\mathscr{J}$ the $\mathbb{R}$-module of the tangent vector fields on $S^n(1)$ of the form πB with $B + {}^{(t)}B = 0$ or of the form πA or of the form πB with $B = {}^{(t)}B$, respectively.

Because of Theorems 4 and 5, we have

$$(5.1) \qquad \mathscr{I} \subset \mathscr{E}^{n-1}_{(1,0)}\,, \quad \mathscr{C} \subset \mathscr{E}^{1}_{(1,0)}\,, \quad \mathscr{J} \subset \mathscr{E}^{n+3}_{(1,0)}\,.$$

Each linear mapping $B\colon V^{n+1} \to V^{n+1}$ may be written, in a unique way, as $B = B^{(1)} + B^{(2)}$ with ${}^{(t)}B^{(1)} = -B^{(1)}$, ${}^{(t)}B^{(2)} = B^{(2)}$, i.e., each vector field of the type πB may be written as $\pi B = V + W$ with $V \in \mathscr{I}$, $W \in \mathscr{J}$.

Theorem 7. *The $\mathbb{R}$-module $\mathscr{I} \oplus \mathscr{J}$ is a Lie algebra, and we have*

$$(5.2) \qquad [\pi B, \pi \tilde{B}] = \pi[B, \tilde{B}]$$

for two linear mappings $B, \tilde{B}\colon V^{n+1} \to V^{n+1}$, the linear mapping $[B, \tilde{B}]\colon V^{n+1} \to V^{n+1}$ being defined by $[B, \tilde{B}](u) = B(\tilde{B}(u)) - \tilde{B}(Bu))$ for each $u \in V^{n+1}$.

Proof. Let $V = V^i v_i$, $W = W^i v_i$ be two tangent vector fields on $S^n(1)$. Then

$$(5.3) \qquad [V, W] = (V^j W^i_{;j} - W^j V^i_{;j})\, v_i\,.$$

We have

$$(5.4) \qquad \pi B = B^i_{n+1} v_i\,, \quad \pi \tilde{B} = \tilde{B}^i_{n+1} v_i\,, \quad \pi\, {}^{(t)}B = \delta^{ij} B^{n+1}_j v_i\,, \quad \pi\, {}^{(t)}\tilde{B} = \delta^{ij} \tilde{B}^{n+1}_j v_i\,.$$

Consider the vector fields

$$(5.5) \quad X^{(1)} := [\pi B, \pi \tilde{B}] = (B^i_{n+1} \tilde{B}^{n+1}_{n+1} - B^j_{n+1} \tilde{B}^i_j - \tilde{B}^i_{n+1} B^{n+1}_{n+1} + \tilde{B}^j_{n+1} B^i_j)\, v_i\,,$$
$$X^{(2)} := [\pi\, {}^{(t)}B, \pi\, {}^{(t)}\tilde{B}] = \delta^{ij}(B^{n+1}_j \tilde{B}^{n+1}_{n+1} - B^{n+1}_k \tilde{B}^k_j - \tilde{B}^{n+1}_j B^{n+1}_{n+1} + \tilde{B}^{n+1}_k B^k_j)\, v_i\,.$$

By a direct calculation,

$$(5.6) \qquad \Delta_0 X^{(1)} = -(n+1)\, X^{(1)} + 2X^{(2)}\,, \quad \Delta_0 X^{(2)} = 2X^{(1)} - (n+1)\, X^{(2)}\,,$$

and we may write

$$X^{(1)} = \tfrac{1}{2}(X^{(1)} - X^{(2)}) + \tfrac{1}{2}(X^{(1)} + X^{(2)}) \quad \text{with}$$
$$X^{(1)} - X^{(2)} \in \mathscr{E}^{n+3}_{(1,0)}\,, \quad X^{(1)} + X^{(2)} \in \mathscr{E}^{n-1}_{(1,0)}\,.$$

Now, for $\bar{B} := [B, \tilde{B}]$, we have $\bar{B}^\beta_\alpha = \tilde{B}^\gamma_\alpha B^\beta_\gamma - B^\gamma_\alpha \tilde{B}^\beta_\gamma$ and

$$\pi[B, \tilde{B}] = \bar{B}^i_{n+1} v_i = (\tilde{B}^\gamma_{n+1} B^i_\gamma - B^\gamma_{n+1} \tilde{B}^i_\gamma)\, v_i = X^{(1)}\,;$$

similarly, $\pi[{}^{-(t)}B, {}^{(t)}\tilde{B}] = X^{(2)}$. Thus

(5.7) $$[\pi B, \pi \tilde{B}] = \tfrac{1}{2}Y^{(1)} + \tfrac{1}{2}Y^{(2)}$$

with

(5.8) $$Y^{(1)} = \pi\{[B, \tilde{B}] - [{}^{(t)}B, {}^{(t)}\tilde{B}]\} \in \mathscr{E}^{n+3}_{(1,0)},$$
$$Y^{(2)} = \pi\{[B, \tilde{B}] + [{}^{(t)}B, {}^{(t)}\tilde{B}]\} \in \mathscr{E}^{n-1}_{(1,0)}.$$ QED.

Of course, $\mathscr{I}$ itself is a Lie algebra. Indeed, let $B, \tilde{B}$: $V^{n+1} \to V^{n+1}$ satisfy ${}^{(t)}B = -B$, ${}^{(t)}\tilde{B} = -\tilde{B}$. Then the general formula ${}^{(t)}[B, \tilde{B}] = [{}^{(t)}\tilde{B}, {}^{(t)}B]$ yields ${}^{(t)}[B, \tilde{B}] = -[B, \tilde{B}]$ and (5.3) implies $[\pi B, \pi\tilde{B}] \in \mathscr{I}$ for $\pi B, \pi\tilde{B} \in \mathscr{I}$.

Theorem 8. *Let $A, \tilde{A} \in V^{n+1}$ be vectors. Then*

(5.9) $$[\pi A, \pi\tilde{A}] = \pi B \in \mathscr{I}$$

with B: $V^{n+1} \to V^{n+1}$ defined by

(5.10) $$B(v) = \langle \tilde{A}, v\rangle A - \langle A, v\rangle \tilde{A}$$

and satisfying

(5.11) $$B + {}^{(t)}B = 0\,.$$

Proof. For $\pi A = A^i v_i$, $\pi\tilde{A} = \tilde{A}^i v_i$, we have

(5.12) $$[\pi A, \pi\tilde{A}] = (A^j\tilde{A}^i_{;j} - \tilde{A}^j A^i_{;j})\, v_i = (A^i\tilde{A}^{n+1} - \tilde{A}^i A^{n+1})\, v_i\,.$$

Let the linear mapping B be defined by (5.10); then $B^\beta_\alpha = \delta_{\alpha\gamma}(\tilde{A}^\gamma A^\beta - A^\gamma\tilde{A}^\beta)$, and πB is exactly equal to (5.12). It is easy to prove (5.11). QED.

Theorem 9. *Let $A \in V^{n+1}$ be a vector and B: $V^{n+1} \to V^{n+1}$ a linear mapping satisfying $B + {}^{(t)}B = 0$. Then*

(5.13) $$[\pi A, \pi B] = -\pi\, B(A)\,.$$

Proof. We have

(5.14) $$[\pi A, \pi B] = (A^j B^i_{n+1;j} - B^j_{n+1}A^i_{;j})\, v_i =$$
$$= (A^i B^{n+1}_{n+1} - A^j B^i_j - A^{n+1}B^i_{n+1})\, v_i = -A^\alpha B^i_\alpha v_i = -\pi\, B(A)$$

because of $B^{n+1}_{n+1} = 0$. QED.

Thus $\mathscr{I} \oplus \mathscr{C}$ is a Lie algebra.

For the sake of completeness, let us study the behavior of $[\pi A, \pi B]$ in the case ${}^{(t)}B = B$. For each linear mapping $\bar{B}$: $V^{n+1} \to V^{n+1}$ of the type $\bar{B} = b.\,\mathrm{id.}$, $b \in \mathbb{R}$, we have $\pi\bar{B} = 0$. Thus, without loss of generality, we may restrict ourselves to linear mappings B satisfying $\operatorname{Tr} B = 0$.

Theorem 10. *Let $A \in V^{n+1}$ be a fixed vector and B: $V^{n+1} \to V^{n+1}$ a linear mapping satisfying ${}^{(t)}B = B$, $\operatorname{Tr} B = 0$. Then we may write*

(5.15) $$[\pi A, \pi B] = \frac{1}{3n(n+3)}\{-3(n+1)(n+2)\, X^{(1)} + 2(n+3)\, X^{(2)} + nX^{(3)}\}$$

with

(5.16)
$$X^{(1)} = \pi\, B(A) \in \mathscr{E}^{1}_{(1,0)}\,,$$

(5.17)
$$X^{(2)} = \pi C \in \mathscr{E}^{2n+1}_{(1,0)}\,,$$

where

$$C(v, w) = \tfrac{1}{2}n\{2\langle v, B(w)\rangle A - \langle A, w\rangle B(v) - \langle A, v\rangle B(w)\} +$$
$$+ \tfrac{1}{2}\{2\langle v, w\rangle B(A) - \langle B(A), v\rangle w - \langle B(A), w\rangle v\} \quad \text{for} \quad v, w \in V^{n+1}\,,$$

(5.18)
$$X^{(3)} = \pi\tilde{C} \in \mathscr{E}^{2n+7}_{(1,0)}\,,$$

where

$$\tilde{C}(v, w) = (n + 3)\left\{\langle v, B(w)\rangle A + \langle A, w\rangle B(v) + \langle A, v\rangle B(w)\right\} -$$
$$- 2\{\langle v, w\rangle B(A) + \langle B(A, v)\rangle w + \langle B(A), w\rangle v\} \quad \textit{for} \quad v, w \in V^{n+1}\,.$$

We have

(5.19)
$$\operatorname{Tr} C = 0\,,$$
$$C(v, w) = C(w, v)\,, \quad \langle C(v, w), u\rangle + \langle C(w, u), v\rangle + \langle C(u, v), w\rangle = 0\,,$$

(5.20)
$$\operatorname{Tr}\tilde{C} = 0\,, \quad \tilde{C}(v, w) = \tilde{C}(w, v)\,, \quad \langle\tilde{C}(v, w), u\rangle = \langle\tilde{C}(u, w), v\rangle$$

for $u, v, w \in V^{n+1}$.

Proof. We have, see (5.14),

(5.21)
$$[\pi A, \pi B] = Y^{(1)} - Y^{(2)} - Y^{(3)}$$

with

(5.22)
$$Y^{(1)} = A^i B^{n+1}_{n+1} v_i\,, \quad Y^{(2)} = A^j B^i_j v_i\,, \quad Y^{(3)} = A^{n+1} B^i_{n+1} v_i\,.$$

Using the conditions $B^\beta_\alpha = B^\alpha_\beta$, $B^\alpha_\alpha = 0$, we find

(5.23)
$$\begin{aligned} \Delta_0 Y^{(1)} &= -(2n + 3)\, Y^{(1)} - 4Y^{(3)}\,, \\ \Delta_0 Y^{(2)} &= 2Y^{(1)} - 3Y^{(2)} + 2(n + 1)\, Y^{(3)}\,, \\ \Delta_0 Y^{(3)} &= -2Y^{(1)} + 2Y^{(2)} - (2n + 3)\, Y^{(3)}\,. \end{aligned}$$

Thus we may write (5.15) with

(5.24)
$$X^{(1)} = Y^{(2)} + Y^{(3)}\,, \quad X^{(2)} = nY^{(1)} + Y^{(2)} - (n - 1)\, Y^{(3)}\,,$$
$$X^{(3)} = (n + 3)\, Y^{(1)} - 2Y^{(2)} + 2(n + 2)\, Y^{(3)}\,,$$

and we see that

(5.25)
$$(\Delta_0 + 1)\, X^{(1)} = 0\,, \quad (\Delta_0 + 2n + 1)\, X^{(2)} = 0\,, \quad (\Delta_0 + 2n + 7)\, X^{(3)} = 0\,.$$

The rest of the proof follows easily. QED.

6. In this section, let us study the differential equation $(\Delta_0 + \lambda)\, x = 0$ for tangent vector fields on a 2-dimensional Riemannian manifold.

Theorem 11. *Let* $(M, \mathrm{d}s^2)$ *be an orientable compact Riemannian manifold,*

$\dim M = 2$, *K its Gauss curvature and x a tangent vector field on M satisfying*

$$(6.1) \qquad (\Delta_0 + l)\, x = 0\,,$$

$l\colon M \to \mathbb{R}$ being a function. If $\max_M l < \min_M K$, *then* $x \equiv 0$.

Proof. Let us write (locally)

$$(6.2) \qquad \mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,;$$

then there is a 1-form ω_1^2 satisfying

$$(6.3) \qquad \mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2\,, \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2\,, \quad \mathrm{d}\omega_1^2 = -K\omega^1 \wedge \omega^2\,,$$

K being the Gauss curvature. The first covariant derivatives of the vector field $x = x^i v_i$ are given by

$$(6.4) \qquad \mathrm{d}x^1 - x^2\omega_1^2 = x_1^1\omega^1 + x_2^1\omega^2\,, \quad \mathrm{d}x^2 + x^1\omega_1^2 = x_1^2\omega^1 + x_2^2\omega^2\,,$$

the second covariant derivatives by

$$(6.5) \qquad \begin{aligned} \mathrm{d}x_1^1 - (x_1^2 + x_2^1)\,\omega_1^2 &= x_{11}^1\omega^1 + x_{12}^1\omega^2\,,\\ \mathrm{d}x_2^1 + (x_1^1 - x_2^2)\,\omega_1^2 &= x_{21}^1\omega^1 + x_{22}^1\omega^2\,,\\ \mathrm{d}x_1^2 + (x_1^1 - x_2^2)\,\omega_1^2 &= x_{11}^2\omega^1 + x_{12}^2\omega^2\,,\\ \mathrm{d}x_2^2 + (x_1^2 + x_2^1)\,\omega_1^2 &= x_{21}^2\omega^1 + x_{22}^2\omega^2\,; \end{aligned}$$

we write simply $x^i_{jk\ldots}$ instead of $x^i_{;jk\ldots}$. Inserting (6.5) into the differential consequences of (6.4), we get

$$(6.6) \qquad x_{21}^1 - x_{12}^1 = Kx^2\,, \quad x_{21}^2 - x_{12}^2 = -Kx^1\,.$$

Introduce the 1-forms

$$(6.7) \qquad \begin{aligned} \varphi_1 &:= \delta_{ij}x^i x_k^j \omega^k = (x^1x_1^1 + x^2x_1^2)\,\omega^1 + (x^1x_2^1 + x^2x_2^2)\,\omega^2\,,\\ \varphi_2 &:= \delta_{ij}x^k x_k^i \omega^j = (x^1x_1^1 + x^2x_2^1)\,\omega^1 + (x^1x_1^2 + x^2x_2^2)\,\omega^2\,,\\ \varphi_3 &:= \delta_{ij}x^i x_k^k \omega^j = (x^1x_1^1 + x^1x_2^2)\,\omega^1 + (x^2x_1^1 + x^2x_2^2)\,\omega^2\,. \end{aligned}$$

Using (6.6) and (6.1), i.e.,

$$(6.8) \qquad x_{11}^1 + x_{22}^1 + lx^1 = 0\,, \quad x_{11}^2 + x_{22}^2 + lx^2 = 0\,,$$

we have

$$(6.9) \qquad \begin{aligned} &\mathrm{d} * (\varphi_1 + \varphi_2 - \varphi_3) =\\ &= \{(x_1^1 - x_2^2)^2 + (x_1^2 + x_2^1)^2 + (K - l)\,[(x^1)^2 + (x^2)^2]\}\,\mathrm{d}v\,, \end{aligned}$$

$*$ being the Hodge operator and $\mathrm{d}v = \omega^1 \wedge \omega^2$ the area element. By the Stokes theorem and the supposition

$$(6.10) \qquad K - l = (K - \min_M K) + (\max_M l - l) + (\min_M K - \max_M l) > 0\,,$$

we have $x^1 = x^2 = 0$, i.e., $x \equiv 0$. QED.

Theorem 12. *Let $(M, \mathrm{d}s^2)$ be an orientable compact Riemannian manifold,* dim $M = 2$, K *its Gauss curvature, and let* $\max_M K < 5 \min_M K$. *Let* x *be a tangent vector field on* M *satisfying* (6.1) *with*

$$\max_M K < \min_M l \leqq \max_M l < 5 \min_M K \,. \tag{6.11}$$

Then $x \equiv 0$.

Proof. The third covariant derivatives of x^i (see the proof of the preceding theorem) are given by

$$\begin{aligned}
&dx^1_{11} - (x^1_{12} + x^1_{21} + x^2_{11})\,\omega^2_1 = x^1_{111}\omega^1 + x^1_{112}\omega^2\,,\\
&dx^1_{12} + (x^1_{11} - x^1_{22} - x^2_{12})\,\omega^2_1 = x^1_{121}\omega^1 + x^1_{122}\omega^2\,,\\
&dx^1_{21} + (x^1_{11} - x^1_{22} - x^2_{21})\,\omega^2_1 = x^1_{211}\omega^1 + x^1_{212}\omega^2\,,\\
&dx^1_{22} + (x^1_{12} + x^1_{21} - x^2_{22})\,\omega^2_1 = x^1_{221}\omega^1 + x^1_{222}\omega^2\,,\\
&dx^2_{11} + (x^1_{11} - x^2_{12} - x^2_{21})\,\omega^2_1 = x^2_{111}\omega^1 + x^2_{112}\omega^2\,,\\
&dx^2_{12} + (x^1_{12} + x^2_{11} - x^2_{22})\,\omega^2_1 = x^2_{121}\omega^1 + x^2_{122}\omega^2\,,\\
&dx^2_{21} + (x^1_{21} + x^2_{11} - x^2_{22})\,\omega^2_1 = x^2_{211}\omega^1 + x^2_{212}\omega^2\,,\\
&dx^2_{22} + (x^1_{22} + x^2_{12} + x^2_{21})\,\omega^2_1 = x^2_{221}\omega^1 + x^2_{222}\omega^2\,.
\end{aligned} \tag{6.12}$$

The differential consequences of (6.5) yield

$$\begin{aligned}
&x^1_{121} - x^1_{112} = K(x^2_1 + x^1_2)\,, \quad x^1_{221} - x^1_{212} = K(x^2_2 - x^1_1)\,,\\
&x^2_{121} - x^2_{112} = K(x^2_2 - x^1_1)\,, \quad x^2_{221} - x^2_{212} = -K(x^2_1 + x^1_2)\,.
\end{aligned} \tag{6.13}$$

From (6.6) we get

$$\begin{aligned}
&x^1_{211} - x^1_{121} = K_1x^2 + Kx^2_1\,, \qquad x^1_{212} - x^1_{122} = K_2x^2 + Kx^2_2\,,\\
&x^2_{211} - x^2_{121} = -K_1x^1 - Kx^1_1\,, \qquad x^2_{212} - x^2_{122} = -K_2x^1 - Kx^1_2\,,
\end{aligned} \tag{6.14}$$

from (6.8),

$$\begin{aligned}
&x^1_{111} + x^1_{221} + l_1x^1 + lx^1_1 = 0\,, \qquad x^2_{111} + x^2_{221} + l_1x^2 + lx^2_1 = 0\,,\\
&x^1_{112} + x^1_{222} + l_2x^1 + lx^1_2 = 0\,, \qquad x^2_{112} + x^2_{222} + l_2x^2 + lx^2_2 = 0\,;
\end{aligned} \tag{6.15}$$

here K_i and l_i are the covariant derivatives of K and l, respectively, defined by

$$dK = K_1\omega^1 + K_2\omega^2\,, \quad dl = l_1\omega^1 + l_2\omega^2\,. \tag{6.16}$$

Using (6.6), (6.8), (6.13)–(6.15), we get

$$\begin{aligned}
&d * [d\{(x^1_1 - x^2_2)^2 + (x^2_1 + x^1_2)^2\} + (r - 2K + 2l)(\varphi_1 + \varphi_2 - \varphi_3)] =\\
&= [(x^1_{11} - x^1_{22} - x^2_{12} - x^2_{21})^2 + (x^2_{11} - x^2_{22} + x^1_{12} + x^1_{21})^2 +\\
&\quad + (4K + r)\{(x^1_1 - x^2_2)^2 + (x^2_1 + x^1_2)^2\} +\\
&\quad + (l - K)(K - l - r)\{(x^1)^2 + (x^2)^2\}]\, dv \quad \text{for} \quad r \in \mathbb{R}\,.
\end{aligned} \tag{6.17}$$

Now, take

$$r = -\tfrac{1}{2}(3 \min_M K + \max_M l)\,; \tag{6.18}$$

then, using (6.11),

$$(6.19)\quad 4K + r = 4(K - \min_M K) + \tfrac{1}{2}(5 \min_M K - \max_M l) > 0 ,$$
$$l - K = (l - \min_M l) + (\max_M K - K) + (\min_M l - \max_M K) > 0 ,$$
$$K - l - r = (K - \min_M K) + (\max_M l - l) + \tfrac{1}{2}(5 \min_M K - \max_M l) > 0 ,$$

and the integral formula based on (6.17) implies $x^1 = x^2 = 0$, i.e., $x \equiv 0$. QED.

Using (6.9) and (6.17), we are in the position to describe the modules $\mathscr{E}^1_{(1,0)}$ and $\mathscr{E}^5_{(1,0)}$ on $S^2(1)$. First of all, let us introduce the following

Definition 5. On $S^2(1)$, let us choose an orientation; the mapping

$$(6.20)\qquad *: T_m(S^2(1)) \to T_m(S^2(1)) , \quad m \in S^2(1) ,$$

associates with the vector $t \in T_m(S^2(1))$ the vector $*t \in T_m(S^2(1))$ such that $\langle t, t\rangle = \langle *t, *t\rangle$, $\langle t, *t\rangle = 0$ and the couple $(t, *t)$ is positively oriented.

Let the frames $\{v_1, v_2, v_3\}$ associated with the points of $S^2(1)$ be chosen in such a way that

$$(6.21)\qquad *v_1 = v_2 , \quad *v_2 = -v_1 .$$

Theorem 13. *On $S^2(1)$, let $x \in \mathscr{E}^1_{(1,0)}$. Then there are vectors $A, \tilde{A} \in V^3$ such that*

$$(6.22)\qquad x = \pi A + *\pi\tilde{A} .$$

Conversely, each vector field of the type (6.22) belongs to $\mathscr{E}^1_{(1,0)}$.

Proof. In our case, we have $K = l = 1$. From the integral formula based on (6.9) we get

$$(6.23)\qquad x^1_1 - x^2_2 = x^2_1 + x^1_2 = 0 ;$$

from (6.5),

$$(6.24)\qquad x^1_{11} - x^2_{21} = x^1_{12} - x^2_{22} = x^1_{21} + x^2_{11} = x^1_{22} + x^2_{12} = 0 .$$

If we write

$$(6.25)\qquad a := x^1_1 = x^2_2 , \quad a' := x^2_1 = -x^1_2 ,$$

the equations (6.4) turn out to be

$$(6.26)\qquad \mathrm{d}x^1 - x^2\omega^2_1 = a\omega^1 - a'\omega^2 , \quad \mathrm{d}x^2 + x^1\omega^2_1 = a'\omega^1 + a\omega^2 .$$

Applying the exterior differentiation and Cartan's lemma, we get the existence of functions $b, b', \ldots, f, f'$ such that

$$(6.27)\qquad \mathrm{d}a = (b - \tfrac{1}{2}x^1)\,\omega^1 - (b' + \tfrac{1}{2}x^2)\,\omega^2 ,$$
$$\mathrm{d}a' = (b' - \tfrac{1}{2}x^2)\,\omega^1 + (b + \tfrac{1}{2}x^1)\,\omega^2 ,$$

$$(6.28)\qquad \mathrm{d}b + b'\omega^2_1 = (c - \tfrac{1}{2}a)\,\omega^1 - (c' + \tfrac{1}{2}a')\,\omega^2 ,$$
$$\mathrm{d}b' - b\omega^2_1 = (c' - \tfrac{1}{2}a')\,\omega^1 + (c + \tfrac{1}{2}a)\,\omega^2 ,$$
$$\mathrm{d}c + 2c'\omega^2_1 = e\omega^1 - e'\omega^2 ,$$

$$\begin{aligned} dc' - 2c\omega_1^2 &= e'\omega^1 + e\omega^2 , \\ de + 3e'\omega_1^2 &= (f + c)\,\omega^1 - (f' - c')\,\omega^2 , \\ de' - 3e\omega_1^2 &= (f' + c')\,\omega^1 + (f - c)\,\omega^2 . \end{aligned}$$

Now,

$$d * d(c^2 + c'^2) = 4(e^2 + e'^2 + c^2 + c'^2)\, dv ; \tag{6.29}$$

applying the Stokes theorem to the 1-form $*d(c^2 + c'^2)$ on $S^2(1)$, we get

$$e = e' = c = c' = 0 , \tag{6.30}$$

and the equations $(6.28_{1,2})$ reduce to

$$db + b'\omega_1^2 = -\tfrac{1}{2}(a\omega^1 + a'\omega^2) , \quad db' - b\omega_1^2 = -\tfrac{1}{2}(a'\omega^1 - a\omega^2) . \tag{6.31}$$

The system (6.26) + (6.27) + (6.31) is completely integrable.

Consider the vectors

$$\begin{aligned} A &= (\tfrac{1}{2}x^1 - b)\, v_1 + (\tfrac{1}{2}x^2 + b')\, v_2 + a v_3 , \\ \tilde{A} &= (\tfrac{1}{2}x^2 - b')\, v_1 - (\tfrac{1}{2}x^1 + b)\, v_2 + a' v_3 ; \end{aligned} \tag{6.32}$$

it is easy to see that $dA = d\tilde{A} = 0$, i.e., A and $\tilde{A}$ are fixed vectors of V^3. Finally, (6.22) is immediate.

Conversely, given two vectors

$$A = A^1 v_1 + A^2 v_2 + A^3 v_3 , \quad \tilde{A} = \tilde{A}^1 v_1 + \tilde{A}^2 v_2 + \tilde{A}^3 v_3 \tag{6.33}$$

of V^3, we have

$$dA^1 - A^2\omega_1^2 = A^3\omega^1 , \quad dA^2 + A^1\omega_1^2 = A^3\omega^2 , \quad dA^3 = -A^1\omega^1 - A^2\omega^2 , \tag{6.34}$$

and analogous equations for $\tilde{A}^\alpha$. The vector field (6.22) then is

$$x \equiv x^1 v_1 + x^2 v_2 = (A^1 - \tilde{A}^2)\, v_1 + (A^2 + \tilde{A}^1)\, v_2 . \tag{6.35}$$

Inserting into (6.4) and using (6.34), we get

$$x_1^1 = A^3 , \quad x_2^1 = -\tilde{A}^3 , \quad x_1^2 = \tilde{A}^3 , \quad x_2^2 = A^3 . \tag{6.36}$$

Thus we have (6.23); this and (6.6) for $K = 1$ imply $x_{11}^1 + x_{22}^1 + x^1 = x_{11}^2 + x_{22}^2 + x^2 = 0$. QED.

Theorem 14. *On $S^2(1)$, let $x \in \mathscr{E}^5_{(1,0)}$. Then there are linear mappings B, $\tilde{B}$: $V^3 \to V^3$ satisfying*

$$B = {}^{(t)}B , \quad \operatorname{Tr} B = 0 ; \quad \tilde{B} = {}^{(t)}\tilde{B} , \quad \operatorname{Tr} B = 0 , \tag{6.37}$$

such that

$$x = \pi B + *\pi\tilde{B} . \tag{6.38}$$

Conversely, each vector field of the type (6.38) belongs to $\mathscr{E}^5_{(1,0)}$.

Proof. In this case, we have $K = 1$, $l = 5$. Using (6.17) with $r = -4$, we get, integrating over $S^2(1)$,

$$x_{11}^1 - x_{22}^1 - x_{12}^2 - x_{21}^2 = 0 , \quad x_{11}^2 - x_{22}^2 + x_{12}^1 + x_{21}^1 = 0 . \tag{6.39}$$

Further we have, of course,

$$x^1_{11} + x^1_{22} + 5x^1 = 0\,, \quad x^2_{11} + x^2_{22} + 5x^2 = 0 \tag{6.40}$$

and

$$x^1_{21} - x^1_{12} - x^2 = 0\,, \quad x^2_{21} - x^2_{12} + x^1 = 0 \tag{6.41}$$

because of (6.6). From (6.29)–(6.31) we get the existence of functions A, A' such that (6.5) become

$$\begin{aligned} dx^1_1 - (x^2_1 + x^1_2)\,\omega^2_1 &= (A' - \tfrac{5}{2}x^1)\,\omega^1 + (A - \tfrac{1}{2}x^2)\,\omega^2\,, \\ dx^1_2 + (x^1_1 - x^2_2)\,\omega^2_1 &= (A + \tfrac{1}{2}x^2)\,\omega^1 - (A' + \tfrac{5}{2}x^1)\,\omega^2\,, \\ dx^2_1 + (x^1_1 - x^2_2)\,\omega^2_1 &= -(A + \tfrac{5}{2}x^2)\,\omega^1 + (A' + \tfrac{1}{2}x^1)\,\omega^2\,, \\ dx^2_2 + (x^2_1 + x^1_2)\,\omega^2_1 &= (A' - \tfrac{1}{2}x^1)\,\omega^1 + (A - \tfrac{5}{2}x^2)\,\omega^2\,. \end{aligned} \tag{6.42}$$

The prolongation yields the existence of functions $B, B', \ldots, E, E'$ such that

$$\begin{aligned} dA + A'\omega^2_1 &= \{B + \tfrac{3}{4}(x^2_1 - x^1_2)\}\,\omega^1 - \{B' + \tfrac{3}{4}(x^1_1 + x^2_2)\}\,\omega^2\,, \\ dA' - A\omega^2_1 &= \{B' - \tfrac{3}{4}(x^1_1 + x^2_2)\}\,\omega^1 + \{B - \tfrac{3}{4}(x^2_1 - x^1_2)\}\,\omega^2\,; \end{aligned} \tag{6.43}$$

$$\begin{aligned} dB + 2B'\omega^2_1 &= (C - A)\,\omega^1 - (C' + A')\,\omega^2\,, \\ dB' - 2B\omega^2_1 &= (C' - A')\,\omega^1 + (C + A)\,\omega^2\,; \end{aligned} \tag{6.44}$$

$$\begin{aligned} &dC + 3C'\omega^2_1 = D\omega^1 - D'\omega^2\,, \quad dC' - 3C\omega^2_1 = D'\omega^1 + D\omega^2\,; \\ &dD + 4D'\omega^2_1 = (E + \tfrac{3}{2}C)\,\omega^1 - (E' - \tfrac{3}{2}C')\,\omega^2\,, \\ &dD' - 4D\omega^2_1 = (E' + \tfrac{3}{2}C')\,\omega^1 + (E - \tfrac{3}{2}C)\,\omega^2\,. \end{aligned} \tag{6.45}$$

Further,

$$d * d(C^2 + C'^2) = 2[2(D^2 + D'^2) + 3(C^2 + C'^2)]\,dv\,; \tag{6.46}$$

integrating over $S^2(1)$ we get $C = C' = D = D' = 0$, and (6.44) reduce to

$$dB + 2B'\omega^2_1 = -A\omega^1 - A'\omega^2\,, \quad dB' - 2B\omega^2_1 = -A'\omega^1 + A\omega^2\,. \tag{6.47}$$

The system (6.4) + (6.42) + (6.43) + (6.47) is completely integrable.

For a linear mapping $B\colon V^3 \to V^3$ given by $B(v_\alpha) = B^\beta_\alpha v_\beta$ we have

$$\begin{aligned} &dB^1_1 - 2B^2_1\omega^2_1 = 2B^3_1\omega^1\,, \quad dB^2_1 + (B^1_1 - B^2_2)\,\omega^2_1 = B^3_2\omega^1 + B^3_1\omega^2\,, \\ &dB^3_1 - B^3_2\omega^2_1 = (B^3_3 - B^1_1)\,\omega^1 - B^2_1\omega^2\,, \quad dB^2_2 + 2B^2_1\omega^2_1 = 2B^3_2\omega^2\,, \\ &dB^3_2 + B^3_1\omega^2_1 = -B^2_1\omega^1 + (B^3_3 - B^2_2)\,\omega^2\,, \quad dB^3_3 = -2B^3_1\omega^1 - 2B^3_2\omega^2 \end{aligned} \tag{6.48}$$

provided

$$B^\beta_\alpha = B^\alpha_\beta\,, \quad B^1_1 + B^2_2 + B^3_3 = 0\,, \tag{6.49}$$

i.e., (6.37).

Returning back, consider the fields of linear mappings $B, \tilde{B}$ defined by

$$\begin{aligned} B^1_1 &= \tfrac{1}{12}(4B' - 5x^1_1 + x^2_2)\,, \quad B^2_2 = -\tfrac{1}{12}(4B' - x^1_1 + 5x^2_2)\,, \\ B^3_3 &= \tfrac{1}{3}(x^1_1 + x^2_2)\,, \\ B^2_1 &= B^1_2 = \tfrac{1}{12}(4B - 3x^2_1 - 3x^1_2)\,, \quad B^3_1 = B^1_3 = -\tfrac{1}{6}(2A' - 3x^1)\,, \end{aligned} \tag{6.50}$$

$$\begin{aligned}
B_2^3 &= \quad B_3^2 = -\tfrac{1}{6}(2A - 3x^2)\,;\\
\tilde{B}_1^1 &= -\tfrac{1}{12}(4B + 5x_1^2 + x_2^1)\,, \quad \tilde{B}_2^2 = \tfrac{1}{12}(4B + x_1^2 + 5x_2^1)\,,\\
\tilde{B}_3^3 &= \quad \tfrac{1}{3}(x_1^2 - x_2^1)\,,\\
\tilde{B}_1^2 &= \quad \tilde{B}_2^1 = \tfrac{1}{12}(4B' + 3x_1^1 - 3x_2^2)\,, \quad \tilde{B}_1^3 = \tilde{B}_3^1 = \tfrac{1}{6}(2A + 3x^2)\,,\\
\tilde{B}_2^3 &= \quad \tilde{B}_3^2 = -\tfrac{1}{6}(2A' + 3x^1)\,.
\end{aligned}$$

By definition, they satisfy (6.37). It is just a matter of patience to verify the equations (6.48) and the analogous equations for $\tilde{B}_\alpha^\beta$. Thus we get two fixed mappings $B, \tilde{B}$: $V^3 \to V^3$. To check (6.38) is easy.

Conversely, let two linear mappings $B, \tilde{B}$: $V^3 \to V^3$ satisfying (6.37), (6.48) and the analogous equations for $\tilde{B}$ be given. Then

(6.51) $$x \equiv x^1 v_1 + x^2 v_2 = (B_1^3 - \tilde{B}_2^3)\, v_1 + (B_2^3 + \tilde{B}_1^3)\, v_2\,.$$

Inserting into (6.4) and using (6.48), we get

(6.52) $$\begin{aligned}
x_1^1 &= \quad B_3^3 - B_1^1 + \tilde{B}_1^2\,, \quad x_2^1 = -B_1^2 + \tilde{B}_2^2 - \tilde{B}_3^3\,,\\
x_1^2 &= -B_1^2 - \tilde{B}_1^1 + \tilde{B}_3^3\,, \quad x_2^2 = \quad B_3^3 - B_2^2 - \tilde{B}_1^2\,;
\end{aligned}$$

from (6.5),

(6.53) $$x_{11}^1 = -4B_1^3 + \tilde{B}_2^3\,, \quad x_{22}^1 = -B_1^3 + 4\tilde{B}_2^3\,, \quad x_{11}^2 = -B_2^3 - 4\tilde{B}_1^3\,,$$
$$x_{22}^2 = -4B_2^3 - \tilde{B}_1^3\,,$$

and we have (6.40). QED.

In the case dim $M \geqq 2$, let us prove just one result.

Theorem 15. *Let $(M, \mathrm{d}s^2)$ be an orientable compact Riemannian manifold. If the quadratic form*

(6.54) $$\{R_{ij} - (n-1)\,\lambda\delta_{ij}\}\,\xi^i\xi^j = Ric(\xi) - (n-1)\,\lambda\langle\xi, \zeta\rangle\,, \quad \lambda \in \mathbb{R}\,,$$

is positive definite at each point $m \in M$, then $\lambda \notin \mathrm{Spec}_{(1,0)}(M, \mathrm{d}s^2)$.

Proof. Let $x = x^i v_i$ be a tangent vector field on M. The first covariant derivatives $x_{;j}^i$ are given by

(6.55) $$\mathrm{d}x^i + x^j \omega_j^i = x_{;j}^i \omega^j$$

with the differential consequences

(6.56) $$(\mathrm{d}x_{;j}^i - x_{;k}^i \omega_j^k + x_{;j}^k \omega_k^i) \wedge \omega^j = -\tfrac{1}{2} x^j R_{jkl}^i \omega^k \wedge \omega^l\,.$$

The second covariant derivatives being given by

(6.57) $$\mathrm{d}x_{;j}^i - x_{;k}^i \omega_j^k + x_{;j}^k \omega_k^i = x_{;jk}^i \omega^k\,,$$

we have

(6.58) $$x_{;jk}^i - x_{;kj}^i = R_{ljk}^i x^l\,.$$

Consider the 1-forms, compare with (6.7),

(6.59) $$\omega_1 = \delta_{ij} x^i x_{;k}^j \omega^k\,, \quad \omega_2 = \delta_{ij} x^j x_{;k}^k \omega^i\,, \quad \omega_3 = \delta_{ij} x^k x_{;k}^j \omega^i\,.$$

Then it is easy to see that, under the supposition

$$(6.60)\qquad \delta^{ij}x^k_{;ij} + \lambda x^k = 0\,,$$

$$(6.61)\qquad \mathrm{d} * \{(n-1)\,\omega_1 - \omega_2 + \omega_3\} = \{\sum_{i<j}(x^i_{;i} - x^j_{;j})^2 + \sum_{i<j}(x^j_{;i} + x^i_{;j})^2 + \\ + (n-2)\sum_{i \neq j}(x^i_{;j})^2 + [R_{ij} - (n-1)\,\lambda\delta_{ij}]\,x^i x^j\}\,\mathrm{d}v\,,$$

where $\mathrm{d}v = \omega^1 \wedge \ldots \wedge \omega^n$ is the volume element. Using the Stokes theorem, we complete the proof. QED.

Theorem 16. *For $n > 2$, we have $\mathscr{E}^1_{(1,0)}(S^n(1)) = \mathscr{C}$.*

Proof. For $M = S^n(1)$, (6.61) reduces to, see (2.4_2),

$$(6.62)\qquad \mathrm{d} * \{(n-1)\,\omega_1 - \omega_2 + \omega_3\} = \{\sum_{i<j}(x^i_{;i} - x^j_{;j})^2 + \sum_{i<j}(x^j_{;i} + x^i_{;j})^2 + \\ + (n-2)\sum_{i \neq j}(x^i_{;j})^2 + (n-1)(1-\lambda)\,\delta_{ij}x^i x^j\}\,\mathrm{d}v\,.$$

Integrating over $S^n(1)$ for $\lambda = 1$, we get

$$(6.63)\qquad x^i_{;j} = 0 \quad \text{for} \quad i \neq j\,; \quad x^i_{;i} = x^j_{;j} \quad \text{(no summation!)} \quad \text{for} \quad i < j\,.$$

Define

$$(6.64)\qquad A^i := x^i\,, \quad A^{n+1} := x^1_{;1} = \ldots = x^n_{;n}\,.$$

Then we we have (2.7_1). Further, for $i \neq j$, (6.58) reduces to (no summation!)

$$(6.65)\qquad x^i_{;ij} = x^i_{;ij} - x^i_{;ji} = \sum_l R^i_{lij}x^l = \sum_l(\delta_{li}\delta^i_j - \delta_{lj}\delta^i_i)\,x^l = -\sum_l \delta_{lj}x^l\,,$$

i.e., we get (2.7_2). Thus $A = A^\alpha v_\alpha$ is a fixed vector and $x = \pi A$. QED.

7. Concerning the whole spectrum of tangent vector fields on the unit hypersphere, I am able to prove just the following

Theorem 17. *Let $S^n(1)$ be a unit hypersphere. Then the numbers*

$$(7.1)\qquad \lambda_0 = 1\,; \quad \lambda^-_p = pn + p^2 - p - 1 \quad \textit{for} \quad p = 1, 2, \ldots\,; \\ \lambda^+_p = pn + p^2 + p + 1 \quad \textit{for} \quad p = 1, 2, \ldots$$

belong to $\mathrm{Spec}_{(1,0)}$ *of $S^n(1)$. Obviously, $\lambda^-_{p+1} - \lambda^+_p = n - 2 \geqq 0$.*

Proof. Let $A \in V^{n+1}$ be a fixed vector; we have (2.7). Further, let $\Omega\colon V^{n+1} \to \mathbb{R}$ be a 1-form; let $\Omega_\alpha := \Omega(v_\alpha)$. From $\mathrm{d}\Omega_\alpha - \Omega_\beta\omega^\beta_\alpha = 0$, we get

$$(7.2)\qquad \mathrm{d}\Omega_i - \Omega_j\omega^j_i = \delta_{ij}\Omega_{n+1}\omega^j\,, \quad \mathrm{d}\Omega_{n+1} = -\Omega_i\omega^i\,,$$

i.e.,

$$(7.3)\qquad \Omega_{i;j} = \delta_{ij}\Omega_{n+1}\,, \quad \Omega_{n+1;i} = -\Omega_i\,.$$

Further, let ${}^{(t)}\Omega \in V^{n+1}$ be the vector defined by

$$(7.4)\qquad \Omega(u) = \langle {}^{(t)}\Omega, u\rangle \quad \text{for} \quad u \in V^{n+1}\,;$$

we have ${}^{(t)}\Omega = \delta^{\alpha\beta}\Omega_\beta v_\alpha$. Let us write

$$(7.5) \qquad v\Omega = \Omega(v_{n+1}) = \Omega_{n+1},$$

v_{n+1} being the unit normal vector field of $S^n(1)$.

Introduce, on $S^n(1)$, the tangent vector fields

$$(7.6) \qquad \begin{aligned} X_{(p)} &= (v\Omega)^p \pi A = (\Omega_{n+1})^p A^i v_i && \text{for } p = 0, 1, \dots, \\ Y_{(p)} &= vA \,.\, (v\Omega)^{p-1} \pi^{(t)}\Omega = \delta^{ij} A^{n+1} (\Omega_{n+1})^{p-1} \Omega_j v_i && \text{for } p = 1, 2, \dots, \\ Z_{(p)} &= (v\Omega)^p \pi^{(t)}\Omega = \delta^{ij} (\Omega_{n+1})^p \Omega_j v_i && \text{for } p = 0, 1, \dots. \end{aligned}$$

We see that $X_{(p)} = \pi F$, where the p-linear mapping F is given by

$$(7.7) \qquad F(u_{(1)}, \dots, u_{(p)}) = \Omega(u_{(1)}) \dots \Omega(u_{(p)}) A \quad \text{for} \quad u_{(1)}, \dots, u_{(p)} \in V^{n+1}.$$

Further, let

$$(7.8) \qquad k := \langle {}^{(t)}\Omega, {}^{(t)}\Omega \rangle = \delta^{\alpha\beta} \Omega_\alpha \Omega_\beta, \quad c := \Omega(A) = \Omega_\alpha A^\alpha.$$

By direct calculation we find, for $p \geqq 2$,

$$(7.9) \qquad \begin{aligned} \Delta_0 X_{(p)} &= (p - p^2 - np - 1) X_{(p)} - 2p Y_{(p)} + p(p-1) k X_{(p-2)}, \\ \Delta_0 Y_{(p)} &= -2X_{(p)} + (1 - p - p^2 - np) Y_{(p)} + (p-1)(p-2) k Y_{(p-2)} + \\ &\quad + 2(p-1) c Z_{(p-2)}, \\ \Delta_0 Z_{(p)} &= -(p^2 + np + p + 1) Z_{(p)} + p(p-1) k Z_{(p-2)}. \end{aligned}$$

Also by a direct calculation,

$$(7.10) \qquad \begin{aligned} \Delta_0 X_{(1)} &= -(n+1) X_{(1)} - 2Y_{(1)}, \quad \Delta_0 Y_{(1)} = -2X_{(1)} - (n+1) Y_{(1)}, \\ \Delta_0 Z_{(1)} &= -(n+3) X_{(1)}, \quad \Delta_0 X_{(0)} = -X_{(0)}, \quad \Delta_0 Z_{(0)} = -Z_{(0)}. \end{aligned}$$

Now, let $V_{(1)}, \dots, V_{(N)}$ be tangent vector fields on $S^n(1)$, and let us have

$$(7.11) \qquad \Delta_0 V_{(A)} = r_A^B V_{(B)}; \quad A, B, \dots = 1, \dots, N; \quad r_A^B \in \mathbb{R}.$$

To exhibit vector fields $V = s^A V_{(A)}$, $s^A \in \mathbb{R}$, satisfying $(\Delta_0 + \lambda) V = 0$, $\lambda \in \mathbb{R}$, we have to solve the well known characteristic equation

$$(7.12) \qquad D := \det \| r_A^B + \lambda \delta_A^B \| = 0,$$

and to proceed as usual.

Introduce the notation

$$(7.13) \qquad \begin{aligned} D_{(p)} &= (\lambda - pn - p^2 + p + 1)(\lambda - pn - p^2 - p - 1), \\ d_{(p)} &= \lambda - p^2 - pn - p - 1. \end{aligned}$$

Let p be odd. Writing, by means of (7.9) and (7.10),

$$\Delta_0 X_{(p)}, \Delta_0 Y_{(p)}, \Delta_0 X_{(p-2)}, \Delta_0 Y_{(p-2)}, \Delta_0 Z_{(p-2)}, \dots \Delta_0 X_{(1)}, \Delta_0 Y_{(1)}, \Delta_0 Z_{(1)}$$

in the form of (7.11), we get, see (7.12),

$$(7.14) \qquad D = D_{(p)} D_{(p-2)} \dots D_{(1)} d_{(p-2)} d_{(p-4)} \dots d_{(1)}.$$

For p even, write

$$\Delta_0 X_{(p)}, \Delta_0 Y_{(p)}, \Delta_0 X_{(p-2)}, \Delta_0 Y_{(p-2)}, \Delta_0 Z_{(p-2)}, \ldots$$
$$\ldots, \Delta_0 X_{(2)}, \Delta_0 Y_{(2)}, \Delta_0 Z_{(2)}, \Delta_0 X_{(0)}, \Delta_0 Z_{(0)}$$

in the form of (7.11); we get

(7.15) $$D = D_{(p)} D_{(p-2)} \ldots D_{(2)} d_{(p-2)} d_{(p-4)} \ldots d_{(2)} (\lambda - 1)^2 .$$

Hence our result follows. QED.

8. We have proved in Theorem 7 that $\mathscr{I} \oplus \mathscr{J}$ is a Lie algebra of vector fields on $S^n(1)$. Of fourse, we are interested in the Lie group of transformations of $S^n(1)$ into itself generating $\mathscr{I} \oplus \mathscr{J}$. The aswer is given by the following

Theorem 18. *Consider $S^n(1) \subset E^{n+1}$. In E^{n+1}, choose an orthonormal system of coordinates (ξ^α) with the origin at the center of $S^n(1)$. Let $a^\alpha \in \mathbb{R}$, and let f_t: $S^n(1) \to \to S^n(1)$ be a 1-parametric group of transformations given by*

(8.1) $$f_t(\xi^1, \ldots, \xi^{n+1}) = \Big(\sum_{\alpha=1}^{n+1} \exp(2a^\alpha t)(\xi^\alpha)^2\Big)^{-1/2} .$$
$$.(\exp a^1 t . \xi^1, \ldots, \exp a^{n+1} t . \xi^{n+1}) , \quad \sum_{\alpha=1}^{n+1} (\xi^\alpha)^2 = 1 .$$

Then the associated tangent vector field

(8.2) $$V(\xi^\alpha) = \frac{\mathrm{d} f_t(\xi^\alpha)}{\mathrm{d} t}$$

belongs to $\mathscr{J}$. Conversely, each vector field of $\mathscr{J}$ may be generated in this way.

Proof. By a direct calculation,

(8.3) $$\frac{\mathrm{d} f_t(\xi^\alpha)}{\mathrm{d} t} = \Big(\sum_{\beta=1}^{n+1} \exp(2a^\beta t)(\xi^\beta)^2\Big)^{-3/2} (\exp a^{\alpha t} . \sum_{\beta=1}^{n+1} (a^\alpha - a^\beta) \exp 2a^\beta . (\xi^\beta)^2 \xi^\alpha) .$$

Further, consider the linear mapping B: $V^{n+1} \to V^{n+1}$ given by

(8.4) $$B(x^\alpha) = (a^\alpha x^\alpha) .$$

The unit normal vector at the point $(x^\alpha) \in S^{n+1}$ being (x^α), we have

(8.5) $$\pi B = \Big(\sum_{\beta=1}^{n+1} (a^\alpha - a^\beta)(x^\beta)^2 . x^\alpha\Big) ;$$

indeed, it is easy to see that $\langle \pi B, (x^\alpha) \rangle = 0$, and we have

(8.6) $$B(x^\alpha) = \pi B + \sum_{\beta=1}^{n+1} a^\beta (x^\beta)^2 . (x^\alpha) .$$

Inserting

(8.7) $$x^\alpha = \Big(\sum_{\beta=1}^{n+1} \exp 2a^\beta t . (\xi^\beta)^2\Big)^{-1/2} \exp a^\alpha t . \xi^\alpha$$

into (8.5), we get

$$(\pi B)(x^\alpha) = \frac{\mathrm{d}f_t(\xi^\alpha)}{\mathrm{d}t}. \tag{8.8}$$
QED.

9. Finally, let us see what happens if we replace the hypersphere by the Veronese surface.

Let (x, y, z) be orthonormal coordinates in E^3 and $(u_1, \ldots, u_5)$ orthonormal coordinates in E^5. Let the mapping $S^2(\sqrt{3}) \to S^4(1)$ be given by

$$\begin{aligned} u_1 &= \tfrac{1}{3}\sqrt{3}\,.\,yz\,, \quad u_2 = \tfrac{1}{3}\sqrt{3}\,.\,xz\,, \quad u_3 = \tfrac{1}{3}\sqrt{3}\,.\,xy\,, \\ u_4 &= \tfrac{1}{6}\sqrt{3}\,.\,(x^2 - y^2)\,, \quad u_5 = \tfrac{1}{6}(x^2 + y^2 - 2z^2)\,; \end{aligned} \tag{9.1}$$

the image of $S^2(\sqrt{3})$ under this mapping is exactly the Veronese surface $\mathscr{V}$. Introducing the usual parameters (α, β) on $S^2(\sqrt{3})$, i.e., writing $S^2(\sqrt{3})$ as

$$x = \sqrt{3}\,.\cos\alpha\cos\beta\,, \quad y = \sqrt{3}\,.\cos\alpha\sin\beta\,, \quad z = \sqrt{3}\,.\sin\alpha\,, \tag{9.2}$$

and considering the orthonormal frames $\{m; v_1, \ldots, v_5\}$ in E^5 with

$$\begin{aligned} v_1 &= (\sin\alpha\cos\beta,\ -\sin\alpha\sin\beta,\ \cos\alpha\cos 2\beta,\ -\cos\alpha\sin 2\beta,\ 0)\,, \\ v_2 &= (\cos 2\alpha\sin\beta,\ \cos 2\alpha\cos\beta,\ -\tfrac{1}{2}\sin 2\alpha\sin 2\beta,\ -\tfrac{1}{2}\sin 2\alpha\cos 2\beta, \\ &\qquad -\tfrac{1}{2}\sqrt{3}\,.\sin 2\alpha)\,, \\ v_3 &= (\cos\alpha\cos\beta,\ -\cos\alpha\sin\beta,\ -\sin\alpha\cos 2\beta,\ \sin\alpha\sin 2\beta,\ 0)\,, \\ v_4 &= \tfrac{1}{2}(\sin 2\alpha\sin\beta,\ \sin 2\alpha\cos\beta,\ (\cos^2\alpha - 2)\sin 2\beta, \\ &\qquad (\cos^2\alpha - 2)\cos 2\beta,\ \sqrt{3}\,.\cos^2\alpha)\,, \\ -m = v_5 &= -\tfrac{1}{2}\sqrt{3}\,.(\sin 2\alpha\sin\beta,\ \sin 2\alpha\cos\beta,\ \cos^2\alpha\sin 2\beta\,, \\ &\qquad \cos^2\alpha\cos 2\beta,\ -\tfrac{1}{3}\sqrt{3}\,.(2\sin^2\alpha - \cos^2\alpha))\,, \end{aligned} \tag{9.3}$$

we get the fundamental equations of $\mathscr{V}$ in the form

$$\begin{aligned} \mathrm{d}m &= \omega^1 v_1 + \omega^2 v_2\,, \\ \mathrm{d}v_1 &= \omega_1^2 v_2 + \tfrac{1}{3}\sqrt{3}\,.(\omega^2 v_3 + \omega^1 v_4) + \omega^1 v_5\,, \\ \mathrm{d}v_2 &= -\omega_1^2 v_1 + \tfrac{1}{3}\sqrt{3}\,.(\omega^1 v_3 - \omega^2 v_4) + \omega^2 v_5\,, \\ \mathrm{d}v_3 &= -\tfrac{1}{3}\sqrt{3}\,.(\omega^2 v_1 + \omega^1 v_2) - 2\omega_1^2 v_4\,, \\ \mathrm{d}v_4 &= -\tfrac{1}{3}\sqrt{3}\,.(\omega^1 v_1 - \omega^2 v_2) + 2\omega_1^2 v_3\,, \\ \mathrm{d}v_5 &= -\omega^1 v_1 - \omega^2 v_2 \end{aligned} \tag{9.4}$$

with

$$\omega^1 = \sqrt{3}\,.\cos\alpha\,\mathrm{d}\beta\,, \quad \omega^2 = \sqrt{3}\,.\,\mathrm{d}\alpha\,, \quad \omega_1^2 = \sin\alpha\,\mathrm{d}\beta\,. \tag{9.5}$$

The Gauss curvature of $\mathscr{V}$ is, of course,

$$K_{\mathscr{V}} = \tfrac{1}{3}\,. \tag{9.6}$$

At each point $m \in \mathscr{V}$ we have the tangent plane $T_m(\mathscr{V})$ spanned by v_1, v_2, the normal plane $N_m(\mathscr{V})$ spanned by v_3, v_4 and the unit normal vector v_5. Obviously, $T_m(\mathscr{V}) \oplus \oplus N_m(\mathscr{V}) = T_m(S^4(1))$; the set of planes $N_m(\mathscr{V})$ will be called the normal bundle of $\mathscr{V}$.

We have to study the sections of the normal bundle $N(\mathscr{V})$ of $\mathscr{V}$. Let us start with general considerations. Let a Riemannian manifold (M, ds^2) be given; over M, let a Euclidean bundle $(\mathscr{B}, \langle\, , \rangle)$ be given, i.e., a vector bundle each fiber of which carries a positive definite symmetric scalar product $\langle\, , \rangle: \mathscr{B}_m \times \mathscr{B}_m \to \mathbb{R}$. On $\mathscr{B}$, let a linear connection D^* be given; using local coordinates, let us assume that $\mathscr{B}$ restricted to a suitable neighborhood $U \subset M$ is trivial over U. Suppose $\dim \mathscr{B} = \dim M + m$. In $\mathscr{B}$ (restricted to U) choose orthonormal sections $(w_1, \ldots, w_m)$, i.e., sections satisfying $\langle w_\alpha, w_\beta \rangle = \delta_{\alpha\beta}$; $\alpha, \beta, \ldots = 1, \ldots, m$. The connection D^* gives rise to 1-forms τ_α^β on U such that

$$D^* w_\alpha = \tau_\alpha^\beta w_\beta \,. \tag{9.7}$$

If $s: U \to \mathscr{B}$, $s = s^\alpha w_\alpha$, is a section, define the covariant derivatives $s^\alpha_{;i}$ by

$$ds^\alpha + s^\beta \tau_\beta^\alpha = s^\alpha_{;i} \omega^i \,; \tag{9.8}$$

for a tangent vector field $V = V^i v_i$ on U, the covariant derivative of s with respect to V then is $D_V^* s = s^\alpha_{;i} V^i w_\alpha$. The connection D^* is said to be Euclidean with respect to $\langle\, , \rangle$ if

$$V\langle s, \tilde{s} \rangle = \langle D_V^* s, \tilde{s} \rangle + \langle s, D_V^* \tilde{s} \rangle \tag{9.9}$$

for each tangent vector field V and any two sections $s, \tilde{s}: U \to \mathscr{B}$. It is easy to see that D^* is Euclidean with respect to $\langle\, , \rangle$ if and only if

$$\tau_\alpha^\beta + \tau_\beta^\alpha = 0 \,. \tag{9.10}$$

The components of the curvature tensor of D^* are defined by

$$d\tau_\alpha^\beta = \tau_\alpha^\gamma \wedge \tau_\gamma^\beta - \tfrac{1}{2} S^\beta_{\alpha ij} \omega^i \wedge \omega^j \,, \quad S^\beta_{\alpha ij} + S^\beta_{\alpha ji} = 0 \,. \tag{9.11}$$

The curvature of D^* at $m \in U$ is the mapping $S: T_m(M) \times T_m(M) \times \mathscr{B}_m \to \mathscr{B}_m$ given by

$$S(V, W)\, s = S^\beta_{\alpha ij} V^i W^j s^\alpha w_\beta \,. \tag{9.12}$$

For the section $s = s^\alpha w_\alpha$, the differential consequences of (9.8) being

$$(ds^\alpha_{;i} - s^\alpha_{;j} \omega_i^j + s^\beta_{;i} \tau_\beta^\alpha) \wedge \omega^i = -\tfrac{1}{2} S^\alpha_{\beta ij} s^\beta \omega^i \wedge \omega^j \,, \tag{9.13}$$

there are functions $s^\alpha_{;ij}$ (the second covariant derivatives of s^α) such that

$$ds^\alpha_{;i} - s^\alpha_{;j} \omega_i^j + s^\beta_{;i} \tau_\beta^\alpha = s^\alpha_{;ij} \omega^j \,, \tag{9.14}$$

$$s^\alpha_{;ij} - s^\alpha_{;ji} = S^\alpha_{\beta ij} s^\beta \,. \tag{9.15}$$

The *Laplacian* of the section $s = s^\alpha w_\alpha$ is then defined by

$$\Delta^* s = \delta^{ij} s^\alpha_{;ij} w_\alpha \,. \tag{9.16}$$

It is obvious how to define the spectrum of a Euclidean bundle $(\mathscr{B}, \langle\, , \rangle)$ over M with a given Euclidean connection D^*.

In the case of the normal bundle $N(\mathscr{V})$ over the Veronese surface $\mathscr{V}$, the connection D^* is given by, see (9.4),

(9.17) $$D^* v_3 = -2\omega_1^2 v_4 \,, \quad D^* v_4 = 2\omega_1^2 v_3 \,.$$

Theorem 19. *Let $\mathscr{V} \subset E^5$ be the Veronese surface, let $A \in V^5$ be a fixed vector. At each point $m \in \mathscr{V}$, consider its decomposition*

(9.18) $$A = \pi A + \pi_N A + vAv_5 \,; \quad \pi A \in T_m(\mathscr{V}) \,, \quad \pi_N A \in N_m(\mathscr{V}) \,, \quad vA \in \mathbb{R} \,.$$

Then, with $K_{\mathscr{V}} = \frac{1}{3}$ as in (9.6),

(9.19) $$(\Delta + 6K_{\mathscr{V}})\, vA = 0 \,, \quad (\Delta_0 + 5K_{\mathscr{V}})\, \pi A = 0 \,, \quad (\Delta^* + 2K_{\mathscr{V}})\, \pi_N A = 0 \,.$$

Proof. Let $A = A^\varrho v_\varrho$; $\varrho, \sigma, \ldots = 1, \ldots, 5$. The vector A being fixed, we have $\mathrm{d}A^\varrho + A^\sigma \omega_\sigma^\varrho = 0$, i.e.,

(9.20) $$\begin{aligned} \mathrm{d}A^1 - A^2\omega_1^2 &= (\tfrac{1}{3}\sqrt{3}\,.\,A^4 + A^5)\,\omega^1 + \tfrac{1}{3}\sqrt{3}\,.\,A^3\omega^2 \,, \\ \mathrm{d}A^2 + A^1\omega_1^2 &= \tfrac{1}{3}\sqrt{3}\,.\,A^3\omega^1 + (A^5 - \tfrac{1}{3}\sqrt{3}\,.\,A^4)\,\omega^2 \,, \\ \mathrm{d}A^3 + 2A^4\omega_1^2 &= -\tfrac{1}{3}\sqrt{3}\,.\,(A^2\omega^1 + A^1\omega^2) \,, \\ \mathrm{d}A^4 - 2A^3\omega_1^2 &= -\tfrac{1}{3}\sqrt{3}\,.\,(A^1\omega^1 - A^2\omega^2) \,, \\ \mathrm{d}A^5 &= -A^1\omega^1 - A^2\omega^2 \,. \end{aligned}$$

Now,

(9.21) $$vA = A^5 \,, \quad \pi A = A^1 v_1 + A^2 v_2 \,, \quad \pi_N A = A^3 v_3 + A^4 v_4 \,.$$

Applying (9.20) and the appropriate definitions of Δ, Δ_0 and Δ^*, we complete the proof. QED.

Let B: $V^5 \to V^5$ be a linear mapping. Analogously to the case of $S^n(1)$, we define

(9.22) $$vB = \langle B(v_5), v_5 \rangle = B_5^5 \,,$$

B being given by $B(v_\varrho) = B_\varrho^\sigma v_\sigma$. The mapping B induces, for each $m \in \mathscr{V}$, linear mappings $v_T B$: $T_m(\mathscr{V}) \to T_m(\mathscr{V})$, $v_N B$: $N_m(\mathscr{V}) \to N_m(\mathscr{V})$ defined as follows:

(9.23) $$(v_T B)(v) = \mathrm{pr}_T\, B(v) \quad \text{for} \quad v \in T_m(\mathscr{V}) \,, \quad \mathrm{pr}_T\colon V^5 \to T_m(\mathscr{V}) \quad \text{an orthogonal projection} \,.$$

$$(v_N B)(s) = \mathrm{pr}_N\, B(s) \quad \text{for} \quad s \in N_m(\mathscr{V}) \,, \quad \mathrm{pr}_N\colon V^5 \to N_m(\mathscr{V}) \quad \text{an orthogonal projection} \,.$$

Further, we have

(9.24) $$\mathrm{Tr}\, v_T B = B_1^1 + B_2^2 \,, \quad \mathrm{Tr}\, v_N B = B_3^3 + B_4^4 \,.$$

Theorem 20. *Let $\mathscr{V} \subset E^5$ be a Veronese surface, let B: $V^5 \to V^5$ be a linear mapping. Then we may write*

(9.25) $$vB = \tfrac{1}{5} f_{(1)} + \tfrac{1}{7} f_{(2)} + \tfrac{3}{35} f_{(3)} \,,$$

where

$$\begin{aligned} f_{(1)} &= \mathrm{Tr}\, B \,, \quad \Delta f_{(1)} = 0 \,; \\ f_{(2)} &= 2vB + \mathrm{Tr}\, v_T B - 2\,\mathrm{Tr}\, v_N B \,, \quad (\Delta + 6K_{\mathscr{V}}) f_{(2)} = 0 \,; \\ f_{(3)} &= 6vB - 4\,\mathrm{Tr}\, v_T B + \mathrm{Tr}\, v_N B \,, \quad (\Delta + 20K_{\mathscr{V}}) f_{(3)} = 0 \,. \end{aligned}$$

Proof. From $\mathrm{d}B^\sigma_\varrho - B^\sigma_\tau\omega^\tau_\varrho + B^\tau_\varrho\omega^\sigma_\tau = 0$ we get

$$\begin{aligned}
(9.26)\qquad \mathrm{d}B^5_5 &= -(B^5_1 + B^1_5)\,\omega^1 - (B^5_2 + B^2_5)\,\omega^2\,,\\
\mathrm{d}B^5_1 - B^5_2\omega^2_1 &= (\tfrac13\sqrt{3}\,.\,B^5_4 + B^5_5 - B^1_1)\,\omega^1 + (\tfrac13\sqrt{3}\,.\,B^5_3 - B^2_1)\,\omega^2\,,\\
\mathrm{d}B^1_5 - B^2_5\omega^2_1 &= (\tfrac13\sqrt{3}\,.\,B^4_5 + B^5_5 - B^1_1)\,\omega^1 + (\tfrac13\sqrt{3}\,.\,B^3_5 - B^1_2)\,\omega^2\,,\\
\mathrm{d}B^5_2 + B^5_1\omega^2_1 &= (\tfrac13\sqrt{3}\,.\,B^5_3 - B^1_2)\,\omega^1 + (B^5_5 - B^2_2 - \tfrac13\sqrt{3}\,.\,B^5_4)\,\omega^2\,,\\
\mathrm{d}B^2_5 + B^1_5\omega^2_1 &= (\tfrac13\sqrt{3}\,.\,B^3_5 - B^2_1)\,\omega^1 + (B^5_5 - B^2_2 - \tfrac13\sqrt{3}\,.\,B^4_5)\,\omega^2\,,\\
\mathrm{d}(B^1_1 + B^2_2) &= \{\tfrac13\sqrt{3}\,.\,(B^1_4 + B^4_1 + B^2_3 + B^3_2) + B^1_5 + B^5_1\}\,\omega^1 +\\
&\quad + \{\tfrac13\sqrt{3}\,.\,(B^1_3 + B^3_1 - B^2_4 - B^4_2) + B^2_5 + B^5_2\}\,\omega^2\,,\\
\mathrm{d}B^3_1 + (2B^4_1 - B^3_2)\,\omega^2_1 &= \{\tfrac13\sqrt{3}\,.\,(B^3_4 - B^2_1) + B^3_5\}\,\omega^1 +\\
&\quad + \tfrac13\sqrt{3}\,.\,(B^3_3 - B^1_1)\,\omega^2\,,\\
\mathrm{d}B^3_2 + (B^3_1 + 2B^4_2)\,\omega^2_1 &= \tfrac13\sqrt{3}\,.\,(B^3_3 - B^2_2)\,\omega^1 +\\
&\quad + \{B^3_2 - \tfrac13\sqrt{3}\,.\,(B^3_4 + B^1_2)\}\,\omega^2\,,\\
\mathrm{d}B^4_1 - (B^4_2 + 2B^3_1)\,\omega^2_1 &= \{\tfrac13\sqrt{3}\,.\,(B^4_4 - B^1_1) + B^4_5\}\,\omega^1 +\\
&\quad + \tfrac13\sqrt{3}\,.\,(B^4_3 + B^2_1)\,\omega^2\,,\\
\mathrm{d}B^4_2 + (B^4_1 - 2B^3_2)\,\omega^2_1 &= \tfrac13\sqrt{3}\,.\,(B^4_3 - B^1_2)\,\omega^1 +\\
&\quad + \{\tfrac13\sqrt{3}\,.\,(B^2_2 - B^4_4) + B^4_5\}\,\omega^2\,.
\end{aligned}$$

Then

$$\begin{aligned}
(9.27)\qquad \Delta B^5_5 &= -4B^5_5 + 2(B^1_1 + B^2_2)\,,\\
\Delta(B^1_1 + B^2_2) &= 4B^5_5 - \tfrac{10}{3}(B^1_1 + B^2_2) + \tfrac43(B^3_3 + B^4_4)\,,\\
\Delta(B^3_3 + B^4_4) &= \tfrac43(B^1_1 + B^2_2) - \tfrac43(B^3_3 + B^4_4)\,,
\end{aligned}$$

and (9.25) easily follows. QED.

In the end, let us prove a simple global result.

Theorem 21. *Let* $(M, \mathrm{d}s^2)$ *be an orientable compact Riemannian manifold,* $(\mathscr{B}, \langle\,,\,\rangle)$ *a Euclidean bundle over* M, D^* *its Euclidean connection, and let* $\dim M = 2$, $\dim \mathscr{B} = 4$. *Let* $K_{\mathscr{B}}$ *be the curvature of* $\mathscr{B}$ (to be defined in the proof). *If there is a non-trivial section* $s\colon M \to \mathscr{B}$ *satisfying*

$$(9.28)\qquad (\Delta^* + \lambda)\,s = 0\,,$$

then

$$(9.29)\qquad \lambda \geqq \max\,(\min_M K_{\mathscr{B}}, -\max_M K_{\mathscr{B}})\,.$$

Proof. Let us restrict ourselves to a coordinate neighborhood $U \subset M$ such that $\mathscr{B}$ is trivial over U; let w_1, w_2 be two orthonormal sections of $\mathscr{B}$ over U. Then D^* is given by

$$(9.30)\qquad D^*w_1 = \tau^2_1 w_2\,,\quad D^*w_2 = -\tau^2_1 w_1\,;$$

see (9.7) and (9.10). The curvature $K_{\mathscr{B}}$ of $\mathscr{B}$ is then defined by

$$(9.31) \qquad d\tau_1^2 = -K_{\mathscr{B}}\omega^1 \wedge \omega^2 ;$$

compare with (9.11).

Let $s = s^1 w_1 + s^2 w_2$ be our section. The covariant derivatives $s^\alpha_{;i}$ are defined by, see (9.8),

$$(9.32) \qquad ds^1 - s^2\tau_1^2 = s^1_{;1}\omega^1 + s^1_{;2}\omega^2 , \quad ds^2 + s^1\tau_1^2 = s^2_{;1}\omega^1 + s^2_{;2}\omega^2 .$$

The equations (9.14), (9.15) read

$$(9.33) \qquad \begin{aligned} ds^1_{;1} - s^1_{;2}\omega_1^2 - s^2_{;1}\tau_1^2 &= s^1_{;11}\omega^1 + s^1_{;12}\omega^2 , \\ ds^1_{;2} + s^1_{;1}\omega_1^2 - s^2_{;2}\tau_1^2 &= s^1_{;21}\omega^1 + s^1_{;22}\omega^2 , \\ ds^2_{;1} + s^2_{;2}\omega_1^2 - s^1_{;1}\tau_1^2 &= s^2_{;11}\omega^1 + s^2_{;12}\omega^2 , \\ ds^2_{;2} - s^2_{;1}\omega_1^2 + s^1_{;2}\tau_1^2 &= s^2_{;21}\omega^1 + s^2_{;22}\omega^2 , \end{aligned}$$

$$(9.34) \qquad s^1_{;21} - s^1_{;12} = K_{\mathscr{B}}s^2 , \quad s^2_{;21} - s^2_{;12} = -K_{\mathscr{B}}s^1 .$$

Consider the 1-forms (here $\varepsilon_{\alpha\alpha} = 0$, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} = 1$ for $\alpha > \beta$)

$$(9.35) \qquad \begin{aligned} \varphi_1 &= *\delta_{\alpha\beta}s^\alpha s^\beta_{;i}\omega^i = -(s^1 s^1_{;2} + s^2 s^2_{;2})\,\omega^1 + (s^1 s^1_{;1} + s^2 s^2_{;1})\,\omega^2 , \\ \varphi_2 &= \varepsilon_{\alpha\beta}s^\alpha s^\beta_{;i}\omega^i = (s^1 s^2_{;1} - s^2 s^1_{;1})\,\omega^1 + (s^1 s^2_{;2} - s^2 s^1_{;2})\,\omega^2 ; \end{aligned}$$

we get

$$(9.36) \qquad \begin{aligned} d\varphi_1 &= \{(s^1_{;1})^2 + (s^2_{;1})^2 + (s^1_{;2})^2 + (s^2_{;2})^2 + s^1(s^1_{;11} + s^1_{;22}) + \\ &\quad + s^2(s^2_{;11} + s^2_{;22})\}\, \omega^1 \wedge \omega^2 , \\ d\varphi_2 &= \{2(s^1_{;1}s^2_{;2} - s^2_{;1}s^1_{;2}) - K_{\mathscr{B}}[(s^1)^2 + (s^2)^2]\}\, \omega^1 \wedge \omega^2 . \end{aligned}$$

Using (9.28), i.e.,

$$(9.37) \qquad s^1_{;11} + s^1_{;22} + \lambda s^1 = 0 , \quad s^2_{;11} + s^2_{;22} + \lambda s^2 = 0 ,$$

we get

$$(9.38) \qquad \begin{aligned} &d(\varphi_1 - \varphi_2) = \\ &\quad = \{(s^1_{;1} - s^2_{;2})^2 + (s^2_{;1} + s^1_{;2})^2 + (K_{\mathscr{B}} - \lambda)\,[(s^1)^2 + (s^2)^2]\}\, \omega^1 \wedge \omega^2 , \\ &d(\varphi_1 + \varphi_2) = \\ &\quad = \{(s^1_{;1} + s^2_{;2})^2 + (s^2_{;1} - s^1_{;2})^2 - (K_{\mathscr{B}} + \lambda)\,[(s^1)^2 + (s^2)^2]\}\, \omega^1 \wedge \omega^2 . \end{aligned}$$

If $\lambda < \min_M K_{\mathscr{B}}$, $s = 0$, then using the Stokes theorem applied to (9.38_1), we get from (9.38_2) that $s = 0$ for $\lambda < -\max_M K_{\mathscr{B}}$. **QED.**

Comparing (9.30) and (9.17), we see that $\tau_1^2 = -2\omega_1^2$ for the normal bundle $N(\mathscr{V})$ of the Veronese surface $\mathscr{V}$. Thus

$$(9.39) \qquad K_{N(\mathscr{V})} = -\tfrac{2}{3} ,$$

and for each non-trivial section $s\colon \mathscr{V} \to N(\mathscr{V})$ satisfying $(\Delta^* + \lambda)\, s = 0$ we have $\lambda \geqq \frac{2}{3} = 2K_{\mathscr{V}}$. Sections with $\lambda = 2K_{\mathscr{V}}$ are realized by the sections of the type $\pi_N A$, see Theorem 19. The Veronese surface is not orientable, but we may use the pull-backs of the forms φ_1, φ_2 under the mapping $S^2(\sqrt{3}) \to \mathscr{V}$ to be able to apply the proof of the last theorem.

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