Luc Vrancken-Mawet; Georges Hansoul

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THE SUBALGEBRA LATTICE OF A HEYTING ALGEBRA

L. VRANCKEN-MAWET and G. HANSOUL, Liège

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In [7], L. Vrancken-Mawet investigates the subalgebra lattice of a finite Heyting algebra. In this paper, we consider infinite Heyting algebras. Minimal (non trivial) and maximal (proper) subalgebras of a Heyting algebra L are determined. This enables to prove that the subalgebra lattice of L is always upper semimodular and that it is atomistic if and only if L is a Stone algebra. Also we characterize those Heyting algebras whose subalgebra lattice is Boolean.

In § 1, we briefly recall Priestley's duality ([5]), adapting it for Heyting algebras. The problems are solved in the dual category and reinterpreted in terms of Heyting algebras in § 3 (Theorem 2.13).

We use standard set theoretic symbols. Note that $\subset$ denotes strict inclusion and $-$ denotes complement (in some given universe).

1. PRIESTLEY'S DUALITY

1.1. Definition. 1) A *Heyting algebra* $L = (L; \vee, \wedge, *, 0, 1)$ is an algebra of type $(2, 2, 2, 0, 0)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice and, for x, y in L, $x * y$ is the relative pseudocomplement of x and y (i.e., $z \leqq x * y$ if and only if $x \wedge z \leqq y$). We are concerned with the subalgebra lattice $\mathrm{Sub}(L)$ of L.

2) If L is a bounded distributive lattice, we denote by $\mathscr{P}(L)$ its dual space (cf. [4]): $\mathscr{P}(L)$ is the topological ordered space of its prime ideals, ordered by inclusion and whose topology is generated by the sets $r(a)$ and $-r(a)$, $a \in L$, where $r(a) = \{P \in \mathscr{P}(L) \mid P \not\ni a\}$.

3) If $(X, \leqq)$ is a partially ordered set, and if $x \in X$, $E \subseteq X$, then $[x)$ is $\{y \mid x \leqq y\}$, $]x)$ is $\{y \mid x < y\}$ and $[E)$ is $\bigcup\{[x) \mid x \in E\}$. We define $(x]$, $(x[$ and $(E]$ dually.

Also, E is *increasing* (resp. *decreasing*) if $E = [E)$ (resp. $E = (E]$). If $X = (X; \tau, \leqq)$ is a topological ordered space, X is said to be *totally order disconnected* (abbreviated t.o.d.) if, whenever $x \nleqq y$, there exists a *clopen* (i.e. closed and open) decreasing subset U of X such that $y \in U$ and $x \notin U$. We denote by $\mathcal{O}(X)$ the (bounded distributive) lattice of all clopen decreasing subsets of X.

1.2. Remark. Priestley's duality states that, if L is a bounded distributive lattice, then $\mathscr{P}(L)$ is compact t.o.d. and L is isomorphic to $\mathcal{O}\,\mathscr{P}(L)$. It is convenient now to recall a few elementary facts concerning compact t.o.d. spaces. Let X be such a space and denote by Min X the set of all minimal elements of X. Then

a) the sets V and $-V$, for $V \in \mathcal{O}(X)$, form a subbasis for the topology on X;

b) the dual $(X; \tau, \geqq)$ of X is also compact t.o.d.;

c) if $Y \subseteq X$, then Y is closed if and only if it is compact t.o.d. with the induced structure;

d) if Y is closed in X, so is $[Y)$;

e) if Y is closed and decreasing and $x \notin Y$, there exists $U \in \mathcal{O}(X)$ such that $U \supseteq Y$ and $U \not\ni x$;

f) for each $x \in X$, there exists some $m \in \text{Min}\, X$ such that $m \leqq x$.

Let us now consider Heyting algebras. It is well known that a bounded distributive lattice L is a Heyting algebra if and only if $X = \mathscr{P}(L)$ satisfies

(H) $\qquad$ if $U \in \mathcal{O}(X)\,, \quad V \in \mathcal{O}(X)\,,$ then $[U - V)$ is open.

By d), this amounts to saying that $-[U - V) \in \mathcal{O}(X)$ whenever $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(X)$. In fact, we have $U * V = -[U - V)$ in $\mathcal{O}(X)$. Note that Min X is closed in this case.

1.3. Definition. A topological ordered space $X = (X; \tau, \leqq)$ is called a *Heyting space* if it is a compact t.o.d. space satisfying (H). By 1.2, it is equivalent to requiring that i) $(X; \tau)$ is a Boolean space, ii) if U is clopen in X, so is $[U)$ and iii) if $x \in X$, then $(x]$ is closed in X. Note that this shows how close Heyting algebras are to closure algebras ([3], p. 119).

If $\phi: L \to L'$ is a $\{0, 1\}$ – lattice homomorphism, then the mapping $\mathscr{P}(\phi): \mathscr{P}(L') \to \mathscr{P}(L)$ defined by $\mathscr{P}(\phi)(P') = \phi^{-1}(P')$ is order preserving and continuous ([5], p. 515). If, moreover, ϕ is a Heyting algebra homomorphism, then $f = \mathscr{P}(\phi)$ satisfies

(M) $$(f(x)] = f((x])$$

for all $x \in \mathscr{P}(L')$. Conversely, if $f: X \to X'$ is a continuous mapping between Heyting spaces satisfying (M), then $\mathcal{O}(f): \mathcal{O}(X') \to \mathcal{O}(X)$ defined by $\mathcal{O}(f)\,(U') = f^{-1}(U')$ is a Heyting algebra homomorphism.

1.4. Definition. Let X, X' be Heyting spaces. A mapping $f: X \to X'$ is said to be a *(Heyting) morphism* if it is continuous and satisfies condition (M). The resulting category is denoted by $\mathscr{X}$, while $\mathscr{H}$ denotes that of Heyting algebras with their usual homomorphisms. Priestley's duality and 1.4 lead to the following fact.

1.5. Theorem. *The functors $\mathscr{P}$ and $\mathcal{O}$ establish a dual equivalence between $\mathscr{H}$ and $\mathscr{X}$. The duality interchanges injectives and surjectives.*

It follows that if $L \in \mathscr{H}$ and $L' \in \text{Sub}\,(L)$, there exists an equivalence θ on $\mathscr{P}(L)$ such that $\mathscr{P}(L)/\theta$ admits a Heyting space stucture which is isomorphic to $\mathscr{P}(L')$.

Such an equivalence is naturally called a congruence. More precisely, we introduce the following definition (in what follows, $\mathrm{Eq}(X)$ is the equivalence lattice on X; for $\theta \in \mathrm{Eq}(X)$, p^θ is the θ-class of p and if $E \subseteq X$, E^θ is $\bigcup\{p^\theta \mid p \in E\}$ and E is θ-*saturated* if $E^\theta = E$).

1.6. Definition. Let $X \in \mathscr{X}$ and let $\theta \in \mathrm{Eq}(X)$. Then θ is a *congruence* on X if X/θ admits a (necessarily unique) Heyting space structure such that the natural mapping $X \to X/\theta$ is a morphism. Note that the topology on X/θ is the usual quotient topology and that $x^\theta \leqq y^\theta$ if and only if there exist $x' \, \theta \, x$ and $y' \, \theta \, y$ with $x' \leqq y'$. It is not difficult to prove that $\theta \in \mathrm{Eq}(X)$ is a congruence if and only if (see [7], p. 83):

i) if $x \leqq y \, \theta \, z$, there exists w such that $x \, \theta \, w \leqq z$;

ii) each θ-class is convex; and

iii) if $x^\theta \nleqq y^\theta$, there exists $U \in \mathcal{O}(X)$ such that $U \ni y$, $U \not\ni x$ and U is θ-saturated.

By 1.3, θ satisfies i), ii), iii) if and only if it satisfies i), ii), iii′), where iii′) is:

iii′) if $x \, \theta \, y$ fails, there exists a clopen θ-saturated U such that $U \ni y$ and $U \not\ni x$.

The set of all congruences on X is denoted by $\mathrm{Con}(X)$. By 1.5, there exists a canonical anti-isomorphism between $\mathrm{Con}(X)$ and $\mathrm{Sub}(\mathcal{O}(X))$. Consequently, $\mathrm{Con}(X)$ is a complete, dually algebraic lattice, with minimum ω (the equality relation) and maximum $\iota (= X \times X)$. Note however that $\mathrm{Con}(X)$ is not a sublattice of $\mathrm{Eq}(X)$. To emphasize this fact, the join in $\mathrm{Eq}(X)$ will be denoted by V_{eq}.

We end this paragraph with some elementary properties of congruences. Note first that a subset Y of a Heyting space X is a sub-Heyting space with the induced structure if and only if Y is closed and decreasing.

1.7. Lemma. *Let $X \in \mathscr{X}$ and $\theta \in \mathrm{Con}(X)$. Then*

a) $\mathrm{Con}(X/\theta)$ *is isomorphic to* $\{\phi \in \mathrm{Con}(X) \mid \phi \geqq \theta\}$;

b) *if Y is closed (resp. decreasing) in X, so is Y^θ;*

c) *if Y is closed and decreasing and $x^\theta \cap Y = \emptyset$, then there exists $U \in \mathcal{O}(X)$ such that $U \supseteq Y$, $U \not\ni x$ and U is θ-saturated;*

d) *if Y is closed and decreasing, then $\theta|_Y \in \mathrm{Con}(Y)$ and for any $\psi \in \mathrm{Con}(Y)$,* $\overline{\psi} = \psi \cup \omega \in \mathrm{Con}(X)$.

Proof. Assertion a) is obvious by duality (a direct proof is also easy to obtain). Assertion b) follows from the definition 1.6 and a standard compactness argument, while c) is just a restatement of 1.2.e) for the quotient X/θ. Finally d) is proved by direct verification.

2. THE CONGRUENCE LATTICE OF A HEYTING SPACE

Let X be a Heyting space.

2.1. Notations. If $E \subseteq X$, we denote by $\theta(E)$ the equivalence on X generated by $E \times E$ and by $\phi(E)$ the equivalence $\theta(E) \cup \theta(-E)$. If $E = \{p, q\}$, we write $\theta(p, q)$ instead of $\theta(\{p, q\})$.

Though we shall first be concerned with the coatoms of $\mathrm{Con}(X)$ (or *coatomic congruences*), let us mention now that, if $\theta(p, q) \in \mathrm{Con}(X)$, then $\theta(p, q)$ is necessarily an *atomic congruence* (i.e. an atom of $\mathrm{Con}(X)$) whenever $p \neq q$, or $\theta(p, q) = \omega$ otherwise. In particular, if $\{p, q\} \subseteq \mathrm{Min}\, X$ and $p \neq q$, then $\theta(p, q)$ is an atomic congruence on X.

2.2. Lemma. *If $\phi \in \mathrm{Con}(X)$, then ϕ is coatomic if and only if it has the form $\phi(U)$ where $U \in \mathcal{O}(X)$, $\emptyset \subset U \subset X$ and either*
1) $U \supseteq \mathrm{Min}\, X$ *or* 2) $-U \in \mathcal{O}(X)$.

Proof. It is clear that the described equivalences are coatomic congruences.

Suppose now $\theta \in \mathrm{Con}(X)$ with $\theta \neq \iota$. If there exists x with $x^\theta \cap \mathrm{Min}\, X = \emptyset$, then Lemma 1.7 c) gives a $U \in \mathcal{O}(X)$ such that $\phi(U)$ is a coatomic congruence of form 1) and $\phi(U) \geqq \theta$. Consider now the case when $(\mathrm{Min}\, X)^\theta = X$. Let x, y be elements of X such that $x^\theta \nleqq y^\theta$. By 1.6 iii), there exists $U \in \mathcal{O}(X)$ such that $U \ni y$, $U \not\ni x$ and U is θ-saturated. Let us show that $-U \in \mathcal{O}(X)$. If $q \in -U$ and $p \leqq q$, then for some $r \in \mathrm{Min}\, X$, one has $p \leqq q\ \theta\ r$ and, by 1.6 i), $p\ \theta\ r$, which implies $p \in -U$. Hence $\phi(U)$ is a coatomic congruence of form 2) such that $\phi(U) \geqq \theta$.

The proof of Lemma 2.2 shows in fact a little more.

2.3. Proposition. *The lattice* $\mathrm{Con}(X)$ *is coatomic (i.e., each θ in* $\mathrm{Con}(X) - \{\iota\}$ *is dominated by a coatom).*

Recall that a lattice is *coatomistic* if each element is the meet of coatoms.

2.4. Proposition. *The lattice* $\mathrm{Con}(X)$ *is coatomistic if and only if*
$(*)$ *for each $x \in X$, there exists a unique $m \in \mathrm{Min}\, X$ such that $m \leqq x$.*

Condition $(*)$ can also be expressed in the following way: each *order-connected component* (abbreviated o.c.c.) of X admits a least element (an o.c.c. of X is a subset of X which is both increasing and decreasing and which is minimal for this property).

Proof. Let p and q be elements of $\mathrm{Min}\, X$ lying in the same o.c.c.. Note that, for any coatom $\phi(U)$ of $\mathrm{Con}(X)$, one has $\phi(U) \geqq \theta(p, q)$. Hence the condition is necessary.

Suppose now condition $(*)$ is satisfied. For $\theta \in \mathrm{Con}(X)$, let $T = \{\phi \in \mathrm{Con}(X) \mid \phi$ coatom and $\phi \geqq \theta\}$. We shall prove $\bigcap T = \theta$, which will imply coatomisticity. It is clear that $\theta \subseteq \bigcap T$. Let $(x, y) \in \bigcap T - \theta$.

If $x^\theta \cap \mathrm{Min}\, X = \emptyset$, we may consider $x^\theta \nleqq y^\theta$ (otherwise interchange x and y). Hence $x^\theta \cap (\mathrm{Min}\, X \cup (y^\theta]) = \emptyset$ and there exists $U \in \mathcal{O}(X)$ such that $U \supseteq \mathrm{Min}\, X \cup \{y\}$, $U \not\ni x$ and U is θ-satured. Thus $\phi(U) \in T$ and $(x, y) \notin \phi(U)$, a contradiction.

Similar arguments hold if $y^\theta \cap \mathrm{Min}\, X = \emptyset$ and it remains to consider the case when both x and y are in $\mathrm{Min}\, X$. By 1.6 iii), there exists $V \in \mathcal{O}(X)$ such that $y \in V$, $x \notin V$ and V is θ-saturated. Let $U = [V)$. Then U is clopen by 1.3 ii) and U is decreasing by condition $(*)$. Moreover, $-U = -[V) \in \mathcal{O}(X)$. Hence $\phi(U) \in T$ and $(x, y) \notin \phi(U)$, a contradiction.

The next lemmas prepare for the characterization of those X for which Con (X) is Boolean. They suggest two questions which we do not solve completely at the present time: when is Con (X) atomistic?, when is Con (X) distributive?

2.5. Lemma. *If* $\theta \in \mathrm{Con}(X)$, *then* θ *is atomic if and only if it has the form* $\theta(p, q)$ *where* $p \neq q$ *and either* 1) $(p[= (q[$, *or* 2) $(p[= (q]$.

Proof. Let θ be an atomic congruence.

1) There exists exactly one θ-class which is not reduced to a singleton. Otherwise, let C_1 and C_2 be θ-classes which are not reduced to a singleton and assume $C_1 \nleqq C_2$. Then there exists $U \in \mathcal{O}(X)$ such that $U \supseteq C_2$, $U \cap C_1 = \emptyset$ and U is θ-saturated. Define ϕ by $\phi = \omega \cup \theta|_U$. Then $\phi \in \mathrm{Con}(X)$ by 1.7 d), and $\omega \subset \phi \subset \theta$.

2) Let E be the unique θ-class which is not reduced to a singleton. If E contains three distinct elements p, q and r, we may assume $p \nleqq q$ and $p \nleqq r$. Hence there exists $U \in \mathcal{O}(X)$ such that $U \supseteq \{q, r\}$ and $U \not\ni p$. Here again, letting $\phi = \omega \cup \theta|_U$, we have $\omega \subset \phi \subset \theta$.

3) Finally, it is routine to prove that $\theta(p, q) \in \mathrm{Con}(X)$ if and only if $(p[= (q[$ or $(p[= (q]$ (use 1.6 i); in fact, if θ is some equivalence for which $\{p \mid \exists q \neq p, p\,\theta\,q\}$ is finite, then $\theta \in \mathrm{Con}(X)$ if and only if it satisfies conditions i) and ii) of 1.6).

An atomic congruence $\theta(p, q)$ is said to be of *type* 1 (resp. *type* 2) if $(p[= (q[$(resp. $(p[= (q]$).

Let us recall that a partially ordered set $(X, \leqq)$ is said to be well-founded if any non-empty subset of X has at least one minimal element.

2.6. Corollary. *If* $(X; \leqq)$ *is well-founded, then* Con (X) *is atomic*

Proof. Let $\theta \in \mathrm{Con}(X)$ be such that $\theta \neq \omega$. Denote by E a θ-class which is not reduced to a singleton and which is minimal for this property. If E contains two minimal elements p and q, then $\theta(p, q)$ is atomic and $\theta(p, q) \subseteq \theta$. Otherwise, E has a least element q. Let p be minimal in $E - \{q\}$. Then again $\theta(p, q)$ is atomic and $\theta(p, q) \subseteq \theta$.

2.7. Lemma. *If* $\phi \in \mathrm{Con}(X)$ *and* θ *is atomic, then* $\phi \vee \theta = \phi \vee_{\mathrm{eq}} \theta$.

Proof. By 2.5, $\theta = \theta(p, q)$ and we assume $(p, q) \notin \phi$. Letting $\psi = \phi \vee_{\mathrm{eq}} \theta$, we must prove $\psi \in \mathrm{Con}(X)$. Condition i) of 1.6 is clearly satisfied. Also, each class is convex except perhaps $p^{\psi} = p^{\phi} \cup q^{\phi}$. Let x, y, z be such that $q\phi x \leqq y \leqq z\phi p$. There exists y' such that $y\phi y' \leqq p$. If $y' = p$, we are done. Otherwise, $y' < p$ and by 2.5, $y' \leqq q$. Hence there exists y'' such that $y'\phi y'' \leqq x$. This proves $x\phi y$ by the convexity of y^{ϕ}. Finally, to separate non ψ-related elements x and y (1.6 iii)), it suffices to consider the possible positions of x and y with respect to p and q.

2.8. Corollary. *The lattice* Con (X) *is always semimodular.*

Proof. Recall that L is semimodular if $\theta \wedge \phi \prec \phi$ implies $\phi \prec \theta \vee \phi$. By the third isomorphism theorem (1.7 a)), we may restrict ourselves to the case $\theta \wedge \phi = \omega$.

Hence, 2.8 is a corollary of 2.7 and the fact that any equivalence lattice is semimodular.

2.9. Lemma. *If* Con (X) *is distributive, then*
1) Min X *contains at most two elements;*
2) *if* $\theta(p, q)$ *is atomic of type* 1, *then* $\{p, q\} \subseteq$ Min X;
3) *if* $\theta(p, q)$ *is atomic of type* 2 *with* $q < p$ *and if* $x \notin$ Min X, *then either* $x \leqq q$ *or* $x \geqq p$.

Proof. 1) Let x_0, x_1 and x_2 be distinct elements of Min X. If $\{i, j\} \subseteq \{0, 1, 2\}$, then $\theta(x_i, x_j) \in$ Con (X) and

$$\theta(x_0, x_1) \vee (\theta(x_0, x_2) \wedge \theta(x_1, x_2)) = \theta(x_0, x_1) \neq \theta(\{x_0, x_1, x_2\}) = = (\theta(x_0, x_1) \vee \theta(x_0, x_2)) \wedge (\theta(x_0, x_1) \vee \theta(x_1, x_2)) .$$

2) Let $\theta = \theta(p, q)$ be atomic of type 1, with $p \notin$ Min X (and therefore $q \notin$ Min X). Let $U_p \in \mathcal{O}(X)$ be such that $U_p \supseteq$ Min $X \cup (p]$ and $U_p \not\ni q$, and define U_q in the same way. Then $\theta \wedge (\phi(U_p) \vee \phi(U_q)) = \theta \neq \omega = (\phi(U_p) \wedge \theta) \vee (\phi(U_q) \wedge \theta)$.

3) Let $\theta = \theta(p, q)$ be atomic of type 2 with $q < p$ and suppose some $x \notin$ Min X satisfies $x \nleqq q$ and $p \nleqq x$. Since $p \notin$ Min $X \cup (q]$ and $x \notin$ Min $X \cup (q]$, there exists $U \in \mathcal{O}(X)$ such that $U \supseteq$ Min $X \cup (q]$ and $U \cap \{p, x\} = \emptyset$. In the same way, there exists $V \in \mathcal{O}(X)$ such that $V \supseteq$ Min $X \cup (q] \cup (x]$ and $V \ni p$. Then $\theta \wedge (\phi(U) \vee \vee \phi(V)) = \theta \neq \omega$ $(\phi(U) \wedge \theta) \vee (\phi(V) \wedge \theta)$.

Lemma 2.9 shows that, if Con (X) is distributive, then X contains at most two o.c.c. If one knows that X contains exactly two o.c.c., it is possible to say more.

2.10. Lemma. *If X contains exactly two o.c.c. and* Con (X) *is distributive, then one of these o.c.c. is reduced to a singleton.*

Proof. By 2.9, each o.c.c. X_i has a least element, say x_i $(i = 0, 1)$. Suppose there exists y_i with $y_i > x_i$, $i = 0, 1$. By 2.9.3), $\theta(x_0, y_0) \notin$ Con (X). Whence $]x_0, y_0[\neq \emptyset$ and there exists z with $x_0 < z < y_0$. Let $U \in \mathcal{O}(X)$ be such that $U \supseteq \{x_0, x_1\}$, $U \not\ni z$ and $U \not\ni y_1$, and let $V \in \mathcal{O}(X)$ be such that $V \supseteq U \cup (z]$, $V \not\ni y_0$, $V \not\ni y_1$. Then $(\phi(X_0) \wedge \phi(U)) \vee \theta(V) = (\phi(X_0) \cap \phi(U)) \vee_{\mathrm{eq}} \theta(V) = \phi(X_0 \cup V) \neq \iota$, whereas $(\phi(X_0) \vee \theta(V)) \wedge (\phi(U) \vee \theta(V)) = \iota$.

In the following proposition, $\oplus$ and $+$ denote ordinal and cardinal sum respectively ([2], p. 199). By the disjoint sum of two partially ordered spaces $(X; \tau, \leqq)$ and $(Y; \tau, \leqq)$, we mean a partially ordered space $X + Y$ whose carrier is the disjoint union of X and Y, whose topology is the topological sum of $(X; \tau)$ and $(Y; \tau)$ and whose order is that of the cardinal sum of $(X; \leqq)$ and $(Y; \leqq)$. Finally, a Boolean chain is a complete chain endowed with its interval topology and such that for all $x \leqq y$, there exists $p \geqq x$, $q \leqq y$ with q covers p (see [4], p. 927).

2.11. Proposition. *If* $(X, \leqq)$ *is well-founded, then* Con (X) *is distributive if and only if X isomorphic to* $(\alpha \oplus 1) + 1$ *or to* $(\alpha + 1) \oplus \beta \oplus 1$ *for some ordinal numbers* α *and* β.

Proof. Suppose first that Con (X) is distributive. If X has a least element, then X is a chain: if (p, q) is minimal in $\{(x, y) \mid x \not\leq y \text{ and } y \not\leq x\}$, then $\theta(p, q)$ is atomic of type 1, which is impossible by 2.9.2). If X has two minimal elements, say Min $X = \{x_0, y_0\}$, then both $[x_0)$ and $[y_0)$ are chains (same proof as above). If X has two o.c.c., then by 2.10, then either $[x_0) = \{x_0\}$ or $[y_0) = \{y_0\}$, whence X is order-isomorphic to $(\alpha \oplus 1) + 1$. Let us consider the case when X is order connected. In this case, the set $[x_0) \cap [y_0)$ is not empty and has a least element m. The either $]x_0, m[$ or $]y_0, m[$ is empty (otherwise one could find x, covering x_0 in $]x_0, m[$ and $]y_0, m[$ should be empty by 2.9.3)). This settles the question of the ordered structure of X. Now it is easy to prove that the interval topology on X is the only one that makes $(X, \leqq)$ into a Heyting space.

Suppose $\alpha \neq 0$ and let $X = (\alpha + 1) \oplus \beta \oplus 1$. Let $\mathscr{P} = \{\theta \in \mathrm{Eq}(\alpha \oplus \beta \oplus 1) \mid$ all θ-classes are bounded intervals$\}$ and $\mathscr{P}_1 = \{\theta \in \mathscr{P} \mid \theta$ separates α from $\beta \oplus 1\}$. Then Con (X) is isomorphic with $\{(\theta, 1) \mid \theta \in \mathscr{P}\} \cup \{(\theta, 0) \mid \theta \in \mathscr{P}_1\}$ endowed with the order relation $(\theta, i) \leqq (\phi, j)$ if and only if $\theta \leqq \phi$ and $i \leqq j$. The distributivity of Con (X) follows from the fact that $\mathscr{P}$ is Boolean. The argument is still more easy in case $X = (\alpha \oplus 1) + 1$.

2.12. Proposition. *The lattice* Con (X) *is Boolean if and only if X is a Boolean chain or the disjoint sum of a Boolean chain and a one point space.*

Proof. Let X be a Boolean chain (hence a Heyting space). Then $\theta \in$ Con (X) if and only if θ corresponds to a partition of X into closed intervals. Hence Con (X) is distributive. If Y is the disjoint sum of X and $\{x_0\}$, then Con (Y) ($\simeq$ Con $(X) \times 2$) is also distributive.

Suppose now X is a Heyting space such that Con (X) is Boolean. Since Con (X) is always complete and coatomic, it is also coatomistic and each o.c.c. has a least element. Moreover, by 2.9, there are at most two o.c.c.. Suppose first X has a least element. For each $U \in \mathcal{O}(X)$, $\phi(U)$ is a coatom. Its complement is an atom, necessarily of type 2), say $\theta(p, q)$ with $q < p$, and $q \in U$, $p \notin U$. If $x \in X$, then by 2.9, either $x \leqq q$ or $x \geqq p$. This prove $U = (q]$ and $-U = [p)$. Now let $y \not\leq x$ in X. There exists $U \in \mathcal{O}(X)$, hence p and q in X, such that $x \in U = (q]$ and $y \in -U = [p)$. Therefore $x < y$ and X is a (Boolean) chain. If X has two o.c.c., one of them is reduced to a singleton $\{x_0\}$, which is necessarily clopen. Hence Con $(X) \simeq$ Con $(X - \{x_0\}) \times 2$ and Con $(X - \{x_0\})$ is Boolean and $X - \{x_0\}$ is a Boolean chain.

All these results about Heyting spaces can be reinterpreted in terms of Heyting algebras. This is done in the following theorem (where Δ denotes symmetric difference).

2.13. Theorem. *Let L be a Heyting algebra and suppose $S \in$* Sub (L). *Then*

1) *S contains a subalgebra isomorphic to a 3-element chain or a 4-element Boolean algebra (provided $S \neq \{0, 1\}$);*

2) *S is maximal (proper) if and only if there exist two distinct prime ideals P and Q such that $-S = P \Delta Q$. Moreover,*

3) Sub (L) *is upper semimodular*;

4) Sub (L) *is atomistic if and only if* L *is a Stone algebra*;

5) Sub (L) *is Boolean if and only if* L *is isomorphic to* C *or* $C \times 2$ *for some bounded chain* C.

Finally, if $\mathcal{P}(L)$ *is well-founded, then*

6) S *is contained in a maximal subalgebra*; *and*

7) Sub (L) *is distributive if and only if* L *is isomorphic to* $(C \times 2) \oplus C'$ *for some chains* C *and* C' *(not both empty).*

Proof. 1) A coatom $\phi(U)$ in Con (X) corresponds to a 3-element chain if $U \supseteq$ Min X, to a 4-element Boolean algebra if $-U \in \mathcal{O}(X)$ (see Lemma 2.2).

2) This is obvious. Note that, by virtue of Lemma 2.5, if P and Q are prime ideals, $-(P \Delta Q) \in$ Sub (L) if and only if, for all x, y in $P - Q$ (resp. $Q - P$), there exists z in $P \cap Q$ with $x \leqq y \vee z$.

3) See Corollary 2.8.

4) Use Lemma 2.5 and [6] p. 129.

5) This is a consequence of proposition 2.12. The "if" part admits a trivial direct proof.

6) and 7) are reinterpretations of 2.6 and 2.11 respectively.

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Authors address: Institut de Mathématique, Université de Liège, Belgique.