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ON TOLERANCE RELATIONS

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A tolerance on a set is a binary relation which is reflexive and symmetric (see [2], [3]). A tolerance T on a semigroup S is called

left compatible if $(x, y) \in T$ implies $(zx, zy) \in T$ for all $z \in S$,
right compatible if $(x, y) \in T$ implies $(xz, yz) \in T$ for all $z \in S$,
weakly compatible if it is left and right compatible,
compatible if $(x, y) \in T$ and $(u, v) \in T$ imply $(ux, vy) \in T$.

In this note we completely characterize the structure of semigroups in which every tolerance is left compatible (or, as the case may be, right compatible, weakly compatible, compatible).

We follow the notation and terminology of [1]. Let

$\boldsymbol{LZ}$ = the class of all left zero semigroups
$\boldsymbol{RZ}$ = the class of all right zero semigroups
$\boldsymbol{Z}$ = the class of all zero semigroups
$\boldsymbol{LC}$ = the class of all semigroups in which every tolerance is left compatible
$\boldsymbol{RC}$ = the class of all semigroups in which every tolerance is right compatible.

If S is a semigroup and x, y are distinct elements of S, then the least tolerance on S which contains the pair (x, y) will be denoted by $T(x, y)$. It is clear that $T(x, y)$ is the tolerance consisting of the pairs (x, y), (y, x), (z, z) for all $z \in S$.

Let $S \in \boldsymbol{LC}$. It is easy to verify that every subsemigroup and every homomorphic image of S is in $\boldsymbol{LC}$. If e, f are distinct idempotents of S then, since $T(e, f)$ is left compatible, $(e, ef) = (ee, ef) \in T(e, f)$ and so we have either $ef = e$ or $ef = f$. This shows, in particular, that the set $E(S)$ of idempotents of S is a band and, for any $e, f \in E(S)$,

$$ef = e \quad \text{or} \quad ef = f . \tag{1}$$

Lemma 1. (i) *Let S be a semigroup. If $|S| \leqq 2$ then every tolerance on S is compatible.*

(ii) *Let Y be a semilattice. If $Y \in \boldsymbol{LC}$ then $|Y| \leqq 2$.*

(iii) *Let S be a rectangular band. If $S \in \boldsymbol{LC}$ then S is either a left zero semigroup or a right zero semigroup. Conversely, if S is either a left zero semigroup or a right zero semigroup then every tolerance on S is compatible.*

Proof. (i) follows from the fact that if $|S| \leqq 2$ then the only tolerances on S are the identity relation and the universal relation.

(ii) Let $Y \in \boldsymbol{LC}$. For any $e, f \in Y$ either $e \geqq f$ or $f \geqq e$, by (1), and therefore Y is a chain. Assume $|Y| \geqq 3$ and let e, f, g be elements of Y such that $e > f > g$. Then $(f, g) = (fe, fg) \in T(e, g)$ implies either $f = e$ or $f = g$, which is a contradiction. Hence $|Y| \leqq 2$, as required.

(iii) Let $S \in \boldsymbol{LC}$. Then every subsemigroup of S is in $\boldsymbol{LC}$.

So, in view of (1), S cannot contain a copy of the 2×2 rectangular band $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ in which $e_{ij}e_{kl} = e_{il}$. Hence S is either a left zero semigroup or a right zero semigroup. To prove the converse let T be any tolerance on S, where $S \in \boldsymbol{LZ}$ or $S \in \boldsymbol{RZ}$, and let $(x, y), (u, v) \in T$. Then $(ux, vy) \in T$, since

$$(ux, vy) = \begin{cases} (u, v) & \text{if } S \in \boldsymbol{LZ}, \\ (x, y) & \text{if } S \in \boldsymbol{RZ}. \end{cases}$$

Hence T is compatible.

Let A, B be semigroups. Define a multiplication on the disjoint union $A \cup B$ by the rule

$$xy = yx = y \quad (x \in A,\ y \in B).$$

This multiplication is associative. We shall denote the resulting semigroup by $S(A, B)$.

The following proposition describes the structure of bands which are in $\boldsymbol{LC}$.

Proposition 2. *Let S be a band. Then S is in $\boldsymbol{LC}$ if and only if S is one of the following:*

(1) *a left zero semigroup;*

(2) *a right zero semigroup;*

(3) *$S(R, L)$, for some $R \in \boldsymbol{RZ}$, $L \in \boldsymbol{LZ}$.*

Proof. Suppose that S is in $\boldsymbol{LC}$. By the Clifford-McLean theorem S is a semilattice Y of rectangular bands E_α, $\alpha \in Y$. Since Y is a homomorphic image of S, $Y \in \boldsymbol{LC}$ and so, by Lemma 1, $|Y| \leqq 2$. If Y is a singleton set, say $Y = \{\alpha\}$, then $S = E_\alpha$ is a rectangular band, So, by Lemma 1, S is either a left zero semigroup or a right zero semigroup. On the other hand, if Y is a two-element set, say $Y = \{\alpha, \beta\}$ with $\alpha > \beta$, then $S = E_\alpha \cup E_\beta$. Now for any $e \in E_\alpha$, $f \in E_\beta$, since $ef, fe \in E_\beta$, (1) implies that $ef = fe = f$. Next, let $e, f \in E_\alpha$ and $g, h \in E_\beta$. Since $(g, gh) = (ge, gh) \in T(e, h)$, we have $gh = g$. Since $(g, ef) = (eg, ef) \in T(g, f)$ and $ef \in E_\alpha$ we get $ef = f$. Hence E_α is a right zero semigroup and E_β is a left zero semigroup. If we put $R = E_\alpha$ and $L = E_\beta$ then it is clear that $S = S(R, L)$.

We now prove the converse part. By Lemma 1, S is in $\boldsymbol{LC}$ whenever S is a left zero semigroup or a right zero semigroup. So suppose that $S = S(R, L)$ for some $R \in \boldsymbol{RZ}$, $L \in \boldsymbol{LZ}$. Then, for any $x, y, z \in S$, we have

$$(zx, zy) = \begin{cases} (x, y) & \text{if } z \in R, \\ (z, z) & \text{if } z \in L. \end{cases}$$

This yields that every tolerance on S is left compatible. Hence $S \in \boldsymbol{LC}$.

For the next proposition we need some notation.

Let A, B be semigroups. Let N be a zero semigroup with zero 0, and let $r: N^* \to B$ be a map, where $N^* = N \setminus \{0\}$. We shall denote by $S(N, r, B)$ the retract extension of B by N determined by the map r. More explicitly, $S(N, r, B) = N^* \cup B$ with multiplication:

$$xy = \begin{cases} (xr)\, y & \text{if } x \in N^*, \quad y \in B, \\ x(yr) & \text{if } x \in B, \quad y \in N^*, \\ (xr)(yr) & \text{if } x, y \in N^*. \end{cases}$$

Extend $r: N^* \to B$ to $S(N, r, B)$ by defining $(x)\, r = x$ for every $x \in B$. Then r is a retraction of $S(N, r, B)$ onto B. We shall denote by $S(A, N, r, B)$ the ideal extension of $S(N, r, B)$ by A^0, where A^0 is the semigroup obtained from A by adjoining a zero 0, $0 \notin A$, determined by the homomorphism

$$x \mapsto (1, r): A \to \Omega S(N, r, B), \quad \text{for all} \quad x \in A.$$

(Here $\Omega S(N, r, B)$ denotes the translational hull of $S(N, r, B)$). More explicitly, $S(A, N, r, B) = A \cup S(N, r, B)$ with multiplication:

$$xy = \begin{cases} y & \text{if } x \in A, \quad y \in S(N, r, B), \\ xr & \text{if } x \in S(N, r, B), \quad y \in A. \end{cases}$$

Similarly, we shall denote by $\bar{S}(A, N, r, B)$ the ideal extension of $S(N, r, B)$ by A^0 determined by the homomorphism

$$x \mapsto (r, 1): A \to \Omega S(N, r, B), \quad \text{for all} \quad x \in A.$$

The following proposition describes the structure of semigroups in $\boldsymbol{LC}$ which are not bands.

Proposition 3. *Let S be a semigroup which is not a band. Then S is in $\boldsymbol{LC}$ if and only if S is one of the following*:

(1) *a cyclic group of order* 2;

(2) *$S(N, r, L)$, for some non-trivial $N \in \boldsymbol{Z}$, $L \in \boldsymbol{LZ}$, $r: N^* \to L$;*

(3) *$S(R, N, r, L)$, for some $R \in \boldsymbol{RZ}$, $N \in \boldsymbol{Z}$, $L \in \boldsymbol{LZ}$, $r: N^* \to L$.*

Proof. Assume that S is in $\boldsymbol{LC}$. Then, for $x \in S$, we have $(x^2, x^3) = (xx, xx^2) \in T(x, x^2)$. Hence we must have

$$\text{either} \quad x^3 = x \quad \text{or} \quad x^3 = x^2 \tag{2}$$

for all $x \in S$.

Suppose there exists an element x in S such that $x^2 \neq x$ and $x^3 = x$. Let $x^2 = e$. Then $e^2 = e = xx$ and $xe = ex = x$ so that $\{e, x\}$ is a cyclic group of order 2. If $y \in S \setminus \{e, x\}$, then

$$(e, xy) = (xx, xy) \in T(x, y) \quad \text{implies} \quad xy = e$$

while

$$(x, xy) = (xe, xy) \in T(e, y) \quad \text{implies} \quad xy = x.$$

Therefore $x = e$, a contradiction. Hence $S = \{e, x\}$. Thus S is a cyclic group of order 2 if S has at least one nonidempotent x such that $x^3 = x$. Consequently, if S is not a cyclic group of order 2 then (2) implies that $x^3 = x^2$ for all $x \in S$.

Suppose now that S is not a cyclic group of order 2. Then $x^3 = x^2$ for all $x \in S$. Since $S \in \boldsymbol{LC}$, the subsemigroup $E(S)$ of idempotents of S is in $\boldsymbol{LC}$. Hence, by Proposition 2, $E(S)$ is either a left zero semigroup or a right zero semigroup or $S(R, L)$ for some $R \in \boldsymbol{RZ}$, $L \in \boldsymbol{LZ}$.

Case 1. Assume $E(S)$ is a left zero semigroup. If x and y are any two elements of S then $(x^2, xy) = (xx, xy) \in T(x, y)$. Consequently, we have $xy = x^2$ for all $x, y \in S$. This implies that $E(S)$ is an ideal in S and the Rees quotient $S/E(S)$ is a zero semigroup. Write $L = E(S)$, $N = S/E(S)$ and define $r: N^* \to L$ by $(x)\, r = x^2$ for all $x \in N^*$. It is then clear that $S = S(N, r, L)$.

Case 2. Assume $E(S)$ is a right zero semigroup. Let $x \in S$ with $x^2 \neq x$. If $e \in E(S)$, then $(e, x^2) = (x^2e, x^2x) \in T(e, x)$ implies $x^2 = e$. Since this is true for every $e \in E(S)$ it follows that $E(S) = \{e\}$ is a singleton set. Now for any two distinct elements $x, y \in S \setminus \{e\}$,

$$(e, xy) = (x^2, xy) = (xx, xy) \in T(x, y) \quad \text{implies} \quad xy = e$$

and

$$(e, yx) = (y^2, yx) = (yy, yx) \in T(y, x) \quad \text{implies} \quad yx = e\,.$$

Hence $xy = yx = e$ for all $x, y \in S$. Write $N = S$ and $L = \{e\}$. Then $S = S(N, r, L)$, where $r: N^* \to L$ is the unique constant map.

Case 3. Assume $E(S) = S(R, L)$ for some $R \in \boldsymbol{RZ}$, $L \in \boldsymbol{LZ}$. Let x be a nonidempotent of S. Then, by (2), $x^2 \in E(S) = R \cup L$; hence $x^2 \in R$ or $x^2 \in L$. We claim that $x^2 \in L$. For, if $x^2 \in R$ then by choosing an element $f \in L$ we see that $(x^2, f) = (x^2x, x^2f) \in T(x, f)$ and hence $x^2 = x$, a contradiction. Thus $x^2 \in L$ for all nonidempotents x of S. We now show that $S \setminus R$ is an ideal of S. Take any $e \in R$ and $x \in S \setminus R$. Since

$$(x^2, xe) = (xx, xe) \in T(x, e)$$

and

$$(ex, e) = (ex, ee) \in T(x, e)\,,$$

it follows that $xe = x^2$ and $ex = x$. In particular, $S \setminus R$ is an ideal of S. Now $S \setminus R$ is in $\boldsymbol{LC}$ and $E(S \setminus R) = L$ is a left zero semigroup. So, by Case 2, $S \setminus R = S(N, r. L)$, where N is the Rees quotient of $S \setminus R$ by the ideal L, and $r: N^* \to L$ is given by $xr = x^2$ for all $x \in N^*$. Clearly $S = S(R, N, r, L)$.

Conversely, let us assume that S is one of the semigroups listed above. If S is a cyclic group of order 2 then, by Lemma 1, $S \in \boldsymbol{LC}$. If $S = S(N, r, L)$ then every tolerance T on S is left compatible, since $(x, y) \in T$ implies $(zx, zy) = (z^2, z^2) \in T$ for all $z \in S$. Hence $S \in \boldsymbol{LC}$. Finally, if $S = S(R, N, r, L)$ then, for any $x, y, z \in S$,

$$(zx, zy) = \begin{cases} (x, y) & \text{if} \quad z \in R\,, \\ (z^2, z^2) & \text{if} \quad z \in S \setminus R\,. \end{cases}$$

This yields that every tolerance on S is left compatible. Hence $S \in \boldsymbol{LC}$.

Propositions 2 and 3 completely characterize the structure of semigroups in which every tolerance is left compatible. Dually, we have

Proposition 4. *Let S be a semigroup. Then every tolerance on S is right compatible if and only if S is one of the following*:

(1) *a left zero semigroup*;

(2) *a right zero semigroup*;

(3) $S(L, R)$, *for some* $L \in \boldsymbol{LZ}$, $R \in \boldsymbol{RZ}$;

(4) *a cyclic group of order* 2;

(5) $S(N, r, R)$, *for some* $N \in \boldsymbol{Z}$, $R \in \boldsymbol{RZ}$, $r: N^* \to R$;

(6) $\bar{S}(L, N, r, R)$, *for some* $L \in \boldsymbol{LZ}$, $R \in \boldsymbol{RZ}$, $N \in \boldsymbol{Z}$, $r: N^* \to R$.

Since a tolerance is weakly compatible if and only if it is both left and right compatible we obtain

Proposition 5. *Let S be a semigroup. Then every tolerance on S is weakly compatible if and only if S is one of the following*:

(1) *a left zero semigroup*;

(2) *a right zero semigroup*;

(3) *a semilattice of order* 2;

(4) *a cyclic group of order* 2;

(5) *a zero semigroup.*

Proof. Suppose that every tolerance on S is weakly compatible.

Case 1. Assume S is a band. If S is a rectangular band then, by Proposition 2, S is either a left zero semigroup or a right zero semigroup. On the other hand, if S is not a rectangular band then it follows from the Clifford-McLean theorem and Lemma 1 that S is a semilattice Y, where $Y = \{\alpha, \beta\}$ with $\alpha > \beta$, of rectangular bands E_α and E_β. By Proposition 2,

$$E_\alpha \in \boldsymbol{RZ} \quad \text{and} \quad E_\beta \in \boldsymbol{LZ}$$

while, by Proposition 4,

$$E_\alpha \in \boldsymbol{LZ} \quad \text{and} \quad E_\beta \in \boldsymbol{RZ}\,.$$

Hence E_α and E_β are singleton sets and so $S = Y$, a semilattice of order 2.

Case 2. Assume S is not a band. Then an easy verification shows that $E(S)$ is a rectangular band. Since $S \in \boldsymbol{LC}$ it follows from Proposition 3 that $E(S) \in \boldsymbol{LZ}$. Since $S \in \boldsymbol{RC}$, Proposition 4 implies that $E(S) \in \boldsymbol{RZ}$. Therefore $E(S)$ is a singleton set. Now by Proposition 3, S is either a cyclic group of order 2 or a zero semigroup.

The Converse follows from the next proposition.

Proposition 6. *If S is one of the semigroups listed in Proposition 5 then every tolerance on S is compatible.*

Proof. The result follows from Lemma 1 once we show that every tolerance on a zero semigroup S is compatible. This is clear, since, for any tolerance T on S, $(u, v), (x, y) \in T$ imply $(u, v)(x, y) = (0, 0) \in T$.

Remark 7. Since a compatible tolerance is weakly compatible, Proposition 5 remains valid if we replace "weakly compatible" by "compatible".

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