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WHITEHEAD PROPERTY OF MODULES

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INTRODUCTION

In the present note, we study relations between the structure of associative rings and extension properties of modules. Let R be an associative ring with unit and R-mod the category of unitary left R-modules. A module $N \in R$-mod is said to have the *Whitehead property* (WP) if either N is injective or, for all $M \in R$-mod, $\operatorname{Ext}_R(M, N) = 0$ implies M is projective.

A given module may or need not have WP according to the extension of ZFC we work in (this happens e.g. if R is a countable Dedekind domain and $N = R$ – see [7] and [4] – or if R is a simple countable non-completely reducible von Neumann regular ring and N is any countable R-module – see Section 2 below). Nevertheless, if we require all R-modules to have WP, we get results on the structure of the ring R, proved in ZFC. Hence, this requirement seems more appropriate for our aims.

Recall that by [2, Appendix A], a ring R such that every left R-module has WP is called a *left T-ring*. By [9] we know that every left T-ring is either left artinian or von Neumann regular. While we have a full description e.g. of left nonsingular left artinian left T-rings (see [9, 4.4 and 6.1]), only little is known about the regular ones. By [10], if R is a simple countable regular ring, then $\operatorname{Ext}_R(M, N) \neq 0$ for all countably generated R-modules M, N such that M is non-projective and N is non-injective. Moreover, assuming $V = L$, every countable R-module has WP (see [10, III.6]).

The present note is divided into three sections. In Section 1, we show that in spite of the facts mentioned above, if R is a simple non-completely reducible regular ring of cardinality $< 2^{\aleph_0}$, then there is an R-module which does not have WP. Hence, R is not a left T-ring. In Section 2, we show that in some models of ZFC, even no countable R-module has WP. Hence, the assertion of [10, III.6] is independent of ZFC. In Section 3, we use the solution of Artin's problem ([6] and [3]) to construct a ring R which is not a left T-ring, but every cyclic R-module has WP.

In what follows, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. Let κ be an infinite cardinal and $E \subseteq \kappa$. Then E is cofinal in κ if $\sup E = \kappa$. Further, E is closed in κ if $\sup F \in E \cup \{\kappa\}$, for every non-empty subset $F \subseteq E$. We say that E is stationary in κ if $E \cap F \neq \emptyset$ for every closed and cofinal subset F of κ. Let G be a filter over κ. Then G is κ-complete if G is closed with respect to intersections of less than κ elements of G. Further, G is normal if for any $g_\alpha \in G$, $\alpha < \kappa$, the set $\{\alpha < \kappa \mid \alpha \in \bigcap_{\beta < \alpha} g_\beta\}$ belongs to G.

In what follows, all rings are associative with unit. If S and T are rings, then $S \boxplus T$ denotes the ring direct sum of S and T. If S is a ring, n is a natural number, $n \geqq 1$, and κ is a cardinal, $\kappa \geqq 1$, then $\mathrm{RFM}_{n \times \kappa}(S)$ denotes the set of all row finite matrices of type $n \times \kappa$ over S.

If S is a ring, then S-mod denotes the category of unitary left S-modules. A unitary left R-module is simply called a *module*. Let R be a left hereditary ring, κ an infinite cardinal and $M \in R$-mod. Then M is κ-free if every submodule of M which is generated by less than κ elements is projective. Moreover, M is strongly κ-free if every submodule A of M which is generated by less than κ elements is contained in a projective submodule A' such that A' is generated by less than κ elements and M/A' is κ-free (see [4, § 18]). If N is a module, then $I(N)$ denotes the injective hull of N and $\mathrm{Soc}(N)$ denotes the left socle of N. A ring R is said to be *completely reducible* if $\mathrm{Soc}(R) = R$. If $N \in R$-mod and $x \in N$, then $\mathrm{Ann}_R(x)$ denotes the left annihilator of x in R.

A module N is said to have a *socle sequence* if there are an ordinal σ and a sequence S_ν, $\nu \leqq \sigma$ of submodules of N such that $S_0 = 0$, $S_{\nu+1}/S_\nu = \mathrm{Soc}(N/S_\nu) \neq 0$ for all $\nu < \sigma$, $S_\nu = \bigcup S_\mu$, $\mu < \nu$ for all limit $\nu \leqq \sigma$ and $S_\sigma = N$. Clearly, if N has a socle sequence, then σ and S_ν, $\nu \leqq \sigma$, are unique.

A sum (direct sum) of submodules is denoted by $\sum$ (by $\dot{\sum}$, respectively). If κ is a cardinal, $\kappa \geqq 1$ and $N \in R$-mod, then $N^{(\kappa)}$ and N^κ denote the direct sum and the direct product of κ copies of N, respectively.

Further concepts and notation can be found e.g. in [1] and [4].

1. REGULAR RINGS AND WP

By [10], the only candidates for non-completely reducible regular left T-rings are rings of the form $(S \boxplus)\, R$, where S is a completely reducible ring and R is a simple regular ring having all left ideals countably generated. Here, in 1.5, we show that, moreover, card $R \geqq 2^{\aleph_0}$. Thus, in 1.6, we obtain a full description of left non-singular left T-rings of cardinality $< 2^{\aleph_0}$.

1.1. Let R be a non-completely reducible regular ring. Let A be a non-empty set of countably generated left ideals of R. For $N \in R$-mod let $f \in \mathrm{Hom}_R(N, N^{\aleph_0}/N^{(\aleph_0)})$ such that $nf = (n_i + N^{(\aleph_0)} \mid i < \aleph_0)$, where $n_i = n$ for all $i < \aleph_0$. Define a sequence S_ν, $\nu \leqq \aleph_1$ of submodules of $N^{\aleph_0}$ by

(i) $S_0 \supseteq N^{(\aleph_0)}$ and $S_0/N^{(\aleph_0)} = (N)f$,

(ii) $S_{\nu+1} = \langle\{n \in N^{\aleph_0} \mid \exists I \in A : In \subseteq S_\nu\}\rangle_R$,

(iii) $S_\nu = \bigcup S_\mu$, $\mu < \nu$ for ν limit.

Put $\overline{N} = S_{\aleph_1}/N^{(\aleph_0)}$.

Lemma. *N is isomorphic to a submodule of $\overline{N}$ and, for all $I \in A$,* $\mathrm{Ext}_R(R/I, \overline{N}) = 0$.

Proof. Obviously, $N \simeq (N)f \subseteq \overline{N}$. The assertion is clear if I is finitely generated. Let $g \in \mathrm{Hom}_R(I, \overline{N})$, where $I = \dot{\sum} Re_j$, $j < \aleph_0$, and $\{e_j \mid j < \aleph_0\}$ is a set of pairwise orthogonal idempotents of R (see [5, § 2]). Let $e_j g = (s_i^j + N^{(\aleph_0)} \mid i < \aleph_0)$, where $e_j s_i^j = s_i^j$ for all $i, j < \aleph_0$. Let $\nu < \aleph_1$ be the smallest ordinal such that $e_j g \in S_\nu/N^{(\aleph_0)}$, for all $j < \aleph_0$. Define an $n = (n_i \mid i < \aleph_0) \in N^{\aleph_0}$ by $n_i = s_i^0 + \ldots + s_i^i$, $i < \aleph_0$. It is easy to see that $n \in S_{\nu+1}$ and $e_j g = e_j n + N^{(\aleph_0)}$ for all $j < \aleph_0$, whence $\mathrm{Ext}_R(R/I, \overline{N}) = 0$.

1.2. Lemma. *Let R be a simple regular ring. Let N_i, $i < \aleph_0$, be a sequence of modules such that N_i is a proper submodule of N_{i+1} for all $i < \aleph_0$. Put $N = \bigcup N_i$, $i < \aleph_0$, and let I be a countably infinitely generated left ideal of R. Then* $\mathrm{Ext}_R(R/I, N) \neq 0$.

Proof. We have $I = \dot{\sum} Re_i$, $i < \aleph_0$, where $\{e_i \mid i < \aleph_0\}$ is a set of pairwise orthogonal idempotents of R. Since R is simple, there is $n_i \in (e_i N_{i+1} - N_i)$, for each $i < \aleph_0$. Now, $g \in \mathrm{Hom}_R(I, N)$, defined by $e_i g = n_i$, is not a restriction of an element of $\mathrm{Hom}_R(R, N)$.

1.3. Proposition. *Let R be a regular left T-ring. If $N \in R$-mod, then $I(N)/N$ has a socle sequence of length $\sigma \leqq \aleph_1$, where either $\sigma = \aleph_1$ or σ is non-limit. Hence, if $M, N \in R$-mod and N is essential in M, then M/N has a socle sequence of length $\leqq \aleph_1$.*

Proof. By [10, II.4], we can use 1.1 with A — the set of all maximal left ideals of R. With regard to 1.2, there is an ordinal $\sigma \leqq \aleph_1$ such that either $\sigma = \aleph_1$ or σ is non-limit, and $S_\nu + N^{(\aleph_0)}/S_0 + N^{(\aleph_0)}$, $\nu \leqq \sigma$ is a socle sequence of $\overline{N}/(N)f$. The rest is clear.

1.4. Lemma. *Let R be a left primitive ring, J a simple faithful module and $K = \mathrm{End}_R(J)$. Then R is a dense subring of $\mathrm{End}_K(J)$ and the following conditions are equivalent:*

(i) *all simple modules are isomorphic,*

(ii) $\dim_K(\bigcap \mathrm{Ker}\, s,\ s \in I) = 1$, *for each maximal left ideal I of R.*

Proof. The density of R is well-known. Assume (i) and let I be a maximal left ideal of R. There is a $j \in J$ with $\mathrm{Ann}_R(j) = I$, i.e. $jK \subseteq \bigcap \mathrm{Ker}\, s$, $s \in I$. By the density, for each $k \in (J - jK)$ there is an $r \in R$ with $rk \neq 0$ and $rj = 0$, whence $k \notin \bigcap \mathrm{Ker}\, s$,

$s \in I$. Assume (ii). Let I and L be maximal left ideals of R and $jK = \bigcap \operatorname{Ker} s$, $s \in I$; $kK = \bigcap \operatorname{Ker} s$, $s \in L$. By the density, there is an $r \in R$ with $rk = j$. Hence, $r \notin L$ and $Ir \subseteq L$, and $\operatorname{Hom}_R(R/I, R/L) \neq 0$.

1.5. Theorem. *Let R be a simple regular ring such that R is not completely reducible. Let J be a simple module and $K = \operatorname{End}_R(J)$. Assume* $\dim_K(J) < 2^{\aleph_0}$ (this holds e.g. if card $R < 2^{\aleph_0}$). *Then there are a non-projective cyclic module M and a non-injective module N such that* $\operatorname{Ext}_R(M, N) = 0$.

Proof. We prove the theorem in two steps.

Step I. Let $2 = \{0, 1\}$ and for $x \in 2^{(\aleph_0)}$, $x = (x_0, \ldots, x_n)$ put $\ln(x) = n$. For $x_i \in 2$ denote by x_i' the binary complement of x_i. By induction, we define a set $\{e_x \mid x \in 2^{(\aleph_0)}\}$ such that

(i) for each $n < \aleph_0$, $\{e_x \mid x \in 2^{(\aleph_0)} \ \& \ \ln(x) = n\}$ is a complete set of pairwise orthogonal idempotents of R;

(ii) if $x, y, z \in 2^{(\aleph_0)}$, $x = (x_0, \ldots, x_n)$, $y = (x_0, \ldots, x_n, 0)$, $z = (x_0, \ldots, x_n, 1)$, then $e_y + e_z = e_x$.

Put $e_0 = e$, $e_1 = 1 - e$, where $e \in R$, $e^2 = e \notin \{0, 1\}$. Then (i) holds for $n = 0$. Assume e_x are defined for all $x \in 2^{(\aleph_0)}$ with $\ln(x) \leqq m$ and (i) holds for all $n \leqq m$ and (ii) for all $n \leqq m - 1$. Let $x, y, z \in 2^{(\aleph_0)}$, $x = (x_0, \ldots, x_m)$, $y = (x_0, \ldots, x_m, 0)$, $z = (x_0, \ldots, x_m, 1)$. Since R is simple, the rings R and e_xRe_x are Morita equivalent and there are orthogonal idempotents $e_y, e_z \in e_xRe_x$ with $e_y + e_z = e_x$ and $e_y \neq e_x \neq e_z$. Then (i) holds for $m \leqq n + 1$ and (ii) for $n \leqq m$. Further, for $u \in 2^{\aleph_0}$, $u = (u_i \mid i < \aleph_0)$ put $w_0 = u_0'$ and $w_{n+1} = (u_0, \ldots, u_n, u_{n+1}')$, $n < \aleph_0$. Let I_u be a maximal left ideal of R containing the set $\{e_{w_n} \mid n < \aleph_0\}$. If $u^0, \ldots, u^m$ are different elements of $2^{\aleph_0}$, let $i < \aleph_0$ be the smallest index such that for all $0 < k \leqq m$ there is a $j \leqq i$ with $u_j^0 \neq u_j^k$. By (i) and (ii), we have $(e_{w_0^0} + \ldots + e_{w_i^0}) \in I_u0$, and for all $0 < k \leqq m$, $1 \in ((e_{w_0^0} + \ldots + e_{w_i^0}) + I_uk)$.

Step II. Assume that, for each cyclic non-projective module M and each non-injective module N, $\operatorname{Ext}_R(M, N) \neq 0$. In particular, $\operatorname{Ext}_R(S, N) \neq 0$ and $\operatorname{Hom}_R(S, I(N)/N) \neq 0$ for each simple module S. Hence, $I(N)/N$ has a socle sequence with factors isomorphic to direct powers of S. Thus, all simple modules are isomorphic. By 1.4, for each $u \in 2^{\aleph_0}$ there is a $j_u \in J$ with $j_uK = \bigcap \operatorname{Ker} x$, $x \in I_u$. We shall show that $P = \{j_u \mid u \in 2^{\aleph_0}\}$ is an independent subset of the right K-module J. On the contrary, let $\{j_{u^0}, \ldots, j_{u^m}\}$ be a dependent subset of P with a smallest number of elements. We have $j_{u^0}k_0 + \ldots + j_{u^m}k_m = 0$ for some $0 \neq k_n \in K$, $n = 0, \ldots, m$. By Step I, $0 = (e_{w_0^0} + \ldots + e_{w_i^0})(j_{u^0}k_0 + \ldots + j_{u^m}k_m) = j_{u^1}k_1 + \ldots + j_{u^m}k_m$, a contradiction. Hence, $\dim_K(J) \geqq 2^{\aleph_0}$, a contradiction.

1.6. Theorem. *Let R be a ring of cardinality $< 2^{\aleph_0}$. Then the following conditions are equivalent:*

(i) *R is a left non-singular left T-ring;*

(ii) *either $R = S$ or $R = T$ or $R = S \boxplus T$, where S is a completely reducible* *ring of cardinality* $<2^{\aleph_0}$ *and there is a division ring* D *of cardinality* $<2^{\aleph_0}$ *such that* T *is Morita equivalent to the upper triangular matrix ring of degree* 2 *over* D.

Proof. By [9, 4.4 and 6.1], [10, II.4] and 1.5.

2. INDEPENDENCE FOR COUNTABLE MODULES

In this section, we use a combinatorial principle due to S. Shelah to prove independence of WP for countable modules over simple countable non-completely reducible regular rings (various examples of such rings can be found e.g. in [5]).

2.1. For $E \subseteq \aleph_1$ consider the assertion: (A_E) Let $(n_\nu \mid \nu \in E)$ be a sequence of strictly increasing $\aleph_0$-sequences such that for each limit $\nu \in E : \sup\limits_{i<\aleph_0} n_\nu(i) = \nu$. Let $(h_\nu \mid \nu \in E)$ be a sequence of functions from $\aleph_0$ to $\aleph_0$. Then there is a function $f\colon \aleph_1 \to \aleph_0$ such that for each limit $\nu \in E$: $\exists j < \aleph_0\ \forall i > j$: $(n_\nu(i))f = (i)\,h_\nu$.

Lemma. *If* ZFC *is consistent, then* ZFC + GCH + "$\exists E \subseteq \aleph_1$: *E stationary in* $\aleph_1$ & (A_E)" *is consistent.*

Proof. Let E be a stationary subset in $\aleph_1$ such that $\aleph_1 - E$ is stationary in $\aleph_1$, too. Take D – a normal $\aleph_1$-complete filter over $\aleph_1$ such that $(\aleph_1 - E) \in D$ – and use [8, 2.1].

2.2. Let R be a non-completely reducible regular ring. Let I be a countably infinitely generated left ideal of R. By [5, § 2], $I = \dot{\sum} Re_i$, $i < \aleph_0$, where e_i, $i < \aleph_0$ are pairwise orthogonal idempotents of R. Let E be a stationary subset in $\aleph_1$ and F the set of limit ordinals from E. Clearly, F is stationary in $\aleph_1$, too. Take a $\nu \in F$. Then either there is a strictly increasing sequence ν_i, $i < \aleph_0$ of limit ordinals less than ν with $\sup\limits_{i<\aleph_0} \nu_i = \nu$, or there is a limit ordinal $\mu < \nu$ with $\nu = \mu + \aleph_0$. In the former case, put $n_\nu(i) = \nu_i + i + 1$, $i < \aleph_0$ and in the latter put $n_\nu(i) = \mu + i + 1$, $i < \aleph_0$. Further, for $\alpha < \aleph_1$ denote by π_α the α-th canonical projection $R^{(\aleph_1)} \to R$. Now, for $\nu \in F$, denote by $g_{i\nu}$ the element of $R^{(\aleph_1)}$ with $\pi_{n_\nu(i)}(g_{i\nu}) = e_i$, $\pi_\nu(g_{i\nu}) = -e_i$, and $\pi_\alpha(g_{i\nu}) = 0$ otherwise. Let $M'_E = \sum Rg_{i\nu}$, $i < \aleph_0$, $\nu \in F$ and put $M_E = R^{(\aleph_1)}/M'_E$.

Theorem. M_E *is a strongly* $\aleph_1$*-free, non-projective module. Moreover,* (A_E) *implies* $\mathrm{Ext}_R(M_E, N) = 0$ *for each countable* $N \in R$-mod.

Proof. For $\alpha < \aleph_1$ let t_α be the element of $R^{(\aleph_1)}$ with $\pi_\alpha(t_\alpha) = 1$ and $\pi_\beta(t_\alpha) = 0$ otherwise. Put $M_0 = 0$ and for $0 < \mu < \aleph_1$ let $M_\mu = \sum R(t_\alpha + M'_E)$, $\alpha < \mu$. Hence, for each limit $\mu < \aleph_1$: $M_\mu = \bigcup M_\nu$, $\nu < \mu$. Further, for each $0 < \mu < \aleph_1$: $M_\mu = \dot{\sum} Rv_\alpha$, $\alpha < \mu$, where

(i) $v_\alpha = (1 - e_i)\,t_\alpha + M'_E$ and $Rv_\alpha \simeq R(1 - e_i)$ provided there are $i < \aleph_0$ and $\sigma \in F$, $\sigma < \mu$ with $\alpha = n_\sigma(i)$,

(ii) $v_\alpha = t_\alpha + M'_E$ and $Rv_\alpha \simeq R$ otherwise.

Hence, for each $\mu < \aleph_1$, M_μ is projective. Moreover, for $v < \mu < \aleph_1$, $M_\mu/M_v \simeq \simeq \dot{\sum} I_\alpha$, $v \leqq \alpha < \mu$, where

(i) $I_\alpha = R(1 - e_i)$ provided there are $i < \aleph_0$ and $\sigma \in F$, $\sigma < \mu$ with $\alpha = n_\sigma(i)$,

(ii) $I_\alpha = R/\dot{\sum}_{i \in A_\alpha} Re_i$ provided $\alpha \in F$, $v \leqq \alpha < \mu$ and $A_\alpha = \{i \mid n_\alpha(i) < v\}$,

(iii) $I_\alpha = R$ otherwise.

Now, if $v \notin F$, then for all μ with $v < \mu < \aleph_1$, all the sets A_α, $\alpha \in F$, $v \leqq \alpha < \mu$ are finite and hence M_μ/M_v is projective. Thus $M_E = \bigcup M_v$, $v < \aleph_1$ is strongly $\aleph_1$-free. On the other hand, if $v \in F$, then $M_{v+1}/M_v \simeq R/I$ is not projective. By [4, 5.1 and § 18], M_E is not projective. To prove the rest, let N be a countable module and $r: N \to \aleph_0$ an injective mapping. Let $p \in \mathrm{Hom}_R(M'_E, N)$. Assume (A_E). Then also (A_F), for $(n_v \mid v \in F)$ defined as above and for $h_v: \aleph_0 \to \aleph_0$ defined by (i) $h_v = (g_{iv})\, pr$, $i < \aleph_0$, $v \in F$. Note that $(g_{iv})\, pr \in (e_i N)\, r$ for all $i < \aleph_0$, $v \in F$. Hence, there is a function $f: \aleph_1 \to \aleph_0$ such that for each $v \in F$ there is a $j_v < \aleph_0$ with $n_v(i) fr^{-1} = (g_{iv})\, p$, for all $j_v < i < \aleph_0$. For each $\alpha \in F$ and each $i \leqq j_\alpha$ put $\delta_{i\alpha} = n_\alpha(i) fr^{-1}$ if there is a $\beta \in F$ such that $j_\beta < i$ and $n_\alpha(i) = n_\beta(i)$, and $\delta_{i\alpha} = 0$ otherwise. Define a $q \in \mathrm{Hom}_R(R^{(\aleph_1)}, N)$ by

(i) $t_\alpha q = (\alpha f)\, r^{-1}$ provided there are $v \in F$ and $i < \aleph_0$ such that $j_v < i$ and $\alpha = n_v(i)$,

(ii) $t_\alpha q = \sum_{i=0}^{j_\alpha} (\delta_{i\alpha} - (g_{i\alpha})\, p)$ provided $\alpha \in F$,

(iii) $t_\alpha q = 0$ otherwise.

Then, for each $i < \aleph_0$, $v \in F$, we have $(g_{iv})\, q = e_i(t_{n_v(i)} q - t_v q) = (g_{iv})\, p$, whence $\mathrm{Ext}_R(M_E, N) = 0$.

2.3. Theorem. *Assume* GCH + "$\exists E \subseteq \aleph_1$: *$E$ stationary in $\aleph_1$* & (A_E)". *Let R be a simple countable non-completely reducible regular ring. Then no non-zero countably generated module has* WP.

Proof. By 2.2 and [10, III.2 and III.4].

2.4. Theorem. *Let R be a simple countable non-completely reducible regular ring. Then the assertion "every countably generated module has* WP" *is independent of* ZFC.

Proof. By [10, III.6] (or by [10, III.4] and [4, 21.6]), the assertion holds if $V = L$ is assumed. The rest follows from 2.1 and 2.3.

3. ARTIN'S PROBLEM AND WP

Recently (see [6] and [3]), Artin's problem for skew field extensions has been solved: for each pair of cardinals (α, β) with $\alpha > 1$, $\beta > 1$, there are division rings S and T such that T is a subring of S, the left dimension of S over T is α and the right dimension is β. Here, in 3.2, we use this fact to construct a matrix ring R such that R is not a left T-ring, but each cyclic module has WP. Our result was announced in [9, 5.4].

Let m be a natural number, $m \geqq 1$, $n = m + 1$, and let S, T be division rings such that T is a subring of $M_{m \times m}(S)$. If κ is a cardinal, $\kappa \geqq 1$, we shall shortly write M_κ and M_κ^+ instead of $\mathrm{RFM}_{m \times \kappa}(S)$ and $\mathrm{RFM}_{n \times \kappa}(S)$, respectively. Note that M_κ (M_κ^+) is a left M_m (M_n^+, respectively)-module. For a matrix $a \in M_\kappa^+$, let $a' \in M_\kappa$ be such that $a'_{ij} = a_{i+1,j+1}$ for all $0 \leqq i, j < m$. Let $R = U(m, S, T)$ be the subring of M_n^+ formed by the set of matrices $a \in M_n^+$ with $a_{10} = \ldots = a_{m0} = 0$ and $a' \in T$. Let $e \in R$ be such that $e_{00} = 1$ and $e_{ij} = 0$ otherwise and put $f = 1 - e$. It is easy to see that $\{e, f\}$ is a basic set of primitive idempotents of R, whence R is a basic ring. Further properties of R can be found e.g. in [9, 5.1].

If κ is a cardinal, $\kappa \geqq 1$ and X (Y) is a subset of M_m (M_κ, respectively), we put

$$X \,.\, Y = \{ \sum_{i=0}^{k} x_i y_i \mid k < \aleph_0,\ x_i \in X,\ y_i \in Y \text{ for all } i = 0, \ldots, k\} .$$

3.1. Lemma. *Let κ be a cardinal, $\kappa \geqq 1$. Then the following conditions are equivalent*:

(i) *there are a non-projective module M and a non-injective module N such that* $\dim(\mathrm{Soc}(N)) = \kappa$ *and* $\mathrm{Ext}_R(M, N) = 0$,

(ii) *there are a finitely generated right T-submodule X of M_m and a proper left T-submodule Y of M_κ such that $X \,.\, Y = M_\kappa$.*

Proof. Denote by A the module $R/\mathrm{Soc}(R)$. Let N be a non-injective module. Using [9, 5.1], it is easy to see that $I(N)/\mathrm{Soc}(N)$, and thus $I(N)/N$, is a direct sum of copies of A. Further, if M is any module, then by [9, 5.1.(i)], there is a projective cover (P, p) of M. By [1, 28.13], there are cardinals $\alpha, \beta, \gamma, \delta$ such that $P = (Re)^{(\alpha)} \dotplus (Rf)^{(\beta)}$ and $\mathrm{Ker}\, p \simeq (Re)^{(\gamma)} \dotplus (Rf)^{(\delta)}$. Since $\mathrm{Ker}\, p$ is superfluous in P, we have $\delta = 0$ and $\mathrm{Ker}\, p \subseteq (Rf)^{(\beta)}$.

Assume (i). Let $x \in \mathrm{Ker}\, p$ be such that Rx is a direct summand of $\mathrm{Ker}\, p$ and $\mathrm{Ann}_R(x) = Rf$. Since $\mathrm{Ext}_R(P/\mathrm{Ker}\, p, N) = 0$, we have $\mathrm{Ext}_R(P/Rx, N) = 0$. Let q be the smallest natural number such that $x \in (Rf)^{(q)}$, i.e. $x = (x_0, \ldots, x_{q-1})$, where $0 \neq x_k \in \mathrm{Soc}(Rf)$ for all $k < q$. Put $G = (Rf)^{(q)}/Rx$. Then G is not projective and $\mathrm{Ext}_R(G, N) = 0$. By [9, 5.1.(ii)], we may assume that $\mathrm{Hom}_R(A, N) = 0$. Hence, by [9, 5.1], we have $I(N) = M_\kappa^+$ and

$$\mathrm{Soc}(N) = \mathrm{Soc}(M_\kappa^+) =$$
$$= \{a \in M_\kappa^+ \mid a_{ij} = 0 \text{ for all } 0 < i < m \text{ and } 0 \leqq j < \kappa\} .$$

Now, put $Y = \{a' \mid a \in N\}$. By [9, 5.1.(vi)], Y is a proper left T-submodule of M_κ. Further, for $0 \leqq i < m$ and $0 \leqq k < q$, let $z_k^i \in M_m$ be such that $(z_k^i)_{ij} = (x_k)_{0,j+1}$ for all $0 \leqq j < m$ and $(z_k^i)_{cj} = 0$ otherwise. Let X be the right T-submodule of M_m generated by $\{z_k^i \mid 0 \leqq i < m \text{ and } 0 \leqq k < q\}$. We shall prove that $X \,.\, Y = M_\kappa$.

Take $u \in M_\kappa$ and let u_i be the i-th row of u, hence $u_i \in S^{(\kappa)}$ for all $0 \leqq i < m$. Clearly, for each $0 \leqq i < m$, there are $v_k^i \in M_\kappa$, $0 \leqq k < q$, such that $\sum_{k=0}^{q-1} x_k v_k^i = u_i$. Let $w_k^i \in M_\kappa^+$ be such that $(w_k^i)' = v_k^i$, $0 \leqq i < m$ and $0 \leqq k < q$. Since $\sum_{k=0}^{q-1} x_k w_k^i \in \mathrm{Soc}(N)$ and $\mathrm{Ext}_R(G, N) = 0$, there are $t_k^i \in M_\kappa^+$, $0 \leqq i < m$ and $0 \leqq k < q$, with $\sum_{k=0}^{q-1} x_k t_k^i = 0$ and $t_k^i + N = w_k^i + N$ for all $0 \leqq i < m$ and $0 \leqq k < q$. Now, put $y_k^i = (w_k^i - t_k^i)'$, $0 \leqq i < m$ and $0 \leqq k < q$. Then $y_k^i \in Y$, for all $0 \leqq i < m$ and $0 \leqq k < q$, and $\sum_{k=0}^{q-1} x_k y_k^i = u_i$, for all $0 \leqq i < m$, whence $\sum_{i=0}^{m-1} \sum_{k=0}^{q-1} z_k^i y_k^i = u$.

Assume (ii). Let N be a submodule of M_κ^+ such that $\mathrm{Soc}(N) = \{a \in M_\kappa^+ \mid a_{ij} = 0$ for all $0 < i < m$ and $0 \leqq j < \kappa\}$ and $Y = \{a' \mid a \in N\}$. Clearly, N is not injective and $I(N) = M_\kappa^+$. Since $\mathrm{Soc}(N) = \mathrm{Soc}(M_\kappa^+)$, [9, 5.1] implies $\dim(\mathrm{Soc}(N)) = \kappa$. Let $\{z_k \mid 0 \leqq k < q\}$ be a finite set of generators of the right T-module X. For each $0 \leqq k < q$, let $x_k \in \mathrm{Soc}(Rf)$ be such that the 0-th row of x_k equals the 0-th row of z_k. Then $\sum_{k=0}^{q-1} x_k N = \mathrm{Soc}(N)$. Let $x = (x_0, \dots, x_{q-1}) \in \sum_{k=0}^{q-1} Rf_k$, where $f_k = f$ for all $0 \leqq k < q$, and put $M = \sum_{k=0}^{q-1} Rf_k/Rx$. We shall prove that $\mathrm{Ext}_R(M, N) = 0$. Take $g \in \mathrm{Hom}_R(M, I(N)/N)$. Then $(f_k + Rx)\, g = u_k + N$, for all $0 \leqq k < q$, where $u_k \in M_\kappa^+$, $0 \leqq k < q$. Since $\sum_{k=0}^{q-1} x_k u_k \in \mathrm{Soc}(N)$, there exist $n_k \in N$, $0 \leqq k < q$, such that $\sum_{k=0}^{q-1} x_k (u_k - n_k) = 0$. Hence, if $h \in \mathrm{Hom}_R(M, I(N))$ is defined by $(f_k + Rx)\, h = u_k - n_k$, $0 \leqq k < q$, then $g = h\pi$, where $\pi: I(N) \to I(N)/N$ is the canonical projection, whence $\mathrm{Ext}_R(M, N) = 0$.

3.2. Theorem. *Let S, T be division rings such that T is a subring of S, the left dimension of S over T is two and the right dimension is infinite. Let $R = U(1, S, T)$. Then $\mathrm{Ext}_R(M, N) \neq 0$ for each non-projective module M and each cyclic non-injective module N, but R is not a left T-ring.*

Proof. By [9, 5.3], R is not a left T-ring (in fact, the proof of [9, 5.3] shows that there are a non-projective 2-generated module M and a non-injective module N such that $\mathrm{Ext}_R(M, N) = 0$). Further, for $\kappa = 1$, we have $M_\kappa = S$ and hence $X \,.\, Y \neq S$, for any finitely generated right T-submodule X of S and any proper left T-submodule Y of S. Now, it is easy to see that each cyclic module is a direct sum of modules N with $\dim(\mathrm{Soc}(N)) = 1$, and it suffices to apply 3.1.

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