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PERIODIC SOLUTIONS TO MAXWELL EQUATIONS IN NONLINEAR MEDIA

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INTRODUCTION

Recently, much attention has been paid to the problem of global or periodic solvability of nonlinear hyperbolic equations. Existence results have been obtained by means of different modifications of Nash's "hard implicit function theorem" technique, provided the data (i.e. the right hand side, initial conditions, if any, etc.) are sufficiently smooth and sufficiently small. From a more or less general point of view, iterative methods in situations when a "loss of derivatives" occurs have been developed in [1], [3], [4], [7], [8], [10], [13], [15], [16], [17], [19], [20], [22], [23], [24], [25], [26] and many others.

In this paper we deal with the existence of classical time-periodic solutions to Maxwell equations in nonlinear media using the ideas of Klainerman [8] and Shibata [25] (let us remark that the local existence of solutions to Maxwell equations in materials of ferromagnetic type has been proved by Milani [14]). As is expected, we assume the right hand side to be sufficiently small. On the other hand, we impose no restrictions concerning the smallness of the nonlinearities and we try to minimize the requirements on the smoothness of data.

The present paper is an elaboration of the author's thesis [9] on small periodic solutions to Maxwell equations in ferromagnetic media. The author is very indebted to Prof. O. Vejvoda for useful suggestions and encouragement.

1. MAXWELL EQUATIONS

Let $N \geqq 5$ be a given integer and let $\Omega \subset \mathbb{R}^3$ be an open bounded domain homeomorphic to a ball in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is of class C^{M+2}, $M = 2N + 1$. We consider the Maxwell equations in the usual form

$$(1.1)\quad \begin{aligned} &\text{(i)}\ \partial_t D - \operatorname{rot} H + J = Q,\\ &\text{(ii)}\ \partial_t B + \operatorname{rot} E = 0,\\ &\text{(iii)}\ \operatorname{div} D = q,\\ &\text{(iv)}\ \operatorname{div} B = 0,\end{aligned}$$

where $Q(t, x)$ is a given vector function defined in $\mathbb{R}^1 \times \Omega$ which is ω-periodic with respect to t, $\omega > 0$ is a given number, the other quantities in (1.1) are to be found. For the physical meaning of these quantities and equations we refer the reader e.g. to [5], [21], [27].

The domain Ω represents a conductive medium which is characterized by the material relations

(1.2) (i) $D = \varepsilon(E)$,
(ii) $J = \sigma(E)$,
(iii) $H = \mu(B)$,

where ε, σ, μ are given vector functions of class C^{M+2} in a neighbourhood of 0. The material is assumed to be "almost isotropic" (details are given in Section 3). On the boundary of Ω we impose the conditions

(1.3) (i) $(B, \nu) = 0$,
(ii) $E \wedge \nu = 0$,

where ν is a normal vector to $\partial\Omega$ and $(\cdot, \cdot)$ denotes the scalar product, $\wedge$ the vector product in $\mathbb{R}^3$.

(1.4) Remark. The relation between B and H is usually considered in the form $B = \tilde{\mu}(H)$. For the sake of simplicity we suppose (1.2) (iii) instead of the invertibility of $\tilde{\mu}$ in a neighbourhood of 0.

2. NOTATION, FUNCTION SPACES AND REGULARIZATION

Throughout the paper, we denote all constants (with several exceptions) whose values depend essentially only on quantities $a, b, \ldots$ by $c_{a,b,\ldots}$. Especially, c_L denotes any constant depending essentially only on L.

For a Banach space V endowed with a norm $|\cdot|_V$ we denote by $L^p(\omega, V)$ the space of ω-periodic measurable functions $u: \mathbb{R}^1 \to V$ with the norm

$$|u|_{\omega,p,V} = \left(\int_\omega |u(t)|_V^p \, dt\right)^{1/p} < \infty, \quad 1 \leqq p < \infty,$$

where $\int_\omega$ denotes the integration over any interval $[t_0, t_0 + \omega]$.

Let $W^{p,L}(\Omega)$, $1 \leqq p < \infty$, $L \geqq 0$ be the usual Sobolev space of functions $u: \Omega \to \mathbb{R}^1$. For $L \geqq 1$ we denote

$$\mathring{W}^{p,L}(\Omega) = \{u \in W^{p,L}(\Omega) \mid u = 0 \text{ on } \partial\Omega\}.$$

For simplicity, we consider only L integer.

We denote by

$$G^L = \{u \in (W^{2,L}(\Omega))^3 \mid u = \operatorname{grad} \varphi, \ \varphi \in \mathring{W}^{2,L+1}(\Omega)\},$$

$$S^L = \left\{u \in (W^{2,L}(\Omega))^3 \mid \forall v \in G^L, \ \int_\Omega (u, v) \, dx = 0\right\}$$

the complementary closed subspaces of $(W^{2,L}(\Omega))^3$ of potential and divergence-free vector functions, respectively.

Further, for $L \geqq 1$ put

$$S_\tau^L = \{u \in S^L \mid u \wedge \nu = 0 \text{ on } \partial\Omega\},$$
$$S_\nu^L = \{u \in S^L \mid (u, \nu) = 0 \text{ on } \partial\Omega\}$$

and for $L \geqq 2$,

$$G_0^L = \{u \in G^L \mid \operatorname{div} u = 0 \text{ on } \partial\Omega\}.$$

Following Bykhovsky [2] we define equivalent norms in S_τ^L, G_0^L by

$$|u|_{S_\tau L} = |\operatorname{rot} u|_{(W^{2,L-1}(\Omega))^3},$$
$$|u|_{G_0 L} = |\operatorname{grad} \operatorname{div} u|_{(W^{2,L-2}(\Omega))^3}.$$

For time-dependent ω-periodic functions we introduce the Banach spaces $Z^L = L^2(\omega, W^{2,L}(\Omega))$, $L \geqq 0$, with a norm $|\cdot|_{Z,L}$. The norm of an element $u = (u_1, u_2, u_3) \in (Z^L)^3 = L^2(\omega, (W^{2,L}(\Omega))^3)$ is $|u|_{Z^3,L} = (\sum_{i=1}^{3} |u_i|^2_{Z,L})^{1/2}$.

For $L \geqq 1$ we put

$$Y^L = \{\Phi \in L^2(\omega, \mathring{W}^{2,L}(\Omega)) \mid \partial_t \Phi \in L^2(\omega, \mathring{W}^{2,L}(\Omega))\}$$

with the norm

$$|\Phi|_{Y,L} = |\Phi|_{Z,L} + |\partial_t \Phi|_{Z,L}.$$

Further,

$$\mathscr{S}^L = \{u \in L^2(\omega, S^L) \mid 0 \leqq K \leqq L \Rightarrow \partial_t^K u \in L^2(\omega, S^{L-K})\}, \quad L \geqq 0,$$
$$\mathscr{G}^0 = L^2(\omega, G^0),$$
$$\mathscr{G}^1 = \{u \in L^2(\omega, G^2) \mid \partial_t u \in L^2(\omega, G^0)\},$$
$$\mathscr{G}^L = \{u \in L^2(\omega, G^L) \mid 1 \leqq K \leqq L-1 \Rightarrow \partial_t^K u \in L^2(\omega, G^{L-K+1}),$$
$$\partial_t^L u \in L^2(\omega, G^0)\}, \quad L \geqq 2,$$
$$\mathscr{S}_\tau^L = \{u \in \mathscr{S}^L \mid 0 \leqq K \leqq L-1 \Rightarrow \partial_t^K u \in L^2(\omega, S_\tau^{L-K})\}, \quad L \geqq 1,$$
$$\mathscr{G}_0^1 = \mathscr{G}^1 \cap L^2(\omega, G_0^2),$$
$$\mathscr{G}_0^L = \{u \in \mathscr{G}^L \cap L^2(\omega, G_0^L) \mid 1 \leqq K \leqq L-1 \Rightarrow \partial_t^K u \in L^2(\omega, G_0^{L-K+1})\}, \quad L \geqq 2.$$

The norms in $\mathscr{S}_\tau^L$, $\mathscr{G}_0^L$ are chosen, respectively, to be

$$|u|^2_{\mathscr{S}_\tau L} = |\partial_t^L u|^2_{Z^3,0} + \sum_{K=0}^{L-1} |\operatorname{rot} \partial_t^K u|^2_{Z^3,L-K-1}, \quad L \geqq 1, \quad \text{and}$$

$$|u|^2_{\mathscr{G}_0 1} = |\partial_t u|^2_{Z^3,0} + |\operatorname{grad} \operatorname{div} u|^2_{Z^3,0},$$
$$|u|^2_{\mathscr{G}_0 L} = |\partial_t^L u|^2_{Z^3,0} + |\operatorname{grad} \operatorname{div} u|^2_{Z^3,L-2} + \sum_{K=1}^{L-1} |\operatorname{grad} \operatorname{div} \partial_t^K u|^2_{Z^3,L-K-1}, \quad L \geqq 2.$$

In fact, the following well-known assertions hold (cf. [2]).

(2.1) **Lemma.** *There exist constants $\varkappa$, ϱ, ϑ such that for $0 \leqq L \leqq M$ and*
(i) *for each $\Phi \in L^2(\omega, \mathring{W}^{2,L+2}(\Omega))$ we have*

$$|\Phi|_{Z,L+2} \leqq \varkappa |\Delta \Phi|_{Z,L};$$

(ii) *for each* $u \in L^2(\omega, S_\tau^{L+1})$ *or* $u \in L^2(\omega, S_\nu^{L+1})$ *we have*

$$|u|_{Z^3,L+1} \leqq \varrho |\mathrm{rot}\, u|_{Z^3,L}\,;$$

(iii) *for each* $u \in L^2(\omega, S_\tau^{L+1} \oplus G^{L+1})$ *we have*

$$|u|_{Z^3,L+1} \leqq \frac{\vartheta}{2}\left(|\mathrm{div}\, u|_{Z,L} + |\mathrm{rot}\, u|_{Z^3,L}\right), \quad \textit{where} \quad \frac{\vartheta}{2} = \max(\varkappa, \varrho)\,.$$

For $1 \leqq p < \infty$ and $L \geqq 0$ we denote

$$W_\omega^{p,L}(\Omega) = \{u \in L^p(\omega, W^{p,L}(\Omega)) \mid 0 \leqq K \leqq L \Rightarrow \partial_t^K u \in L^p(\omega, W^{p,L-K}(\Omega))\}$$

with the norm

$$|u|_{\omega,p,L} = \left(\sum_{K=0}^{L} |\partial_t^K u|^p_{\omega,p,W^{p,L-K}(\Omega)}\right)^{1/p},$$

and

$$W^L = (W_\omega^{2,L}(\Omega))^3\,, \quad L \geqq 0\,,$$

$$X^L = \mathscr{G}_0^L \oplus \mathscr{S}_\tau^L\,, \quad L \geqq 1$$

with the norms, respectively,

$$|u|^2_{W,L} = \sum_{K=0}^{L} |\partial_t^K u|^2_{Z^3,L-K}\,, \qquad L \geqq 0\,,$$

$$|u|^2_{X,L} = |\partial_t^L u|_{W,0} + |\mathrm{grad\, div}\, u|^2_{Z^3,L-2} + $$
$$+ \sum_{K=1}^{L-1} |\mathrm{grad\, div}\, \partial_t^K u|^2_{Z^3,L-K-1} + \sum_{K=0}^{L-1} |\mathrm{rot}\, \partial_t^K u|^2_{Z^3,L-K-1}\,, \quad L \geqq 2\,,$$

and

$$|u|^2_{X,1} = |\partial_t u|^2_{W,0} + |\mathrm{grad\, div}\, u|^2_{W,0} + |\mathrm{rot}\, u|^2_{W,0}\,.$$

Obviously, X^L is continuously embedded into W^L for $L \geqq 1$.

Further, let $C_\omega^L(\bar{\Omega})$ be the space of all functions u: $\mathbb{R}^1 \times \bar{\Omega} \to \mathbb{R}^1$ continuously differentiable in $\mathbb{R}^1 \times \bar{\Omega}$ up to the order L, ω-periodic with respect to t, with the norm

$$|u|_{\omega,\infty,L} = \sup\{|\partial^\lambda u(t,x)|,\ t \in \mathbb{R}^1,\ x \in \bar{\Omega},\ \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3),\ |\lambda| \leqq L\}\,,$$
$$\text{where} \quad \partial^\lambda = \partial_t^{\lambda_0}\, \partial_{x_1}^{\lambda_1}\, \partial_{x_2}^{\lambda_2}\, \partial_{x_3}^{\lambda_3}\,.$$

Let $\mathfrak{M}^L$ denote the space of all (3×3) matrices

$$A(t,x) = \{A_{ij}(t,x)\}_{i,j=1}^3\,, \quad A_{ij} = A_{ji} \in C_\omega^L(\bar{\Omega})\,,$$

with the norm

$$\|A\|_L^2 = \sum_{i,j=1}^{3} |A_{ij}|^2_{\omega,\infty,L}\,.$$

For $\delta_0 > 0$ put

$$U(\delta_0) = \{z \in \mathbb{R}^3 \mid |z| = \left(\sum_{i=1}^{3} |z_i|^2\right)^{1/2} < \delta_0\}\,,$$

and let $C_{\delta_0}^L = C^L(\overline{U(\delta_0)})$ be the space of all functions ξ: $\overline{U(\delta_0)} \to \mathbb{R}^1$ continuously differentiable in $\overline{U(\delta_0)}$ up to the order L with the norm

$$|\xi|_{\delta_0,L} = \sup\{|\partial^\lambda \xi(z)|,\ z \in U(\delta_0),\ \lambda = (\lambda_1, \lambda_2, \lambda_3),\ |\lambda| \leqq L\}\,.$$

Now, we state without proof two lemmas. The first is an easy consequence of the Nirenberg inequality (cf. [18]), the second is a slight modification of Moser's lemma the proof of which can be found e.g. in [16] or [25].

(2.2) **Lemma** (Nirenberg).

(i) *There exist constants* $c_{p,L}$, $1 \leqq p \leqq \infty$, $L \geqq 0$ *such that the inequality*

$$|u|_{\omega,p,K} \leqq c_{p,L} |u|^{K/L}_{\omega,p,L} \, |u|^{1-K/L}_{\omega,p,0}$$

holds for every sufficiently smooth function u and $0 \leqq K \leqq L$.

(ii) *Let* $0 \leqq J \leqq K \leqq I \leqq L$, $1 \leqq p$, $q \leqq \infty$. *Then there exist constants* $c_{p,q,L}$ *such that the inequality*

$$|u|_{\omega,p,K} \, |v|_{\omega,q,L-K} \leqq c_{p,q,L}\big(|u|_{\omega,p,J} \, |v|_{\omega,q,L-J} + |u|_{\omega,p,I} \, |v|_{\omega,q,L-I}\big)$$

holds for every sufficiently smooth functions u, v.

(2.3) Remark. The assertion of Lemma (2.2) remains valid if any of the norms $|\cdot|_{\omega,p,L}$, $|\cdot|_{\omega,q,L}$ is replaced by $|\cdot|_{W^{p,L}(\Omega)}$. $|\cdot|_{Z,L}$, $|\cdot|_{Z^3,L}$, $|\cdot|_{W,L}$, or $\|\cdot\|_L$.

(2.4) **Lemma** (Moser). *For* $\delta_0 > 0$ *and* $L \geqq 0$ *there exists a constant* $c_{\delta_0,L}$ *such that for every* $\xi \in C^L_{\delta_0}$ *and* $u \in (C^L_\omega(\bar\Omega))^3$, $|u|_{(C_\omega{}^0(\bar\Omega))^3} \leqq \delta_0$ *we have* $\xi(u) \in C^L_\omega(\bar\Omega)$ *and* $|\xi(u)|_{\omega,\infty,L} \leqq c_{\delta_0,L} |\xi|_{\delta_0,L}(1 + |u|_{(C_\omega{}^L(\bar\Omega))^3})$.

(2.5) **Corollary.** *For* $\xi = (\xi_1, \xi_2, \xi_3) \in (C^L_{\delta_0})^3$ *let* $D\,\xi(z)$ *denote the Jacobi matrix of* ξ *at the point* $z \in U(\delta_0)$, *i.e.* $(D\,\xi(z))_{i,j} = \partial_{z_j}\xi_i(z)$, *and let* $D^2\,\xi(z)(r,s)$ *denote the second Gâteaux derivative of* ξ *at the point* $z \in U(\delta_0)$ *in the directions r, s, i.e.* $D^2\,\xi(z)(r,s) = \sum_{j,k=1}^{3} \partial_{z_j}\partial_{z_k}\xi(z)\, r_j s_k$. *Put* $|||\xi|||_L = \sum_{i=1}^{3} |\xi_i|_{\delta_0,L}$. *Then for all* $u \in (C^L_\omega(\bar\Omega))^3$, $|u|_{(C_\omega{}^0(\bar\Omega))^3} < \delta_0$, $v, w \in (W^{4,L}_\omega(\Omega))^3$ *we have*

(i) $\|D\,\xi(u)\|_L \leqq c_{\delta_0,L} |||\xi|||_{L+1}(1 + |u|_{(C_\omega{}^L(\bar\Omega))^3})$,

(ii) $|D^2\,\xi(u)(v,w)|_{W,L} \leqq c_{\delta_0,L} |||\xi|||_{L+2} \cdot$

$$\cdot \sum_{\substack{\lambda=(\lambda_1,\lambda_2,\lambda_3)\\ |\lambda|=L}} (1 + |u|_{(C_\omega{}^{\lambda_1}(\bar\Omega))^3}) \, |v|_{(W_\omega{}^{4,\lambda_2}(\Omega))^3} |w|_{(W_\omega{}^{4,\lambda_3}(\Omega))^3}.$$

Next, for $M = 2N + 1$, $N \geqq 5$ we introduce the smoothening operators following [7], [8], [17], [25]. Let $P_{p,M}: (W^{p,L}_\omega(\Omega))^3 \to (W^{p,L}_\omega(R^3))^3$ be the continuous linear prolongation operators, $1 \leqq p \leqq \infty$, $0 \leqq L \leqq M + 1$ (a simple construction employing the partition of unity in a neighbourhood of $\partial\Omega$ is presented in [9] following the idea of Hestenes [6]; here, it is omitted). Further, let $\varphi_M \in C^\infty(\mathbb{R}^1)$ be a function with its support in $]-1, 1[$ such that (cf. [25])

(2.6) (i) $\int_{-\infty}^{\infty} \varphi_M(s)\,\mathrm{d}s = 1$,

(ii) $\int_{-\infty}^{\infty} s^k\,\varphi_M(s)\,\mathrm{d}s = 0$, $k = 1, 2, \ldots, M$.

Let $r > 1$ be a fixed real number. For $u \in W^L$ and $n \geqq 0$ we put

$$(2.7)\quad (S_n u)(t,x) = \int_{\mathbb{R}^4} r^{4n} \varphi_M(r^n(t-s)) \prod_{j=1}^{3} \varphi_M(r^n(x_j - y_j))\,(P_{2,M}u)(s,y)\,\mathrm{d}y\,\mathrm{d}s\,.$$

Using (2.2), (2.3), (2.6), (2.7) we can directly check that the following proposition holds.

(2.8) **Proposition.** *For each* $L \geqq 0$, $K \geqq 0$, $u \in W^K$ *we have* $S_n u \in W^L$ *and there exist constants* c_L *such that*

(i) $|S_n u|_{W,L} \leqq c_L\, r^{(L-K)n} |u|_{W,K}$, $L \geqq K$, $n \geqq 0$,

(ii) $|(I - S_n)\, u|_{W,L} \leqq c_L\, r^{(L-K)n} |u|_{W,K}$, $L \leqq K \leqq M + 1$, $n \geqq 0$,

where I *is the identity mapping.*

3. MAIN THEOREM

(3.1) **Theorem.** *Let* $N \geqq 5$, $\delta_0 > 0$ *be given and put* $M = 2N + 1$. *Let* $\varepsilon, \mu, \sigma \in (C^{M+2}_{\sigma_0})^3$ *be such that the matrices* $D\varepsilon, D\mu, D\sigma$ *are symmetric and that there exist functions* $\varepsilon_0, \mu_0, \sigma_0 \in C^{M+1}_{\delta_0}$ *satisfying*

(i) $\inf\{\varepsilon_0(z),\ z \in U(\delta_0)\} \geqq \alpha_0 > 0$,

$\inf\{\mu_0(z),\ z \in U(\delta_0)\} \geqq \beta_0 > 0$,

$\inf\{\sigma_0(z),\ z \in U(\delta_0)\} \geqq \zeta_0 > 0$,

(ii) $\sup\{\|D\,\varepsilon(z) - \varepsilon_0(z)\,.\,I\|_0,\ z \in U(\delta_0)\} \leqq \min\left\{\dfrac{\alpha_0}{2\varkappa}, \dfrac{\alpha_0}{2}\right\}$,

$\sup\{\|D\,\mu(z) - \mu_0(z)\,.\,I\|_0,\ z \in U(\delta_0)\} \leqq \min\left\{\dfrac{\beta_0}{8}, \dfrac{\beta_0}{4\varrho}\right\}$,

$\sup\{\|D\,\sigma(z) - \sigma_0(z)\,.\,I\|_0,\ z \in U(\delta_0)\} \leqq \min\left\{\dfrac{\zeta_0}{4}, \dfrac{\zeta_0}{2\vartheta}, \dfrac{\beta_0}{4\vartheta}, \dfrac{\zeta_0}{4\varkappa}\right\}$,

where $\varkappa, \varrho, \vartheta$ *are defined in* (2.1) *and* I *is the* 3×3 *identity matrix.*

Then there exists $\delta_N > 0$ *such that for each* $Q \in W^M$, $|Q|_{W,M} < \delta_N$, *there exists at least one solution* $u \in X^{N+1}$ *to the equation*

$$\partial_t(\varepsilon(-\partial_t u + \operatorname{grad}\operatorname{div} u)) - \operatorname{rot}(\mu(\operatorname{rot} u)) + \sigma(-\partial_t u + \operatorname{grad}\operatorname{div} u) = Q\,. \tag{3.2}$$

(3.3) Remarks.

(i) By embedding theorems we have $u \in (C^3_\omega(\bar\Omega))^3$ (even $u \in (C^4_\omega(\bar\Omega))^3$), hence we are concerned with a classical solution to (3.2).

(ii) The assertion of Theorem (3.1) implies the existence of a classical solution to the system (1.1), (1.2), (1.3). Indeed, if we put $B = \operatorname{rot} u$, $E = -\partial_t u + \operatorname{grad}\operatorname{div} u$, $q = \operatorname{div}\varepsilon(-\partial_t u + \operatorname{grad}\operatorname{div} u)$, then using (1.2) as definitions of D, H, J we check easily that the relations (1.1), (1.3) are satisfied.

Conversely, we can transform (1.1), (1.2), (1.3) into (3.2). By (1.1) (iv) and (1.3) (i) (cf. [2]) there exists a vector function v, $B = \operatorname{rot} v$, $v \wedge \nu = 0$ on $\partial\Omega$. From (1.1) (ii) and (1.3) (ii) it follows that there exists a function ψ such that $\partial_t v + E = -\operatorname{grad}\psi$, $\psi = \text{const.}$ on $\partial\Omega$. We can choose const. $= 0$. The functions v, ψ are not uniquely determined. More precisely, the same relations are fulfilled with $u = v + \operatorname{grad} \tilde{\psi}$, $\varphi = \psi - \partial_t \tilde{\psi}$, where $\tilde{\psi}$ is an arbitrary smooth function, $\tilde{\psi} = 0$ on $\partial\Omega$. Thus we may require u, φ to be subject to the "Lorentz – type" condition $\varphi + \operatorname{div} u = 0$ which leads paradoxically to the heat equation for $\tilde{\psi}$: $\partial_t \tilde{\psi} - \Delta \tilde{\psi} = \psi + \operatorname{div} v$. Thus, (1.1), (1.2), (1.3) imply (3.2) together with the boundary conditions $u \wedge v = 0$, $\operatorname{div} u = 0$ on $\partial\Omega$.

(iii) The assumptions of Theorem (3.1) contain the "isotropic case", i.e., the case when there exist functions $\tilde{\varepsilon}, \tilde{\mu}, \tilde{\sigma} \in C^{M+2}([-\delta_0^2, \delta_0^2])$ such that $\varepsilon(z) = \tilde{\varepsilon}(|z|^2) \, . \, z$, $\mu(z) = \tilde{\mu}(|z|^2) \, . \, z$, $\sigma(z) = \tilde{\sigma}(|z|^2) \, . \, z$, $\tilde{\varepsilon}(0) > 0$, $\tilde{\mu}(0) > 0$, $\tilde{\sigma}(0) > 0$. By taking $\delta_0 > 0$ sufficiently small we obtain the conditions (3.1) (i), (ii).

The proof of Theorem (3.1) is based on Nash's iteration scheme following [7], [8], [25]. In fact, we evoke the situation of [10] wiht $N_+ = N_\infty = 3$, $N_0 = 2$, $N_q = N_{\dot{q}} = N_- = 1$.

Put

$$F(u) = \partial_t(\varepsilon(-\partial_t u + \operatorname{grad} \operatorname{div} u)) - \operatorname{rot}(\mu(\operatorname{rot} u)) + \\ + \sigma(-\partial_t u + \operatorname{grad} \operatorname{div} u)) \, . \tag{3.4}$$

The mapping F: $D_L(F) \subset (C_\omega^{L+3}(\bar{\Omega}))^3 \to (C_\omega^L(\bar{\Omega}))^3$, where $D_L(F) = \{u \in (C_\omega^{L+3}(\bar{\Omega}))^3 \cap W^5 | \, |u|_{W,5} < \delta_+\}$, is continuous for $0 \leqq L \leqq M + 1$ and twice Fréchet differentiable in $D_L(F)$ for $0 \leqq L \leqq M - 1$. We put

$$\delta_+ = \delta_0 / c \tag{3.5}$$

where c is the constant of the embedding $W^3 \hookrightarrow (C_\omega^0(\bar{\Omega}))^3$.

Using (2.5) (ii) and the embedding theorems for Sobolev spaces we get for $0 \leqq L \leqq M - 1$, $u_1, u_2, v \in W^{L+6}$, $|v|_{W,5} < \delta_+$ the inequality

$$|F''(v)(u_1, u_2)|_{W,L} \leqq c_L \sum_{|\lambda| = L+1} (1 + |v|_{W,\lambda_1+5}) \, |u_1|_{W,\lambda_2+3} \, |u_2|_{W,\lambda_3+3} \, . \tag{3.6}$$

(F', F'' denote the first and the second Fréchet derivative of F, respectively).

The equation $F(u) = Q$ is substituted by the infinite system:

(3.7) $\quad F'(0) \, u_0 = Q$,

$(3.8)_0$ $\quad F'(S_0 u_0) \, w_0 = h_0$,

$\vdots$

$(3.8)_{n+1}$ $\quad F'(S_{n+1} u_{n+1}) \, w_{n+1} = h_{n+1}$,

$\vdots$

where $\{S_n\}_{n=0}^\infty$ is the sequence of smoothening operators (2.7), $u_0, w_0, w_1, \ldots$ are unknown functions and

(3.9) (i) $$u_{n+1} = u_0 + \sum_{k=0}^{n} w_k \, ,$$

(ii) $$h_0 = S_0 e_0 \, , \quad e_0 = F'(0) \, u_0 - F(u_0) = \int_0^1 (1 - \chi) \, F''(\chi u_0)(u_0, u_0) \, d\chi \, ,$$

(iii)

$$h_{n+1} = S_{n+1} e_{n+1} + (S_{n+1} - S_n) \sum_{k=0}^{n} e_k ,$$
$$e_{n+1} = f_{n+1} + g_{n+1} ,$$
$$f_{n+1} = F'(u_n) w_n - F(u_{n+1}) + F(u_n) =$$
$$= - \int_0^1 (1 - \chi) F''(u_n + \chi w_n)(w_n, w_n) \, d\chi ,$$
$$g_{n+1} = (F'(S_n u_n) - F'(u_n)) w_n =$$
$$= - \int_0^1 F''(u_n - \chi(I - S_n) u_n)((I - S_n) u_n, w_n) \, d\chi .$$

4. LINEAR EQUATIONS

In this section we deal with the linear equation

(4.1)
$$\partial_t(A(t, x)(\partial_t u - \text{grad div } u)) + \text{rot}(B(t, x) \text{ rot } u) + $$
$$+ S(t, x)(\partial_t u - \text{grad div } u) = h ,$$

where $A, B, S \in \mathfrak{M}^{M+1}$ are given matrices satisfying

(4.2) **Assumptions.** There exist functions $\alpha, \beta, \zeta \in C^{M+1}_{\omega}(\bar{\Omega})$ such that:

$$\inf \alpha \geqq \alpha_0 > 0 ,$$
$$\inf \beta \geqq \beta_0 > 0 ,$$
$$\inf \zeta \geqq \zeta_0 > 0 ,$$
$$\|\tilde{A}\|_0 \leqq \min \left\{ \frac{\alpha_0}{2\varkappa}, \frac{\alpha_0}{2} \right\} ,$$
$$\|\tilde{B}\|_0 \leqq \min \left\{ \frac{\beta_0}{8}, \frac{\beta_0}{4\varrho} \right\} ,$$
$$\|\tilde{S}\|_0 \leqq \min \left\{ \frac{\zeta_0}{4}, \frac{\zeta_0}{2\vartheta}, \frac{\beta_0}{4\vartheta}, \frac{\zeta_0}{4\varkappa} \right\} ,$$
$$\|\partial_t B\|_0 \leqq \frac{\beta_0}{(8M + 4) a} ,$$
$$\|\partial_t A\|_0 \leqq \min \left\{ \frac{\zeta_0}{8M + 4}, \frac{\zeta_0}{4\varkappa} \right\} ,$$
$$\sum_{i=0}^{3} \|\partial_{x_i} S\|_0 \leqq \min \left\{ \frac{\zeta_0}{2\vartheta}, \frac{\beta_0}{8\vartheta} \right\} ,$$

where $\tilde{A} = A - \alpha I$, $\tilde{B} = B - \beta I$, $\tilde{S} = S - \zeta I$, $a = 4b/\zeta_0$, $b = \|A\|_1 + \|B\|_1 + \|S\|_0$, $\partial_{x_0} = \partial_t$ and $\alpha_0, \beta_0, \zeta_0, \varkappa, \varrho, \vartheta$ are the constants from (2.1), (3.1).

(4.3) **Theorem.** *Let the assumptions* (4.2) *be satisfied. Then for each* $h \in W^M$

there exists a unique solution $u \in X^{M+1}$ *to* (4.1). *Moreover, this solution satisfies the inequality*

$$(4.4) \qquad |u|_{X,L+1} \leqq c_{L,b}(|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L)\,|h|_{W,0}), \quad 0 \leqq L \leqq M.$$

Before proving the theorem we need two lemmas.

(4.5) **Lemma.** *Let* (4.2) *be satisfied. Then for every* L, $0 \leqq L \leqq M - 1$ *and for every* $H \in Z^L$ *there exists a unique solution* $\Phi \in Y^{L+2}$ *to the equation*

$$(4.6) \qquad \partial_t \operatorname{div} A \operatorname{grad} \Phi + \operatorname{div} S \operatorname{grad} \Phi = H$$

and this solution satisfies the inequality

$$(4.7) \qquad |\Phi|_{Y,L+2} \leqq c_L\{|H|_{Z,L} + \|A\|_{L+1}\,|\partial_t\Phi|_{Z,1} + \|A\|_1\,|\partial_t\Phi|_{Z,L+1} + (\|A\|_{L+2} + \|S\|_{L+1})\,|\Phi|_{Z,1} + (\|A\|_2 + \|S\|_1)\,|\Phi|_{Z,L+1}\}.$$

(4.8) Remark. Here, the expression "Φ is a solution to (4.6)" means that (4.6) is satisfied a.e. in $\mathbb{R}^1 \times \Omega$.

Proof of Lemma (4.5). For $\theta \in [0, 1]$ put

$$T_\theta\Phi = \partial_t \operatorname{div}(\theta A + (1-\theta)\,\alpha_0 I) \operatorname{grad} \Phi + \operatorname{div}(\theta S + (1-\theta)\,\zeta_0 I) \operatorname{grad} \Phi.$$

The proof consists in verifying that T_θ: $Y^{L+2} \to Z^L$ is an isomorphism for every $\theta \in [0, 1]$ and $0 \leqq L \leqq M - 1$. For $\theta = 0$ this follows from the well-known results on the regularity of solutions to the Poisson equation, cf. [12]. For $\theta > 0$ we derive the inequality

$$(4.9) \qquad \tfrac{1}{2}|T_0\Phi|_{Z,L} \leqq |T_\theta\Phi|_{Z,L} + \theta c_L\{\|A\|_{L+1}\,|\partial_t\Phi|_{Z,1} + \|A\|_1\,|\partial_t\Phi|_{Z,L+1} + (\|A\|_{L+2} + \|S\|_{L+1})\,|\Phi|_{Z,1} + (\|A\|_2 + \|S\|_1)\,|\Phi|_{Z,L+1}\}, \quad \Phi \in Y^{L+2}, \quad 0 \leqq L \leqq M - 1.$$

For this purpose we choose some $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $|\lambda| = L$. Then we have

$$(4.10) \qquad \partial^\lambda T_\theta\Phi = (\theta\alpha + (1-\theta)\,\alpha_0)\,\partial^\lambda\,\partial_t\,\Delta\Phi + (\theta\zeta + (1-\theta)\,\zeta_0)\,\partial^\lambda\,\Delta\Phi + \theta\sum_{i,j=1}^{3}\{\tilde{A}_{ij}\,\partial^\lambda\,\partial_t\,\partial_{x_i}\,\partial_{x_j}\,\Phi + \tilde{S}_{ij}\,\partial^\lambda\,\partial_{x_i}\,\partial_{x_j}\,\Phi + (\partial_t A_{ij})\,\partial^\lambda\,\partial_{x_i}\,\partial_{x_j}\,\Phi\} + \theta R_\lambda\Phi,$$

where

$$\partial^\lambda = \partial_{x_1}^{\lambda_1}\,\partial_{x_2}^{\lambda_2}\,\partial_{x_3}^{\lambda_3},$$

and

$$(4.11) \qquad |R_\lambda\Phi|_{Z,0} \leqq c_L \sum_{K=1}^{L+1} (\|A\|_K\,|\partial_t\Phi|_{Z,L-K+2} + \|A\|_{K+1}\,|\Phi|_{Z,L-K+2} + \|S\|_K\,|\Phi|_{Z,L-K+2}).$$

By the assumption (4.2) we have

$$\Big|\sum_{i,j=1}^{3} \tilde{S}_{ij}\, \partial^\lambda\, \partial_{x_i}\, \partial_{x_j}\Phi\Big|_{Z,0} \leqq \frac{\zeta_0}{4}\, |\Delta\Phi|_{Z,L}\,,$$

$$\Big|\sum_{i,j=1}^{3} \tilde{A}_{ij}\, \partial^\lambda\, \partial_t\, \partial_{x_i}\, \partial_{x_j}\Phi\Big|_{Z,0} \leqq \frac{\alpha_0}{2}\, |\partial_t\, \Delta\Phi|_{Z,L}\,,$$

$$\Big|\sum_{i,j=1}^{3} (\partial_t A_{ij})\, \partial^\lambda\, \partial_{x_i}\, \partial_{x_j}\Phi\Big|_{Z,0} \leqq \frac{\zeta_0}{4}\, |\Delta\Phi|_{Z,L}\,.$$

By using the interpolation inequalities (2.2), (2.3) we obtain (4.9) directly from (4.10).

It follows from (2.2), (2.3) that for an arbitrarily small $\delta > 0$ we can find a constant c_δ such that

(4.12) $$|\Phi|_{Y,L+1} \leqq \delta|\Phi|_{E,L+2} + c_\delta|\Phi|_{Y,1} \leqq c_L\delta|T_0\Phi|_{Z,L} + c_\delta|\Phi|_{Y,1}\,.$$

We see immediately that

$$\int_\omega\int_\Omega T_\theta\Phi\,.\,\Phi\,\mathrm{d}x\,\mathrm{d}t \geqq c|\Phi|^2_{Z,1}\,,$$

$$\int_\omega\int_\Omega T_\theta\Phi\,.\,\partial_t\Phi\,\mathrm{d}x\,\mathrm{d}t \geqq c|\partial_t\Phi|^2_{Z,1} - c'|\Phi|^2_{Z,1}\,,$$

so that we have

(4.13) $$|\Phi|_{Y,1} \leqq c|T_\theta\Phi|_{Z,0}\,,$$

where the constant c is independent of θ. Combining the inequalities (4.9),(4.12), (4.13) we obtain

(4.14) $$|\Phi|_{Y,L+2} \leqq c_{L,A,S}|T_\theta\Phi|_{Z,L}\,,\quad \theta\in[0,1]\,,\quad 0 \leqq L \leqq M-1\,.$$

For proving that $T_\theta\colon Y^{L+2} \to Z^L$ is an isomorphism we proceed by induction (the analogous method was used by Ladyzhenskaya and Uraltseva [12] for proving the regularity of general elliptic equations). Let us assume this assertion to be proved for some $\theta \in [0, 1]$. Then

$$T_{\theta+\delta}\Phi = T_\theta\Phi + \delta(\partial_t \operatorname{div}(A - \alpha_0 I)\operatorname{grad}\Phi + \operatorname{div}(S - \zeta_0 I)\operatorname{grad}\Phi)\,.$$

The equation $T_{\theta+\delta}\Phi = H$ is equivalent to

(4.15) $$\Phi = \delta T_\theta^{-1}(\partial_t \operatorname{div}(A - \alpha_0 I)\operatorname{grad}\Phi + \operatorname{div}(S - \zeta_0 I)\operatorname{grad}\Phi) + T_\theta^{-1}H\,.$$

If we choose $\delta > 0$ sufficiently small, the solvability of the equation (4.15) is ensured e.g. by the Banach contraction principle. From (4.14) it follows that $T_{\theta+\delta}$ is an isomorphism $Y^{L+2} \to Z^L$, $0 \leqq L \leqq M - 1$.

The relation (4.14) implies that the norm $\|T_\theta^{-1}\|_{Z^L\to Y^{L+2}}$ is independent of θ. Hence, the choice of δ can be made independently of θ, thus after a finite number of steps we verify that, in particular, $T_1\colon Y^{L+2} \to Z^L$ is an isomorphism. The inequality (4.7) follows directly from (4.9) for $\theta = 1$. The lemma is proved. □

(4.16) **Lemma.** *Let* (4.2) *be satisfied. For every* L, $0 \leqq L \leqq M - 1$, *and for every* $f \in L^2(\omega, S^L)$ *there exists a unique solution* $w \in L^2(\omega, S_\tau^{L+2})$ *to the equation*

$$\operatorname{rot}(B \operatorname{rot} w) = f \tag{4.17}$$

and this solution satisfies the inequality

$$|w|_{Z^3,L+2} \leqq c_L(|f|_{Z^3,L} + \|B\|_{L+1} |w|_{Z^3,1} + \|B\|_1 |w|_{Z^3,L+1}) . \tag{4.18}$$

Proof. The proof is analogous to that of Lemma (4.5). For $\theta \in [0, 1]$ we put

$$\Lambda_\theta(w) = \operatorname{rot}(\theta B + (1 - \theta) \beta_0 I) \operatorname{rot} w .$$

By [2] and Theorem 7.1 of [11] the mapping $\Lambda_0: L^2(\omega, S_\tau^{L+2}) \to L^2(\omega, S^L)$ is an isomorphism for $0 \leqq L \leqq M - 1$. The inequality corresponding to (4.9) takes the form

$$\tfrac{1}{2}|\Lambda_0(w)|_{Z^3,L} \leqq |\Lambda_\theta(w)|_{Z^3,L} + \theta c_L(\|B\|_{L+1} |w|_{Z^3,1} + \|B\|_1 |w|_{Z^3,L+1}) \tag{4.19}$$

and the proof follows as above. □

Proof of Theorem (4.3). In X^{M+1} we choose an orthogonal basis in the following way: let $\{g_k\}_{k=1}^\infty$ and $\{s_k\}_{k=1}^\infty$ be the bases of G_0^{M+1} and S_τ^{M+1}, respectively, where we put $g_k = \operatorname{grad} \varphi_k$, $\Delta\varphi_k = \lambda_k\varphi_k$, $\int_\Omega \varphi_k^2 \, dx = 1$, $k = 1, 2, \ldots$. The basis of X^{M+1} is formed by all functions $\sin(2j\pi t/\omega) \, g_k(x)$, $\cos(2j\pi t/\omega) \, g_k(x)$, $\sin(2j\pi t/\omega) \,.$ $. \, s_k(x)$, $\cos(2j\pi t/\omega) \, s_k(x)$, $j = 0, 1, 2, \ldots$, $k = 1, 2, \ldots$. We arrange these functions into a sequence $\{w_k\}_{k=-\infty}^{+\infty}$ in order to obtain, for any integer k,

(4.20) (i) $\partial_t w_k(t, x) = c_k \, w_{-k}(t, x)$,
(ii) $\operatorname{grad} \operatorname{div} w_k(t, x) = d_k \, w_k(t, x)$.

We use the standard Galerkin procedure. Put

$${}_m u(t, x) = \sum_{k=-m}^{m} v_k \, w_k(t, x) , \quad m \geqq 1 . \tag{4.21}$$

The constant vector $(v_k)_{k=-m}^m$ is required to satisfy

$$\int_\omega \int_\Omega \{-(A(t, x) (\partial_t \, {}_m u - \operatorname{grad} \operatorname{div} {}_m u), \partial_t w_k(t, x)) + \tag{4.22}$$
$$+ (S(t, x) (\partial_t \, {}_m u - \operatorname{grad} \operatorname{div} {}_m u), w_k(t, x)) + (B(t, x) \operatorname{rot} {}_m u, \operatorname{rot} w_k(t, x))\} \, dx \, dt =$$
$$= \int_\omega \int_\Omega (h(t, x), w_k(t, x)) \, dx \, dt , \quad k = -m, \ldots, m ,$$

which is a linear algebraic system of the type $Ev = f$, where E is the square matrix $\{E_{jk}\}_{j,k=-m}^m$,

$$E_{jk} = \int_\omega \int_\Omega \{-(A(\partial_t w_j - \operatorname{grad} \operatorname{div} w_j), \partial_t w_k) +$$
$$+ (S(\partial_t w_j - \operatorname{grad} \operatorname{div} w_j), w_k) + (B \operatorname{rot} w_j, \operatorname{rot} w_k)\} \, dx \, dt .$$

First, let us derive a priori estimates for ${}_mu$. We multiply the k-th equation in (4.22) by v_k and sum over $k = -m, \ldots, m$. We get

$$(4.23) \qquad \int_\omega \int_\Omega \{-(A(\partial_t\, {}_mu - \text{grad div}\, {}_mu), \partial_t\, {}_mu) - ((\tfrac{1}{2}\,\partial_t S)\, {}_mu, {}_mu) +$$

$$+ \text{div}\, S\, {}_mu\, \text{div}\, {}_mu + (B\, \text{rot}\, {}_mu, \text{ rot}\, {}_mu)\}\, \text{d}x\, \text{d}t = \int_\omega \int_\Omega (h, {}_mu)\, \text{d}x\, \text{d}t\,.$$

Analogously, we multiply the k-th equation in (4.22) by $a(c_{-k}v_{-k} - d_k v_k)$ and sum again over $k = -m, \ldots, m$, where a is defined in (4.2) and c_k, d_k are from (4.20). Now, we get

$$(4.24) \qquad \int_\omega \int_\Omega a\{((S + \tfrac{1}{2}\partial_t A)(\partial_t\, {}_mu - \text{grad div}\, {}_mu), \partial_t\, {}_mu - \text{grad div}\, {}_mu) -$$

$$- ((\tfrac{1}{2}\partial_t B)\, \text{rot}\, {}_mu, \text{ rot}\, {}_mu)\}\, \text{d}x\, \text{d}t = \int_\omega \int_\Omega a(h, \partial_t\, {}_mu - \text{grad div}\, {}_mu)\, \text{d}x\, \text{d}t\,.$$

We check immediately that

$$(4.25) \qquad \left|\int_\omega \int_\Omega (A(\partial_t\, {}_mu - \text{grad div}\, {}_mu), \partial_t\, {}_mu)\, \text{d}x\, \text{d}t\right| \leqq$$

$$\leqq \|A\|_0 \left(|\partial_t\, {}_mu|_{W,0}^2 + |\text{grad div}\, {}_mu|_{W,0}^2\right)$$

and

$$(4.26) \qquad \int_\omega \int_\Omega \text{div}\, S\, {}_mu\, \text{div}\, {}_mu\, \text{d}x\, \text{d}t \geqq \zeta_0 \int_\omega \int_\Omega (\text{div}\, {}_mu)^2\, \text{d}x\, \text{d}t\, -$$

$$- \vartheta\|\tilde{S}\|_0 \int_\omega \int_\Omega |\text{rot}\, {}_mu|^2 + (\text{div}\, {}_mu)^2\, \text{d}x\, \text{d}t - \sum_{i=1}^{3} \|\partial_{x_i} S\|_0 \int_\omega \int_\Omega |{}_mu| \cdot |\text{div}\, {}_mu|\, \text{d}x\, \text{d}t,$$

where ϑ is from (2.1) (iii).
After adding (4.23) to (4.24) and employing (4.25), (4.26) we obtain (notice that $\vartheta \geqq 1$)

$$(4.27) \quad \left(a\zeta_0 - a\|\tilde{S}\|_0 - \frac{a}{2}\|\partial_t A\|_0 - \|A\|_0\right)\left(|\partial_t\, {}_mu|_{W,0}^2 + |\text{grad div}\, {}_mu|_{W,0}^2\right) +$$

$$+ \left\{\beta_0 - \|\tilde{B}\|_0 - \frac{a}{2}\|\partial_t B\|_0 - \vartheta(\|\tilde{S}\|_0 + \sum_{i=0}^{3} \|\partial_{x_i} S\|_0)\right\} |\text{rot}\, {}_mu|_{W,0}^2 +$$

$$+ \{\zeta_0 - \vartheta(\|S\|_0 + \sum_{i=0}^{3} \|\partial_{x_i} S\|_0)\}\, |\text{div}\, {}_mu|_{Z,0}^2 \leqq$$

$$\leqq \int_\omega \int_\Omega |h|\, (a|\partial_t\, {}_mu - \text{grad div}\, {}_mu| + |{}_mu|)\, \text{d}x\, \text{d}t\,.$$

The inequality (4.27) and the assumptions (4.2) yield

$$(4.28) \qquad |{}_mu|_{X,1} \leqq c_b |h|_{W,0}\,.$$

We obtain estimates of higher order in a similar way. By induction over L we derive the relations

$$(4.29)\qquad (|\partial_t^{L+1}\,{}_m u|_{W,0}^2 + |\text{grad div}\,\partial_t^L\,{}_m u|_{W,0}^2 + |\text{rot}\,\partial_t^L\,{}_m u|_{W,0}^2)^{1/2} \leqq$$
$$\leqq c_{L,b}(|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L)\,|h|_{W,0})\,, \quad 0 \leqq L \leqq M\,.$$

Indeed, for $L = 0$, (4.29) is nothing else than (4.28). Let us assume now that (4.29) is proved for all $K = 0, 1, \ldots, L-1$. We multiply the k-th equation in (4.22) successively by $c_k^L c_{-k}^L v_k$ and by $a(c_k^L c_{-k}^{L+1} v_{-k} - d_k c_k^L c_{-k}^L v_k)$ and we sum over $k = -m, \ldots, m$. We find, respectively,

$$(4.30)\qquad \int_\omega\int_\Omega \{-(\partial_t^L(A(\partial_t\,{}_m u - \text{grad div}\,{}_m u)), \partial_t^{L+1}\,{}_m u) +$$
$$+ (\partial_t^L(S(\partial_t\,{}_m u - \text{grad div}\,{}_m u)), \partial_t^L\,{}_m u) + (\partial_t^L(B\,\text{rot}\,{}_m u), \text{rot}\,\partial_t^L\,{}_m u)\}\,\text{d}x\,\text{d}t =$$
$$= \int_\omega\int_\Omega (\partial_t^L h, \partial_t^L\,{}_m u)\,\text{d}x\,\text{d}t$$

and

$$(4.31)\qquad \int_\omega\int_\Omega a\{(\partial_t^{L+1}(A(\partial_t\,{}_m u - \text{grad div}\,{}_m u)), \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u) +$$
$$+(\partial_t^L(S(\partial_t\,{}_m u - \text{grad div}\,{}_m u)), \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u) -$$
$$- (\partial_t^{L+1}(B\,\text{rot}\,{}_m u), \text{rot}\,\partial_t^L\,{}_m u)\}\,\text{d}x\,\text{d}t =$$
$$= \int_\omega\int_\Omega a(\partial_t^L h, \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u)\,\text{d}x\,\text{d}t\,.$$

After an elementary computation, (4.30) and (4.31) yield, respectively,

$$(4.32)\int_\omega\int_\Omega \Big[-(A(\partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u), \partial_t^{L+1}\,{}_m u) + (B\,\text{rot}\,\partial_t^L\,{}_m u, \text{rot}\,\partial_t^L\,{}_m u) +$$
$$+ \sum_{K=1}^{L}\binom{L}{K}\{-(\partial_t^K A(\partial_t^{L-K+1}\,{}_m u - \text{grad div}\,\partial_t^{L-K}\,{}_m u), \partial_t^{L+1}\,{}_m u) +$$
$$+ (\partial_t^K B\,\text{rot}\,\partial_t^{L-K}, \text{rot}\,\partial_t^L\,{}_m u)\} +$$
$$+ \sum_{K=0}^{L}\binom{L}{K}(\partial_t^K S(\partial_t^{L-K+1}\,{}_m u - \text{grad div}\,\partial_t^{L-K}\,{}_m u), \partial_t^L\,{}_m u)\Big]\,\text{d}x\,\text{d}t =$$
$$= \int_\omega\int_\Omega (\partial_t^L h, \partial_t^L\,{}_m u)\,\text{d}x\,\text{d}t$$

and

$$(4.33)\qquad \int_\omega\int_\Omega a\Big[((S + (L + \tfrac{1}{2})\,\partial_t A)(\partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u),$$
$$\partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u) - (L + \tfrac{1}{2})(\partial_t B\,\text{rot}\,\partial_t^L\,{}_m u, \text{rot}\,\partial_t^L\,{}_m u) +$$

$$+\sum_{K=2}^{L+1}\binom{L+1}{K}\{(-\partial_t^K B(\text{rot}\,\partial_t^{L-K+1}\,{}_m u),\ \text{rot}\,\partial_t^L\,{}_m u) +$$
$$+(\partial_t^K A(\partial_t^{L-K+2}\,{}_m u - \text{grad div}\,\partial_t^{L-K+1}\,{}_m u),\ \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u)\} +$$
$$+\sum_{K=1}^{L}\binom{L}{K}(\partial_t^K S(\partial_t^{L-K+1}\,{}_m u - \text{grad div}\,\partial_t^{L-K}\,{}_m u),\ \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u)\Big]\,dx\,dt =$$
$$=\int_\omega\int_\Omega a(\partial_t^L h,\ \partial_t^{L+1}\,{}_m u - \text{grad div}\,\partial_t^L\,{}_m u)\,dx\,dt\,.$$

We add again (4.32) to (4.33) and by using (4.25), (4.2), (2.1)(iii) we obtain

$$|\partial_t^{L+1}\,{}_m u|_{W,0}^2 + |\text{grad div}\,\partial_t^L\,{}_m u|_{W,0}^2 + |\text{rot}\,\partial_t^L\,{}_m u|_{W,0}^2 \leqq$$
$$\leqq c_{L,b}\Big[\sum_{K=1}^{L}\{(\|A\|_{K+1} + \|S\|_K)(|\partial_t^{L-K+1}\,{}_m u|_{W,0} + |\text{grad div}\,\partial_t^{L-K}\,{}_m u|_{W,0})\,.$$
$$.\,(|\partial_t^{L+1}\,{}_m u|_{W,0} + |\text{grad div}\,\partial_t^L\,{}_m u|_{W,0}) +$$
$$+\|B\|_{K+1}\,|\text{rot}\,\partial_t^{L-K}\,{}_m u|_{W,0}\,|\text{rot}\,\partial_t^L\,{}_m u|_{W,0}\} +$$
$$+|\partial_t^L h|_{W,0}(|\partial_t^{L+1}\,{}_m u|_{W,0} + |\text{grad div}\,\partial_t^L\,{}_m u|_{W,0} + |\text{rot}\,\partial_t^L\,{}_m u|_{W,0})\Big]$$

and hence

$$(|\partial_t^{L+1}\,{}_m u|_{W,0}^2 + |\text{grad div}\,\partial_t^L\,{}_m u|_{W,0}^2 + |\text{rot}\,\partial_t^L\,{}_m u|_{W,0}^2)^{1/2} \leqq$$
$$\leqq c_{L,b}\{|h|_{W,L} + \sum_{K=1}^{L}(\|A\|_{K+1} + \|B\|_{K+1} + \|S\|_K)(|\partial_t^{L-K+1}\,{}_m u|_{W,0}^2 +$$
$$+|\text{grad div}\,\partial_t^{L-K}\,{}_m u|_{W,0}^2 + |\text{rot}\,\partial_t^{L-K}\,{}_m u|_{W,0}^2)^{1/2}\,.$$

We transform the right hand side of the last inequality by using the induction assumption and the interpolation inequalities (2.2), (2.3) and we obtain just (4.29).

The a priori estimates (4.29) imply that the matrix E on the left hand side of (4.22) is nonsingular, hence the solutions ${}_m u$ of (4.22) exist for all $m = 1, 2, \ldots$ and satisfy (4.29). Especially, the sequences $\{\partial_t^{L+1}\,{}_m u\}_{m=1}^\infty$, $\{\text{grad div}\,\partial_t^L\,{}_m u\}_{m=1}^\infty$, $\{\text{rot}\,\partial_t^L\,{}_m u\}_{m=1}^\infty$, $0 \leqq L \leqq M$, are L^2-bounded and therefore there exists $u \in X^1$ such that $\partial_t^{L+1} u$, $\text{rot}\,\partial_t^L u$, $\text{grad div}\,\partial_t^L u$ are square integrable for $0 \leqq L \leqq M$, and a subsequence $\{{}_n u\}$ of $\{{}_m u\}$ such that $\partial_t^{L+1}\,{}_n u \rightharpoonup \partial_t^{L+1} u$, $\text{grad div}\,\partial_t^L\,{}_n u \rightharpoonup \text{grad div}\,\partial_t^L u$, $\text{rot}\,\partial_t^L\,{}_n u \rightharpoonup \text{rot}\,\partial_t^L u$ as $n \to \infty$, $0 \leqq L \leqq M$, where the symbol $\rightharpoonup$ denotes the weak convergence in $W^0 = L^2(\omega, L^2(\Omega))$. Passing to the weak limit in (4.22) we conclude that for arbitrary $v \in X^1$ the function u satisfies the relation

$$(4.34)\qquad \int_\omega\int_\Omega\{(\partial_t(A(\partial_t u - \text{grad div}\,u)) + S(\partial_t u - \text{grad div}\,u),\ v) +$$
$$+(B\,\text{rot}\,u,\ \text{rot}\,v)\}\,dx\,dt = \int_\omega\int_\Omega (h, v)\,dx\,dt\,.$$

Now, put

$$f = h - \partial_t(A(\partial_t u - \text{grad div}\,u)) - S(\partial_t u - \text{grad div}\,u)\,.$$

Then $f \in L^2(\omega, S^0)$. By Lemma (4.16) we find the unique $w \in L^2(\omega, S^2_\tau)$ such that rot B rot $w = f$. From (4.34) it follows that $w = u$, hence $u \in X^2$ and the equation (4.1) is satisfied a.e. in $\mathbb{R}^1 \times \Omega$. Moreover,

$$(4.35) \qquad |u|_{X,2} \leqq c_b\{|h|_{W,1} + (\|A\|_2 + \|B_2\|_2 + \|S\|_1)\,|h|_{W,0}\}\,,$$

so that (4.4) holds for $L \leqq 1$.

Thus, it remains to prove the regularity of u and the estimates (4.4) for $L \geqq 2$. For this purpose we proceed again by induction with respect to L using Lemmas (4.5) and (4.16). Let us assume

(4.36) (i) $u \in X^L$ for some $L \geqq 2$ and (4.4) holds for $0 \leqq K \leqq L - 1$;
(ii) for all $K' \leqq K$, $1 \leqq K \leqq L - 1$ we have

$$|\text{grad div } \partial_t^{L-K'} u|_{Z^3,K'} + |\text{rot } \partial_t^{L-K'} u|_{Z^3,K'} \leqq$$
$$\leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L)\,|h|_{W,0}\}\,.$$

From (4.35) we see that (4.36) (i) is satisfied for $L = 2$ and (4.29) yields (4.36) (ii) for every L, $0 \leqq L \leqq M$ and $K = 1$.

Further, put

$$(4.37) \qquad H = \sum_{J=1}^{L-K-1} \binom{L-K-1}{J} (-\partial_t \text{ div } \partial_t^J A \text{ grad div } \partial_t^{L-K-J-1} u -$$
$$- \text{div } \partial_t^J S \text{ grad div } \partial_t^{L-K-J-1} u) + \text{div } A\, \partial_t^{L-K+1} u +$$
$$+ \sum_{J=1}^{L-K} \binom{L-K}{J} \text{div } \partial_t^J A\, \partial_t^{L-K-J+1} u + \partial_t^{L-K-1} \text{div } S\, \partial_t u - \partial_t^{L-K-1} \text{div } h\,.$$

Then $H \in Z^{K-1}$ and

$$|H|_{Z,K-1} \leqq c_L\{\sum_{I=1}^{L-1} (\|A\|_I\, |\text{grad div } \partial_t u|_{W,L-I-1} +$$
$$+ (\|A\|_{I+1} + \|S\|_I)\, |\text{grad div } u|_{W,L-I-1}) + \sum_{I=0}^{K} \|A\|_I\, |\partial_t^{L-K+1} u|_{Z^3,K-I} +$$
$$+ \sum_{I=1}^{L} (\|A\|_I + \|S\|_{I-1})\, |u|_{W,L-I+1} + |h|_{W,L-1}\} \leqq$$
$$\leqq c_{L,b}\{\sum_{I=1}^{L-1} (\|A\|_{I+1} + \|S\|_I)\, |u|_{X,L-I+1} + |\partial_t^{L-K+1} u|_{Z^3,K} + |h|_{W,L-1}\}\,.$$

The inequality

$$|\partial_t^{L-K+1} u|_{Z^3,K} \leqq c_K(|\text{grad div } \partial_t^{L-K+1} u|_{Z^3,K-1} + |\text{rot } \partial_t^{L-K+1} u|_{Z^3,K-1})$$

and the assumption (4.36) (ii) imply

$$(4.38) \qquad |H|_{Z,K-1} \leqq c_{L,b}[|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L)\,|h|_{W,0} +$$
$$+ \sum_{I=1}^{L-1} (\|A\|_{I+1} + \|S\|_I)\{|h|_{W,L-I} +$$
$$+ (\|A\|_{L-I+1} + \|B\|_{L-I+1} + \|S\|_{L-I})\,|h|_{W,0}\}] \leqq$$
$$\leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L)\,|h|_{W,0}\}\,.$$

Now, let $\Phi \in Y^{K+1}$ be the solution of (4.6) with the right hand side (4.37). Putting $v = \operatorname{grad} \partial_t^{L-K-1} \varphi$ in (4.34), where $\varphi \in \mathring{W}_\omega^{2,L-K+1}(\Omega)$ is arbitrarily chosen, we see that $\Phi = \operatorname{div} \partial_t^{L-K-1} u$, and consequently

$$\begin{aligned} |\Phi|_{Z,K} &\leqq c_L |u|_{X,L}, \\ |\partial_t \Phi|_{Z,K} &= c_L |u|_{X,L}, \\ |\Phi|_{Z,1} &\leqq c_L |u|_{X,L-K}, \\ |\partial_t \Phi|_{Z,1} &\leqq c_L |u|_{X,L-K+1}. \end{aligned}$$

On the other hand, Φ satisfies (4.7) for $L = K - 1$, hence (4.36) yields

$$\begin{aligned} (4.39) \qquad |\Phi|_{Y,K+1} \leqq c_{L,b}[&|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0} + \\ &+ \|A\|_K \{|h|_{W,L-K} + (\|A\|_{L-K+1} + \|B\|_{L-K+1} + \|S\|_{L-K}) |h|_{W,0}\} + \\ &+ (\|A\|_2 + \|S\|_1) \{|h|_{W,L-1} + (\|A\|_L + \|B\|_L + \|S\|_{L-1}) |h|_{W,0}\} + \\ &+ (\|A\|_{K+1} + \|S\|_K) \{|h|_{W,L-K-1} + \\ &+ (\|A\|_{L-K} + \|B\|_{L-K} + \|S\|_{L-K-1}) |h|_{W,0}\}]. \end{aligned}$$

In (4.38) we use tacitly (2.2), (2.3). Similarly, from (4.39) we obtain

$$\begin{aligned} (4.40) \qquad &|\operatorname{grad} \operatorname{div} \partial_t^{L-K} u|_{Z^3,K} + |\operatorname{grad} \operatorname{div} \partial_t^{L-K-1} u|_{Z^3,K} \leqq \\ &\leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0}\}. \end{aligned}$$

Analogously, put

$$\begin{aligned} (4.41) \qquad f = \partial_t^{L-K} h - &\sum_{J=1}^{L-K} \binom{L-K}{J} \operatorname{rot} (\partial_t^J B \operatorname{rot} \partial_t^{L-K-J} u) - \\ &- \partial_t^{L-K+1}(A(\partial_t u - \operatorname{grad} \operatorname{div} u)) - \partial_t^{L-K}(S(\partial_t u - \operatorname{grad} \operatorname{div} u)). \end{aligned}$$

By the induction assumption (4.36) we have $f \in L^2(\omega, S^{K-1})$ and

$$\begin{aligned} (4.42) \qquad |f|_{Z^3,K-1} \leqq c_L\{ &\sum_{I=2}^{L} (\|A\|_I + \|S\|_{I-1}) |u|_{X,L-I+2} + \\ &+ |\operatorname{grad} \operatorname{div} \partial_t^{L-K+1} u|_{Z^3,K-1} + |\partial_t^{L-K+2} u|_{Z^3,K-1} + |h|_{W,L-1} + \\ &+ \sum_{I=1}^{L} \|B\|_I |u|_{W,L-I+1}\} \leqq \\ &\leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0}\}. \end{aligned}$$

By Lemma (4.16) we find the solution $w \in L^2(\omega, S_\tau^{K+1})$ to the equation (4.17). We check again that $w = \partial_t^{L-K} u$. From (4.18) it follows that

$$\begin{aligned} (4.43) \qquad &|\operatorname{rot} \partial_t^{L-K} u|_{Z^3,K} \leqq c_{L,b}\{\|B\|_K \operatorname{rot} \partial_t^{L-K} u|_{Z^3,0} + \\ &+ \|B\|_1 |\operatorname{rot} \partial_t^{L-K} u|_{Z^3,K-1} + |h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0}\} \leqq \\ &\leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0}\}. \end{aligned}$$

By the same argument, putting

$$(4.44)\qquad f = \partial_t^{L-K-1} h - \sum_{J=1}^{L-K-1} \binom{L-K-1}{J} \operatorname{rot}(\partial_t^J B \operatorname{rot} \partial_t^{L-K-J-1} u) -$$

$$- \partial_t^{L-K}(A(\partial_t u - \operatorname{grad} \operatorname{div} u)) - \partial_t^{L-K-1}(S(\partial_t u - \operatorname{grad} \operatorname{div} u))$$

and using (4.40) we see that the solution w to the equation $\operatorname{rot}(B \operatorname{rot} w) = f$ is an element of $L^2(\omega, S_\tau^{K+2})$ and equals to $\partial_t^{L-K-1} u$. Applying (4.18) we get

$$(4.45)\qquad |\operatorname{rot} \partial_t^{L-K-1} u|_{Z^3,K+1} \leqq c_{L,b}\{|h|_{W,L} + (\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L) |h|_{W,0}\} .$$

Hence, by the induction argument using (4.29), (4.36), (4.40), (4.43), (4.45) we conclude that $u \in X^{M+1}$ and (4.4) holds. Thus, Theorem (4.3) is proved. □

5. PROOF OF THEOREM (3.1)

As a consequence of Theorem (4.3) we have

(5.1) **Proposition.** *Let the assumptions of Theorem* (3.1) *be satisfied and let* F *be the mapping* (3.4). *Then there exists* $\delta_- > 0$ *such that for every* $h \in W^M$ *and* $v \in W^{M+6}$, $|v|_{W,6} < \delta_-$, *there exists a unique solution* $u \in X^{M+1}$ *to the equation*

$$(5.2)\qquad F'(v)\, u = h$$

and this solution satisfies

$$(5.3)\qquad |u|_{X,L+1} \leqq c_L\{|h|_{W,L} + |v|_{W,L+6}\, |h|_{W,0}\} , \quad 0 \leqq L \leqq M .$$

Proof. The equation (5.2) is of the type (4.1), where

$$A = D\varepsilon(-\partial_t v + \operatorname{grad} \operatorname{div} v) ,$$
$$B = D\mu(\operatorname{rot} v) ,$$
$$S = D\sigma(-\partial_t v + \operatorname{grad} \operatorname{div} v) .$$

Remember that these expressions are defined for

$$(5.4)\qquad |v|_{W,5} < \delta_+ ,$$

where δ_+ is introduced in (3.5). We have

$$\|\partial_t A\|_0 \leqq |||\varepsilon|||_2\, |v|_{W,6} ,$$
$$\|\partial_t B\|_0 \leqq |||\mu|||_2\, |v|_{W,5} ,$$
$$\sum_{i=0}^{3} \|\partial_{x_i} S\|_0 \leqq |||\sigma|||_2\, |v|_{W,6} .$$

By (2.5) (i) we have

$$b = \|A\|_1 + \|B\|_1 + \|S\|_0 \leqq c(|||\varepsilon|||_2 + |||\mu|||_2 + |||\sigma|||_1)(1 + |v|_{W,6}) .$$

Further, put $\alpha = \varepsilon_0(-\partial_t v + \operatorname{grad} \operatorname{div} v)$, $\beta = \mu_0(\operatorname{rot} v)$, $\zeta = \sigma_0(-\partial_t v + \operatorname{grad} \operatorname{div} v)$.

Taking

$$(5.5) \qquad |v|_{W,6} < \delta_-$$

we obtain b independent of v and for $\delta_- > 0$ sufficiently small we see that the assumptions (4.2) are verified. Moreover, (2.5) (i) yields

$$\|A\|_{L+1} + \|B\|_{L+1} + \|S\|_L \leqq$$
$$\leqq c_L(|||\varepsilon|||_{L+2} + |||\mu|||_{L+2} + |||\sigma|||_{L+1})(1 + |v|_{W,L+6}),$$

hence the inequality (5.3) is a consequence of (4.4) and the proof of (5.1) follows easily. □

Finally, we use (5.1) for the construction of the approximating sequence $\{u_n, n = 0, 1, \ldots\}$ by solving consecutively the equations (3.7), $(3.8)_n$, $n = 0, 1, \ldots$.

The solution $u_0 \in X^{M+1}$ of (3.7) satisfies

$$(5.6) \qquad |u_0|_{X,M+1} \leqq c_M |Q|_{W,M}.$$

The sufficient conditions for the solvability of $(3.8)_0$,

$$(5.7) \qquad \begin{aligned} |S_0 u_0|_{W,5} &< \delta_+ , \\ |u_0|_{W,5} &< \delta_+ , \\ |S_0 u_0|_{W,6} &< \delta_- , \end{aligned}$$

are fulfilled provided Q is taken small enough, say $|Q|_{W,M} < \delta_1$. On the other hand, by (2.8) (i) and (3.6) we have

$$(5.8) \qquad |h_0|_{W,L} = |S_0 e_0|_{W,L} \leqq c_L \int_0^1 |F''(\chi u_0)(u_0, u_0)|_{W,0} \, d\chi \leqq$$
$$\leqq c_L |u_0|^2_{W,4} (1 + |u_0|_{W,6}).$$

The unique solution $w_0 \in X^{M+1}$ of $(3.8)_0$ satisfies

$$(5.9) \qquad |w_0|_{X,L+1} \leqq c_L(|h_0|_{W,L} + |u_0|_{W,0} \, |h_0|_{W,0}), \quad 0 \leqq L \leqq M.$$

For the present, let $\eta > 0$ be arbitrarily chosen. From (5.6), (5.8), (5.9) it follows that we can find δ_η, $0 < \delta_\eta < \eta$, such that if

$$(5.10) \qquad |Q|_{W,M} < \delta_\eta,$$

then

$$(5.11)_0 \qquad |w_0|_{X,L+1} \leqq \eta, \quad 0 \leqq L \leqq M.$$

Put

$$(5.12) \qquad \gamma = N + \tfrac{1}{3}.$$

Our next goal is to choose η in $(5.11)_0$ in such a way that the inequalities

$$(5.11)_k \qquad |w_k|_{X,L+1} \leqq \eta r^{(-\gamma+L)k}$$

hold for arbitrary integers $k \geqq 0$ and $0 \leqq L \leqq M$. The constant $r > 1$ is introduced in (2.7).

We proceed by induction over k. We adopt the following assumption.

(5.13) For some $\eta > 0$ and $n \geqq 0$ there exists a sequence $\{w_k\}_{k=0}^n \subset X^{M+1}$ of solutions of $\{(3.8)_k\}_{k=0}^n$ satisfying $(5.11)_k$, $k = 0, 1, \ldots, n$, $0 \leqq L \leqq M$.

Assuming (5.13) we derive (for technical details, see [10], cf. also [8], [25])

(5.14) **Estimates.**

(i) $|u_{n+1}|_{X,L+1} \leqq c_L\eta$, $0 \leqq L \leqq N$,

(ii) $|u_{n+1}|_{X,L+1} \leqq c_L\eta r^{(-\gamma+L)(n+1)}$, $N < L \leqq M$,

(iii) $|u_n|_{X,L+1} \leqq c_L\eta$, $0 \leqq L \leqq N$,

(iv) $|u_n|_{X,L+1} \leqq c_L\eta r^{(-\gamma+L)(n+1)}$, $N < L \leqq M$,

(v) $|(I - S_{n+1})\, u_{n+1}|_{W,L+1} \leqq c_L\eta r^{(-\gamma+L)(n+1)}$, $0 \leqq L \leqq M$,

(vi) $|(I - S_n)\, u_n|_{W,L+1} \leqq c_L\eta r^{(-\gamma+L)(n+1)}$, $0 \leqq L \leqq M$,

(vii) $|f_{n+1}|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L \leqq M - 5$,

(viii) $|g_{n+1}|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L \leqq M - 5$,

(ix) $|e_{n+1}|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L \leqq M - 5$,

(x) $|\sum_{k=0}^{n} e_k|_{W,L} \leqq c_L\eta^2$, $0 \leqq L < M - 5$,

(xi) $|\sum_{k=0}^{n} e_k|_{W,M-5} \leqq c_L\eta^2 r^{(n+1)/3}$,

(xii) $|(I - S_n)\sum_{k=0}^{n} e_k|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L \leqq M - 5$,

(xiii) $|(I - S_{n+1})\sum_{k=0}^{n} e_k|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L \leqq M - 5$,

(xiv) $|h_{n+1}|_{W,L} \leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)}$, $0 \leqq L$.

By (5.1) and (3.9), sufficient conditions for the solvability of $(3.8)_{n+1}$ are

(5.15)
$$\begin{aligned} |S_{n+1}u_{n+1}|_{W,6} &< \delta_- , \\ |u_n|_{W,5} &< \delta_+ , \\ |u_{n+1}|_{W,5} &< \delta_+ , \\ |S_n u_n|_{W,5} &< \delta_+ . \end{aligned}$$

Since $N \geqq 5$, we see by (2.8), (5.14) (i) and (iii) that (5.15) is satisfied provided η is taken sufficiently small. Following (5.1), (5.3) the solution $w_{n+1} \in X^{M+1}$ to $(3.8)_{n+1}$ fulfils the inequality

$$\begin{aligned} |w_{n+1}|_{X,L+1} &\leqq c_L(|h_{n+1}|_{W,L} + |S_{n+1}u_{n+1}|_{W,L+6}\,|h_{n+1}|_{W,0}) \leqq \\ &\leqq c_L(|h_{n+1}|_{W,L} + r^{L(n+1)}|u_{n+1}|_{W,6}\,|h_{n+1}|_{W,0})\,, \quad 0 \leqq L \leqq M\,. \end{aligned}$$

Using (5.14) (i), (xiv) we obtain the estimate

(5.16)
$$\begin{aligned} |w_{n+1}|_{X,L+1} &\leqq c_L\eta^2 r^{(-2\gamma+L+5)(n+1)} \\ &c_L\eta^2 r^{(-\gamma+L)(n+1)}\,, \quad 0 \leqq L \leqq M\,. \end{aligned}$$

The constants c_L in (5.16) are independent of n. Hence the choice

$$(5.17) \qquad \eta < (\max\{c_L,\ 0 \leqq L \leqq M\})^{-1}$$

yields $(5.11)_{n+1}$.

By induction over n we conclude that we can construct the infinite sequence $\{w_n,\ n = 0, 1, \ldots\} \subset X^{M+1}$ of solutions to $\{(3.8)_n,\ n = 0, 1, \ldots\}$ provided $\delta_\eta = \delta_N$ is taken sufficiently small (so that (5.7), $(5.11)_0$, (5.15), (5.17) are fulfilled) and each w_n satisfies the corresponding inequality $(5.11)_n$. Since the series

$$\sum_{n=0}^{\infty} |w_n|_{X,N+1} \leqq \eta \sum_{n=0}^{\infty} r^{-n/3}$$

is convergent, we see that $\{u_n\}_{n=0}^{\infty}$ is a fundamental sequence in X^{N+1} and hence $u_n \to u$ in X^{N+1}. By the embedding theorem, $u_n \to u$ in $(C_\omega^3(\bar{\Omega}))^3$. By continuity, $F(u_n) \to F(u)$ in $(C_\omega^0(\bar{\Omega}))^3$. On the other hand, (3.7), (3.8), (3.9) yield

$$(5.18) \qquad F(u_{n+1}) = Q - e_{n+1} - (I - S_n) \sum_{k=0}^{n} e_k\,.$$

Using (5.14) (ix), (xii) and (5.18) we obtain

$$|F(u_{n+1}) - Q|_{(C_\omega{}^0(\bar{\Omega}))^3} \leqq c|F(u_{n+1}) - Q|_{W,3} \leqq$$
$$\leqq c(|e_{n+1}|_{W,3} + |(I - S_n) \sum_{k=0}^{n} e_k|_{W,3}) \leqq c\eta^2 r^{-3(n+1)}\,,$$

hence $F(u) = Q$, and the proof of Theorem (3.1) is complete.

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