Zbigniew Lipecki
On common extensions of two quasi-measures

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 3, 489–494

Persistent URL: http://dml.cz/dmlcz/102109

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

ON COMMON EXTENSIONS OF TWO QUASI-MEASURES

ZBIGNIEW LIPECKI, Wrocław

(Received June 21, 1985)

1. INTRODUCTION

Throughout the paper X stands for an arbitrary (non-empty) set and $\boldsymbol{M}$ and $\boldsymbol{N}$ stand for algebras of subsets of X. We denote by $S(\boldsymbol{M})$ the linear space spanned by the characteristic functions 1_M, $M \in \boldsymbol{M}$. We note that

$$(1) \qquad S(\boldsymbol{M}) \cap S(\boldsymbol{N}) = S(\boldsymbol{M} \cap \boldsymbol{N}).$$

The closure of $S(\boldsymbol{M})$ in the Banach space of all real-valued bounded functions on X with the supremum norm $\|\cdot\|$ is denoted by $B(\boldsymbol{M})$. The dual of $B(\boldsymbol{M})$ can be identified with the Banach space $ba(\boldsymbol{M})$ of all real-valued quasi-measures, i.e., bounded additive set functions, on $\boldsymbol{M}$ with the total variation norm also denoted by $\|\cdot\|$ (see, e.g., [7], Corollary 4.7.5). The unique element of $B(\boldsymbol{M})^*$ corresponding to $\mu \in ba(\boldsymbol{M})$ is denoted by I_μ ([7], Theorem 4.7.4).

We note that

$$(2) \qquad B(\boldsymbol{M}) \cap B(\boldsymbol{N}) \supset B(\boldsymbol{M} \cap \boldsymbol{N})$$

and the equality holds in case $\boldsymbol{M}$ and $\boldsymbol{N}$ are σ-algebras, but not in general. Indeed, for $\boldsymbol{M}$ and $\boldsymbol{N}$ as specified in Example 1 below the identity function on $[0, 1)$ is a counter-example.

We are concerned with the following problem:
Given $\mu \in ba(\boldsymbol{M})$ and $\nu \in ba(\boldsymbol{N})$ which are *consistent*, i.e.,

$$\mu \mid \boldsymbol{M} \cap \boldsymbol{N} = \nu \mid \boldsymbol{M} \cap \boldsymbol{N},$$

when does there exist $\varphi \in ba(\boldsymbol{F})$, where $\boldsymbol{F}$ stands for the algebra generated by $\boldsymbol{M} \cup \boldsymbol{N}$, with $\varphi \mid \boldsymbol{M} = \mu$ and $\varphi \mid \boldsymbol{N} = \nu$ (called in the sequel a *common extension of* μ *and* ν)?

This problem has been suggested by the papers by Guy [1] and Pták [6]. The former gave a complete solution to a version of the problem with μ, ν and φ positive (see also [2], [4] and [7], Theorem 3.6.1). The latter dealt with the general question of extending simultaneously two continuous linear functionals defined on subspaces of a locally convex space. We also note that, with the boundedness condition dropped, the problem admits an easy affirmative solution ([4], Corollary 2.1, [7], Theorem 3.6.2).

We shall present below two negative examples and three affirmative partial solutions to the problem*). The first two solutions (Propositions 1 and 2) are of global character, i.e., they involve assumptions on $\mathbf{M}$ and $\mathbf{N}$ only, while the third (Corollary) imposes some strong conditions on one of the quasi-measures to be extended. The global solutions are related to some results of [6] (see Remark 2 below). Reasonable necessary and sufficient conditions in order that the answer to the problem be affirmative individually, i.e., in terms of μ and ν, seem hard to find. The proofs of Propositions 1 and 2 are based on the Hahn-Banach theorem.

2. NEGATIVE RESULTS

The following examples show that the condition that $\mathbf{M} \cap \mathbf{N} = \{\emptyset, X\}$ is not sufficient even in the case when μ and ν are positive. In the first example μ and ν are additionally two-valued, while in the second $\mathbf{M}$ and $\mathbf{N}$ are σ-algebras generated by countable partitions and μ and ν are measures.

Example 1. Suppose $\mathbf{M} \cap \mathbf{N} = \{\emptyset, X\}$ and the following condition holds:
(3) There exist $M_n \in \mathbf{M}$ and $N_n \in \mathbf{N}$ with $\emptyset \neq M_1 \subset N_1 \subset M_2 \subset N_2 \subset \ldots \neq X$ (e.g., $X = [0, 1)$ and $\mathbf{M}$ and $\mathbf{N}$ are generated by the families

$$\{[0, a): 0 < a < 1 \text{ is rational}\},$$

$$\{[0, a): 0 < a < 1 \text{ is irrational}\},$$

respectively). Extend $\{M_n: n = 1, 2, \ldots\}$ to a maximal ideal $\mathbf{I}$ in $\mathbf{M}$ and put

$$\mu(M) = 0 \quad \text{if} \quad M \in \mathbf{I} \quad \text{and} \quad \mu(M) = 1 \quad \text{if} \quad M \in \mathbf{M} \setminus \mathbf{I}.$$

Choose $x \in N_1$ and for $N \in \mathbf{N}$ put

$$\nu(N) = 0 \quad \text{if} \quad x \notin N \quad \text{and} \quad \nu(N) = 1 \quad \text{if} \quad x \in N.$$

Observe that every common additive extension φ of μ and ν to $\mathbf{F}$ is unbounded. Indeed, $N_n \setminus M_n$ are pairwise disjoint and

$$\varphi(N_n \setminus M_n) = \nu(N_n) - \mu(M_n) = 1.$$

Example 2. Let X be the set of all natural numbers and let $\mathbf{M}$ and $\mathbf{N}$ be the σ-algebras of subsets of X generated by the partitions

$$\{1\}, \{2, 3\}, \ldots, \{2n - 2, 2n - 1\}, \ldots,$$

$$\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}, \ldots,$$

respectively. Clearly, $\mathbf{M} \cap \mathbf{N} = \{\emptyset, X\}$. Let (a_n) and (b_n) be sequences of positive

*) Some of these results were announced at the 13th Winter School on Abstract Analysis, Srní (in the Šumava Mountains), 1985; see Suppl. Rend. Circ. Mat. Palermo (2), to appear.

real numbers such that

$$a_n \searrow 0\,, \quad \sum_{n=1}^{\infty} a_n = \infty\,, \quad b_{n+1} > a_n - a_{n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty$$

(e.g., $a_n = 1/n$, $b_{n+1} = 1/n^2$). Put

$$\mu(\{1\}) = b_1 \quad \text{and} \quad \mu(\{2n-2, 2n-1\}) = b_n\,,$$

$$\nu(\{1, 2\}) = b_1 + a_1 \quad \text{and} \quad \nu(\{2n-1, 2n\}) = b_n + (a_n - a_{n-1})\,.$$

Clearly, μ and ν extend uniquely to (positive) measures on $\boldsymbol{M}$ and $\boldsymbol{N}$, respectively, which we also denote by μ and ν. Moreover, $\mu(X) = \nu(X)$ since $a_n \searrow 0$. Let φ be a common additive extension of μ and ν to $\boldsymbol{F}$. We have

$$\varphi(\{2n\}) = \nu(\{1, \ldots, 2n\}) - \mu(\{1, \ldots, 2n-1\}) = a_n\,.$$

Hence φ is unbounded.

3. AFFIRMATIVE RESULTS AND COMMENTS

We say that $\boldsymbol{M}$ and $\boldsymbol{N}$ are *weakly independent* if, given two partitions $\{M_1, \ldots, M_m\} \subset \boldsymbol{M}$ and $\{N_1, \ldots, N_n\} \subset \boldsymbol{N}$ of X into non-empty sets, the set

$$\{(i,j)\colon\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n \ \text{and}\ M_i \cap N_j \neq \emptyset\}$$

contains a row $\{(i_0, j)\colon 1 \leqq j \leqq n\}$ and a column $\{(i, j_0))\ 1 \leqq i \leqq m\}$ of the matrix $\{(i,j)\colon\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$.

Clearly, this condition implies $\boldsymbol{M} \cap \boldsymbol{N} = \{\emptyset, X\}$. Moreover, it is implied by the independence of $\boldsymbol{M}$ and $\boldsymbol{N}$ in the sense of Marczewski ([5], p. 220), i.e., the condition that for every pair of non-empty sets $M \in \boldsymbol{M}$ and $N \in \boldsymbol{N}$ we have $M \cap N \neq \emptyset$. The latter implication cannot be reversed as shown by the following simple

Example 3. Let X be the set of all natural numbers and let $\boldsymbol{M}$ and $\boldsymbol{N}$ be algebras of subsets of X generated by the even and the odd singletons, respectively. Then $\boldsymbol{M}$ and $\boldsymbol{N}$ are weakly indepedent but not indepedent.

Lemma 1. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be weakly independent and let $g \in S(\boldsymbol{M})$ and $h \in S(\boldsymbol{N})$. Then there exists a real number c such that*

$$3\|g + h\| \geqq \|g - c\| + \|h + c\|\,.$$

Proof. Let a_i, $i = 1, \ldots, m$, and b_j, $j = 1, \ldots, n$, be all the values of g and h, respectively. Let i_0, j_0 be such that

$$g^{-1}(a_{i_0}) \cap h^{-1}(b_j) \neq \emptyset \quad \text{and} \quad g^{-1}(a_i) \cap h^{-1}(b_{j_0}) \neq \emptyset$$

for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. Then $|b_j + a_{i_0}| \leqq \|g + h\|$ and

$$|a_i - a_{i_0}| \leqq |a_i + b_{j_0}| + |a_{i_0} + b_{j_0}| \leqq 2\|g + h\|\,.$$

Thus we may take $c = a_{i_0}$.

The following is a partial extension of a result of Marczewski ([5], Theorem I).

Proposition 1. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be weakly independent and let $\mu \in ba(\boldsymbol{M})$ and $\nu \in ba(\boldsymbol{N})$ be consistent. Then there exists $\varphi \in ba(\boldsymbol{F})$ which is a common extension of μ and ν and satisfies $\|\varphi\| \leqq 3 \max(\|\mu\|, \|\nu\|)$.*

Proof. By (1), $S(\boldsymbol{M}) \cap S(\boldsymbol{N})$ consists of constant functions only. Hence the formula $J(g + h) = I_\mu(g) + I_\nu(h)$ defines unambigously a linear functional J on $S(\boldsymbol{M}) + S(\boldsymbol{N})$. In view of Lemma 1,

$$\|J\| \leqq 3 \max(\|\mu\|, \|\nu\|) .$$

Hence, by the Hahn-Banach theorem, J extends to a continuous linear functional K on $S(\boldsymbol{F})$ with $\|K\| = \|J\|$. Then φ defined on $\boldsymbol{F}$ by $\varphi(F) = K(1_F)$ is as desired.

It follows from [3], Example 1, that the constant "3" in Proposition 1 is best possible even in the case where μ and ν are two-valued.

Lemma 2. *If $f \in B(\boldsymbol{M})$ has finite range, then $f \in S(\boldsymbol{M})$.*

Proof. Clearly, it is enough to prove that if Z_j, $j = 1, \dots, n$, are non-empty and pairwise disjoint and $Z_j \notin \boldsymbol{M}$ for some j, then

$$\|\sum_{j=1}^{n} b_j 1_{Z_j} + g\| \geqq \tfrac{1}{2} \min\{|b_j|, |b_k - b_l|: 1 \leqq j, k, l \leqq n;\ k \neq l\}$$

whenever b_j, $j = 1, \dots, n$, are (non-zero distinct) real numbers and $g \in B(\boldsymbol{M})$. We may and do assume that $g \in S(\boldsymbol{M})$. Accordingly, let $g = \sum_{i=1}^{m} a_i 1_{M_i}$, where $M_i \in \boldsymbol{M}$, $i = 1, \dots, m$, are non-empty and pairwise disjoint. Denote by δ the right-hand side of the above inequality.

Suppose, to get a contradiction, that

$$\|\sum_{j=1}^{n} b_j 1_{Z_j} + \sum_{i=1}^{m} a_i 1_{M_i}\| < \delta .$$

Since $|b_j| \geqq \delta$, we have

(a) $Z_j \subset \bigcup_{i=1}^{m} M_i$ for $j = 1, \dots, n$.

Moreover,

(b) $M_i \cap Z_j \neq \emptyset$ implies $M_i \subset Z_j$.

Indeed, first observe that $M_i \cap Z_k = \emptyset$ for all $k \neq j$. Otherwise $|b_j + a_i|, |b_k + a_i| < \delta$, which implies $|b_j - b_k| < 2\delta$, a contradiction with the definition of δ. If $M_i \setminus Z_j \neq \emptyset$, it follows that $|a_i| < \delta$. Since, moreover, $|b_j + a_i| < \delta$, we get $|b_j| < 2\delta$, which contradicts the definition of δ.

From (a) and (b) we infer that for each j

$$Z_j = \bigcup_{i \in T_j} M_i , \quad \text{where} \quad T_j = \{1 \leqq i \leqq n:\ M_i \cap Z_j \neq \emptyset\} ,$$

which contradicts the assumption that $Z_j \notin \boldsymbol{M}$ for some j.

The following is a partial generalization of Proposition 3 of [3].

Proposition 2. *Let $\boldsymbol{N}$ be finite and let $\mu \in ba(\boldsymbol{M})$ and $\nu \in ba(\boldsymbol{N})$ be consistent. Then there exists $\varphi \in ba(\boldsymbol{F})$ which is a common extension of μ and ν.*

Proof. In view of (2) and Lemma 2,

$$B(\boldsymbol{M}) \cap B(\boldsymbol{N}) = B(\boldsymbol{M} \cap \boldsymbol{N}).$$

Hence the formula $J(g + h) = I_\mu(g) + I_\nu(h)$ defines unambigously a linear functional on $B(\boldsymbol{M}) + B(\boldsymbol{N})$. Clearly, the restrictions of J to $B(\boldsymbol{M})$ as well as to $B(\boldsymbol{N})$ are continuous. Since $B(\boldsymbol{M})$ is complete and $B(\boldsymbol{N})$ is finite-dimensional, it is not hard to see that J itself is continuous. Now, applying the Hahn-Banach theorem as in the proof of Proposition 1, we get the assertion.

We shall need the following notation. For $\nu \in ba(\boldsymbol{N})$ we put

$$\mathcal{N}(\nu) = \{N \in \boldsymbol{N}: \nu(S) = 0 \text{ for all } N \supset S \in \boldsymbol{N}\}.$$

In case $\mathcal{N}(\nu)$ is a hereditary family of subsets of X, ν is called *(Lebesgue) complete.*

Corollary. *Let $\mu \in ba(\boldsymbol{M})$ and $\nu \in ba(\boldsymbol{N})$ be consistent and let ν be complete and have finite range. Then there exists $\varphi \in ba(\boldsymbol{F})$ which is a common extension of μ and ν.*

Proof. First we note that if $N \in \mathcal{N}(\nu)$ and $N \supset M \in \boldsymbol{M}$, then $\mu(M) = 0$. (Indeed, $M \in \mathcal{N}(\nu)$, and so $\nu(M) = 0$.) Hence, by [3], Proposition 1, μ extends to a real-valued quasi-measure μ' on $\boldsymbol{M}'$, where $\boldsymbol{M}'$ stands for the algebra generated by $\boldsymbol{M} \cup \cup \mathcal{N}(\nu)$, such that

$$\mu'(M \dot{-} Z) = \mu(M) \quad \text{for all} \quad M \in \boldsymbol{M} \quad \text{and} \quad Z \in \mathcal{N}(\nu).$$

We claim that μ' and ν are consistent. Indeed, if $M \dot{-} Z = N$ with $M \in \boldsymbol{M}$, $Z \in \mathcal{N}(\nu)$ and $N \in \boldsymbol{N}$, then $M = N \dot{-} Z$. Hence $\mu(M) = \nu(N)$, and so $\mu'(N) = \nu(N)$.

Let $\boldsymbol{N}'$ be a finite subalgebra of $\boldsymbol{N}$ such that for every $N \in \boldsymbol{N}$ there exists $N' \in \boldsymbol{N}'$ with $N \dot{-} N' \in \mathcal{N}(\nu)$. Then $\boldsymbol{F}$ coincides with the algebra generated by $\boldsymbol{M}' \cup \boldsymbol{N}'$. Put $\nu' = \nu \mid \boldsymbol{N}'$. Clearly, μ' and ν' are consistent, whence, by Proposition 2, there exists $\varphi \in ba(\boldsymbol{F})$ which is a common extension of μ' and ν'. Since $\mathcal{N}(\nu) \subset \mathcal{N}(\varphi)$ and $\varphi \mid \boldsymbol{N}' = \nu'$, we have $\varphi \mid \boldsymbol{N} = \nu$.

We note that both the above assumptions on ν are essential as is shown by Examples 1 and 2, respectively.

Remark 1. Condition (3) of Example 1 admits the following strenghtening:

$$(\forall M \in \boldsymbol{M},\ M \neq X)\,(\exists N \in \boldsymbol{N},\ N \neq \emptyset)$$
$$(\forall N \in \boldsymbol{N},\ N \neq X)\,(\exists M \in \boldsymbol{M},\ M \neq \emptyset) \quad [M \cap N = \emptyset].$$

The latter might be called the *total dependence* of $\boldsymbol{M}$ and $\boldsymbol{N}$. Unfortunately, it is much stronger than just the negation of the weak independence of $\boldsymbol{M}$ and $\boldsymbol{N}$. This sheds some light on the dimension of the gap which exists between the negative Example 1 and the affirmative Proposition 1.

We shall present another strengthening of condition (3).

Proposition 3. *If $B(\boldsymbol{M}) \cap B(\boldsymbol{N})$ contains a non-constant function f, then* (3) *holds.*

Proof. Fix $a \in f(X)$. We first show that given $\varepsilon > 0$, we can find $M \in \boldsymbol{M}$ such that

$$|a - f(x)| < 2\varepsilon \quad \text{for all} \quad x \in M\,, \quad |a - f(x)| > \varepsilon \quad \text{for all} \quad x \in X \setminus M\,.$$

Indeed, let $S_1, \ldots, S_n \in \boldsymbol{M}$ be a partition of X with

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad x, y \in S_i\,, \quad i = 1, \ldots, n$$

(see [7], Proposition 4.7.2). Put

$$T = \{1 \leqq i \leqq n: |a - f(x)| \leqq \varepsilon \text{ for some } x \in S_i\} \quad \text{and} \quad M = \bigcup_{i \in T} S_i\,.$$

Fix $\varepsilon > 0$ with $2^{-2}\varepsilon < \sup\{|a - f(x)|: x \in X\}$. By what we have just proved, there exist $M_n \in \boldsymbol{M}$ and $N_n \in \boldsymbol{N}$ with $|a - f(x)| < 2^{-2n}\varepsilon$ for all $x \in M_n$, $|a - f(x)| > 2^{-(2n+1)}\varepsilon$ for all $x \in X \setminus M_n$, $|a - f(x)| < 2^{-(2n+1)}\varepsilon$ for all $x \in N_n$, $|a - f(x)| > 2^{-(2n+2)}\varepsilon$ for all $x \in X \setminus N_n$, $n = 1, 2, \ldots$. Then $M_1 \neq X$ and $f^{-1}(a) \subset M_n$. Moreover, $(X \setminus M_n) \cap N_n = \emptyset$, and so $N_n \subset M_n$. Analogously, $M_{n+1} \subset N_n$. Thus $X \setminus M_n$ and $X \setminus N_n$ satisfy (3).

Remark 2 (H. Weber). The existence of a common extension $\varphi \in ba(\boldsymbol{F})$ for every consistent pair $\mu \in ba(\boldsymbol{M})$ and $\nu \in ba(\boldsymbol{N})$ is equivalent to the conjunction of the conditions:

(i) $B(\boldsymbol{M}) \cap B(\boldsymbol{N}) \subset B(\boldsymbol{M} \cap \boldsymbol{N})$,

(ii) $B(\boldsymbol{M}) + B(\boldsymbol{N})$ is closed in $B(\boldsymbol{F})$.

This follows from [6], Theorems 2.1 and 2.4, and (1). In case $\boldsymbol{M} \cap \boldsymbol{N} = \{\emptyset, X\}$, the necessity of (i) also follows from Proposition 3 and Example 1 above. Finally, note that (i) $\not\Rightarrow$ (ii) (see Example 2).

Added in proof. Lemma 2 above is essentially identical with Lemma 2 of Dierolf, P., Dierolf, S., Drewnowski, L.: Remarks and examples concerning unordered Baire-like and ultrabarrelled spaces, Colloq. Math. 39, 109—116 (1978). The proof given there is somewhat simpler than ours.

References

[1] *Guy, D. L.:* Common extensions of finitely additive measures, Portugal. Math. *20*, 1—5 (1961).

[2] *Lembcke, J.:* Gemeinsame Urbilder endlich additiver Inhalte, Math. Ann. *198*, 239—258 (1972).

[3] *Lipecki, Z.:* Conditional and simultaneous extensions of group-valued quasi-measures, Glas. Mat. *19* (*39*), 49—58 (1984).

[4] *Maharam, D.:* Consistent extensions of linear functionals and of probability measures, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Berkeley and Los Angeles, vol. *2*, 127—147. University of California Press 1972.

[5] *Marczewski, E.:* Measures in almost independent fields, Fund. Math. *38*, 217—229 (1951).

[6] *Pták, V.:* Simultaneous extensions of two functionals, Czechoslovak Math. J. *19* (*94*), 553—566 (1969).

[7] *Rao, K. P. S. Bhaskara, Rao, M. Bhaskara:* Theory of Charges. A Study of Finitely Additive Measures. London and New York. Academic Press 1983.

Author's address: Institute of Mathematics, Polish Academy of Sciences, Kopernika 18, 51-617 Wrocław, Poland.