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BOUNDEDNESS OF SOLUTIONS OF THE THIRD ORDER DIFFERENTIAL EQUATION WITH OSCILLATORY RESTORING AND FORCING TERMS

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1. In this paper we study the behaviour of solutions of the equation

$$x''' + ax'' + bx' + h(x) = p(t)\,, \tag{1}$$

where $a > 0$, $b > 0$ are constants with $a^2 > 4b$, the functions $h(x)$, $p(t)$ have their first derivatives continuous for all real values of their arguments and are oscillatory in the following sense:

for each argument u there exist such numbers $\beta_1 > \alpha_1 > u > \alpha_{-1} > \beta_{-1}$ that

$$f(\alpha_1) < 0\,,\quad f(\beta_1) > 0\,,\quad f(\alpha_{-1}) < 0\,,\quad f(\beta_{-1}) > 0\,,$$

where f is either $h(x)$ or $p(t)$, u is either x or t and all roots of the restoring term $h(x)$ are isolated.

2. Our main tool for attacking the equation (1) will be the well-known *Cauchy formula* for the particular solution of nonhomogeneous linear differential equations with constant coefficients.

Lemma 1. *If there exist such positive constants H, P that for all $x \in \mathscr{R}^1$ and $t \geqq 0$ the inequalities*

$$1)\ |h(x)| \leqq H\,,\quad 2)\ |p(t)| \leqq P$$

hold, then each solution $x(t)$ of the equation (1) *satisfies the inequalities*

$$\limsup_{t\to\infty} |x'(t)| \leqq (H + P)/b := D'\,, \tag{2}$$

$$\limsup_{t\to\infty} |x''(t)| \leqq 2(H + P)/a := D''\,.$$

Proof. Substituting $y := x'$, we get from (1) the equation

$$y'' + ay' + by = p(t) - h(x(t)) \tag{3}$$

with solutions of the form

$$/x'(t) = /y(t) = C_1 e^{\varrho_1 t} + C_2 e^{\varrho_2 t} + \int_0^t \frac{e^{\varrho_1(t-\tau)} - e^{\varrho_2(t-\tau)}}{\varrho_1 - \varrho_2} [p(\tau) - h(x(\tau))] \, d\tau ,$$

where $\varrho_{1,2} = (-a \pm \sqrt{(a^2 - 4b)})/2$ and C_1, C_2 are arbitrary constants.

Hence by virtue of 1), 2), for $t \geqq 0$ we have not only

$$\left| \int_0^t \frac{e^{\varrho_1(t-\tau)} - e^{\varrho_2(t-\tau)}}{\varrho_1 - \varrho_2} [p(\tau) - h(x(\tau))] \, d\tau \right| \leqq \frac{H + P}{b} \left(1 + \frac{\varrho_2 e^{\varrho_1 t} - \varrho_1 e^{\varrho_2 t}}{\varrho_1 - \varrho_2} \right),$$

but also

$$\limsup_{t \to \infty} |x'(t)| \leqq (H + P)/b . \tag{4}$$

Furthermore, putting $z := y'$, we get from (3) the equation

$$z' + az = p(t) - b\, x'(t) - h(x(t))$$

with solutions of the form

$$/x''(t) = /z(t) = C e^{-at} + \int_{T_x}^t e^{-a(t-\tau)} [p(\tau) - b\, x'(\tau) - h(x(\tau))] \, d\tau ,$$

where C is an arbitrary constant and T_x a great enough number.

Thus by virtue of 1), 2) and (4), for $t \geqq T_x$ we have not only

$$\left| \int_{T_x}^t e^{-a(t-\tau)} [p(\tau) - b\, x'(\tau) - h(x(\tau))] \, d\tau \right| \leqq 2(H + P + |o(T_x)|) \int_{T_x}^t e^{-a(t-\tau)} \, d\tau \leqq$$

$$\leqq \frac{2}{a} (H + P + |o(T_x)|) (1 - e^{-a(t-T_x)}) ,$$

but also

$$\limsup_{t \to \infty} |x''(t)| \leqq 2(H + P)/a , \quad \text{q.e.d.}$$

Lemma 2. *Under the assumptions of Lemma* 1, *if*

1') $\qquad |h'(x)| \leqq H'$ *for all* $x \in \mathscr{R}^1$, $\quad$ 3) $\left| \int_0^\infty p(t) \, dt \right| < \infty$,

where H' is a suitable constant, then every bounded solution $x(t)$ of the equation (1) *either satisfies the relation*

$$\lim_{t \to \infty} x(t) = \bar{x} , \quad \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = 0 \quad (h(\bar{x}) = 0) \tag{5}$$

or there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. Substituting a fixed bounded solution $x(t)$ of (1) into (1) and integrating the result from T_x to t (T_x — a great enough number, whose magnitude will be specified later in (9)), we get the identity

$$(6)\qquad \int_{T_x}^{t} h(x(\tau))\,d\tau = -\{b[x(t)-x(T_x)] + a[x'(t)-x'(T_x)] + x''(t) - x''(T_x)\} + \\ + \int_{T_x}^{t} p(\tau)\,d\tau \ (:\equiv I(t)).$$

Therefore, by virtue of the condition 3), the assertion of Lemma 1 and the boundedness of $x(t)$, there exists such a constant M_x that for $t \geqq T_x$ the relation

$$(7)\qquad |I(t)| \leqq M_x \quad \text{i.e.} \quad \left|\int_{T_x}^{t} h(x(\tau))\,d\tau\right| \leqq M_x$$

is satisfied.

Now let us assume that $x(t)$ does not converge to any root $\bar{x}$ of $h(x)$: i.e.,

$$(8)\qquad \limsup_{t\to\infty} |x(t) - \bar{x}| > 0$$

and simultaneously, for $t \geqq T_x$,

$$(9)\qquad h(x(t)) \geqq 0 \quad \text{or} \quad h(x(t)) \leqq 0\,.$$

Then

$$H(t) :\equiv \int_{T_x}^{t} h(x(\tau))\,d\tau \quad (\text{for } t \geqq T_x)$$

evidently is a composed monotone function with a finite or infinite limit for $t \to \infty$. Since (7) implies that the "divergent case" can be disregarded, it follows from (9) that not only

$$(7')\qquad \lim_{t\to\infty} \int_{T_x}^{t} |h(x(\tau))|\,d\tau = \lim_{t\to\infty} \left|\int_{T_x}^{t} h(x(\tau))\,d\tau\right| \leqq M_x$$

but also

$$(8')\qquad \liminf_{t\to\infty} |x(t) - \bar{x}| = 0$$

holds, because otherwise (i.e. if

$$\liminf_{t\to\infty} |x(t) - \bar{x}| > 0)$$

(9) together with the fact that the roots of $h(x)$ are isolated would yield

$$\liminf_{t\to\infty} |h(x(t))| = \liminf_{t\to\infty} |h(x(t)) - h(\bar{x})| > 0\,,$$

a contradiction to (7').

Thus (8) and (8') imply

$$\limsup_{t\to\infty} |h(x(t))| = \limsup_{t\to\infty} |h(x(t)) - h(\bar{x})| > 0 = \liminf_{t\to\infty} |h(x(t))|$$

and consequently there exists such a sequence $\{t_i\} \geqq T_x$ and such a constant $\tilde{H} > 0$

that (in what follows, $\mathrm{d}(x, y)$ denotes the distance between x and y)

$$\alpha) \quad \liminf_{i\to\infty /\Rightarrow t_i\to\infty/} \mathrm{d}(t_i, t_{i-1}) > 0\,, \quad \beta) \ |h(x(t_i))| \geqq \tilde{H}$$

hold. Hence

$$M_x \geqq \lim_{t\to\infty} \int_{t_1}^{t} |h(x(\tau))|\,\mathrm{d}\tau = \sum_{i=2}^{\infty} \int_{t_{i-1}}^{t_i} |h(x(\tau))|\,\mathrm{d}\tau \Rightarrow \limsup_{i\to\infty/\Rightarrow t_i\to\infty/} \int_{t_{i-1}}^{t_i} |h(x(t))|\,\mathrm{d}t = 0$$

or (cf. α), β))

$$H' \limsup_{t\to\infty} |x'(t)| \geqq \limsup_{t\to\infty} \left|\frac{\mathrm{d}h(x(t))}{\mathrm{d}x(t)}\, x'(t)\right| = \limsup_{t\to\infty} \left|\frac{\mathrm{d}h(x(t))}{\mathrm{d}t}\right| = \infty\,.$$

But according to the assertion of Lemma 1, this is impossible and that is why $(x(t) - \bar{x})$ necessarily oscillates.

The remaining part of our lemma follows immediately from the assertion

(10) $$x(t) \in \mathfrak{C}^{(n)}\langle 0, \infty)\,, \quad \limsup_{t\to\infty} |x^{(n)}(t)| < \infty\,,$$

$$\lim_{t\to\infty} |x(t)| < \infty \Rightarrow \lim_{t\to\infty} x^{(k)}(t) = 0\,,$$

$$(\text{where } n \geqq 2 \text{ is a natural number and } k = 1, \ldots, (n-1))\,,$$

whose proof can be found e.g. in [1, p. 161]. This completes the proof.

Lemma 3. *Under the assumptions of Lemma 2 and if*

$$2') \quad |p'(t)| \leqq P' \quad \textit{for all} \quad t \geqq 0\,, \qquad 2'') \ \limsup_{t\to\infty} |p(t)| > 0$$

hold, where P' is a suitable constant, then for every bounded solution $x(t)$ of the equation (1) there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. If Lemma 3 does not hold, then according to Lemma 2 (5) holds and the fourth derivative of $x(t)$ satisfies

$$x''''(t) = p'(t) - ax'''(t) - bx''(t) - h'(x)\,x'(t)\,.$$

But it can be readily checked that, by the ultimate boundedness of $x'(t)$, $x''(t)$, $x'''(t)$ (see (2)) and 1'), 2'), there exists such a constant D_4 that

$$\limsup_{t\to\infty} |x''''(t)| \leqq D_4\,,$$

which according to (10) gives the relations

$$\lim_{t\to\infty} x(t) = \bar{x}/ \Rightarrow \lim_{t\to\infty} h(x(t)) = h(\bar{x}) = 0/\,, \quad \lim_{t\to\infty} x^{(j)}(t) = 0 \quad j = 1, 2, 3$$

or

$$\limsup_{t\to\infty} |p(t)| = \limsup_{t\to\infty} |x'''(t) + a\,x''(t) + b\,x'(t) + h(x(t))| = 0\,,$$

a contradiction to $\limsup_{t\to\infty} |p(t)| > 0$ (cf. 2'')), q.e.d.

3. Now we can give the principal result of our paper.

Theorem. *If there exist such positive constants H, H', P, P', P_0, R that for $|x| > R$ and $t \geqq 0$ the following conditions are satisfied:*

1) $|h(x)| \leqq H,\ |h'(x)| \leqq H'$,

2) $|p(t)| \leqq P,\ |p'(t)| \leqq P',\ \left|\int_0^t p(\tau)\,\mathrm{d}\tau\right| \leqq P_0,\ \limsup_{t\to\infty} |p(t)| > 0$,

3) $\min\left[\mathrm{d}(\bar{x}_k, \bar{x}_{k+1}), \mathrm{d}(\bar{x}_k, \bar{x}_{k-1})\right] > \dfrac{2(H+P)}{b}\left(\dfrac{2}{a} + \dfrac{a}{b}\right) + \dfrac{P_0}{b}$,

where $\bar{x}_k$ are roots of $h(x)$ with $h'(\bar{x}_k) > 0$ and $\bar{x}_{k-1}, \bar{x}_{k+1}$ denote the couple of adjacent roots of $\bar{x}_k$ $(k = 0, \pm 2, \pm 4, \ldots)$, then all solutions $x(t)$ of the equation (1) are bounded and for each of them there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. Let us assume, on the contrary, that $x(t)$ is an unbounded solution of (1); i.e., for example, $\limsup_{t\to\infty} x(t) = \infty$.

Lemma 1 implies the existence of such a number $T_0 \geqq 0$ great enough that for $t \geqq T_0$

$$|x'(t)| \leqq D' + \varepsilon_1\,, \quad |x''(t)| \leqq D'' + \varepsilon_2\,,$$

with $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ small enough constants.

Let $T_1 \geqq T_0$ be the last point with $x(T_1) = \bar{x}_k$ (k-even) and $T_2 > T_1$ be the first point with $x(T_2) = \bar{x}_{k+1}$. If we integrate (1) from T_1 to t, $T_1 \leqq t \leqq T_2$, we come to

$$[x'(t) - x''(T_1)] + a[x'(t) - x'(T_1)] + b[x(t) - x(T_1)] + \\ + \int_{T_1}^t h(x(\tau))\,\mathrm{d}\tau = \int_{T_1}^t p(\tau)\,\mathrm{d}\tau\,. \tag{11}$$

However, for $T_1 \leqq t \leqq T_2$ we have $h(x(t))\operatorname{sgn} x(t) \geqq 0$, whence we can obtain (multiplying (11) by $\operatorname{sgn} x$)

$$|x(t)| \leqq |x(T_1)| + \frac{2}{b}\left[D'' + aD' + \tfrac{1}{2}P_0\right] + \varepsilon\,,$$

where $\varepsilon > 0$ is an arbitrarily small constant, a contradiction to $x(T_2) = \bar{x}_{k+1}$ with respect to 3).

Since the remaining part of our theorem immediately follows from Lemma 3, the proof is complete.

4. In the end, let us note that in [2] we have dealt also with the case

$$\int_0^\infty |p(t)|\,\mathrm{d}t < \infty\,.$$

References

[1] *W. A. Coppel:* Stability and Asymptotic Behavior of Differential Equations, D. C. Heath, Boston, 1975.

[2] *J. Andres:* Asymptotic properties of solutions of a certain third order differential equation with the oscillatory restoring term, to appear.

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