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ON LATTICES WITH ISOMORPHIC INTERVAL LATTICES

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Let A be a lattice. Denote by $\mathrm{Int}(A)$ the lattice of all intervals of A (including the empty set). Let B be a lattice and f an isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$. The one-element intervals of A are just atoms of $\mathrm{Int}(A)$ and so the isomorphism f induces a bijective mapping f' of A onto B defined by: $f'(a) = b$ iff $f([a, a]) = [b, b]$. The aim of this paper is to give answers to the following two questions (see [1], Problem I.10):

1) Under what conditions is f' an isomorphism or a dual isomorphism?

2) Under what conditions does $\mathrm{Int}(A)$ determine A up to isomorphism or dual isomorphism?

Recall that if $I = [a, b]$ and $J = [c, d]$ are intervals of A, then $I \vee J = [a \wedge c, b \vee d]$ and $I \wedge J = I \cap J = [a \vee c, b \wedge d]$ or the empty set if $a \vee c \nleq b \wedge d$.

Lemma 1. *For all $x, y \in A$,*

$$f([x \wedge y, x \vee y]) = [f'(x) \wedge f'(y), f'(x) \vee f'(y)]\,.$$

Proof. $f([x \wedge y, x \vee y]) = f([x, x] \vee [y, y]) = f([x, x]) \vee f([y, y]) = [f'(x), f'(x)] \vee [f'(y), f'(y)] = [f'(x) \wedge f'(y), f'(x) \vee f'(y)]$.

Lemma 2. *If $f'(x) \leqq f'(y)$ for all $x, y \in A$ such that $x \leqq y$, then f' is an isomorphism. If $f'(x) \geqq f'(y)$ for all $x, y \in A$ such that $x \leqq y$, then f' is a dual isomorphism.*

Proof. It is enough to show that if $f'(x) \leqq f'(y)$ then $x \leqq y$. Suppose that $f'(x) \leqq f'(y)$. By Lemma 1, $f([x \wedge y, x \vee y]) = [f'(x), f'(y)]$. Since $x \wedge y \in [x \wedge y, x \vee y]$, $f'(x \wedge y) \geqq f'(x)$. By assumption, $f'(x) \geqq f'(x \wedge y)$ and so $f'(x) = f'(x \wedge y)$. Thus $x = x \wedge y$, i.e. $x \leqq y$. Dually one can obtain the rest.

1. BOUNDED LATTICES

In this section let A be a bounded lattice with the least element 0_A and the greatest element 1_A. Let B be a lattice and f an isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$. Then, evidently, $B = f([0_A, 1_A])$ is a bounded lattice with the extreme elements 0_B and 1_B.

If $f'(0_A) < f'(1_A)$, then $f'(0_A) = 0_B$, $f'(1_A) = 1_B$ and $f([0_A, x]) = [0_B, f'(x)]$ for all $x \in A$. Thus f' is an isomorphism. Dually, if $f'(0_A) > f'(1_A)$, the mapping f' is a dual isomorphism.

Now suppose that f' is neither an isomorphism nor a dual isomorphism. Then the elements $f'(0_A)$ and $f'(1_A)$ are incomparable and, by Lemma 1, $f'(0_A) \wedge f'(1_A) = 0_B$ and $f'(0_A) \vee f'(1_A) = 1_B$.

Lemma 3. *The elements $r, s \in A$ such that $f'(r) = 0_B$ and $f'(s) = 1_B$ are incomparable and $r \vee s = 1_A$ and $r \wedge s = 0_A$.*

Proof. By Lemma 1, $f([r \wedge s, r \vee s]) = [f'(r) \wedge f'(s), f'(r) \vee f'(s)] = [0_B, 1_B] = f([0_A, 1_A])$. So $r \wedge s = 0_A$ and $r \vee s = 1_A$.

Lemma 4. *For all $x \in A$, $x = (x \wedge r) \vee (x \wedge s)$ and $x = (x \vee r) \wedge (x \vee s)$.*

Proof. By Lemma 1, $f([x \wedge r, x \vee r]) = [f'(r), f'(x)]$ and $f([x \wedge s, x \vee s]) = [f'(x), f'(s)]$. Since the interval $[(x \wedge r) \vee (x \wedge s), x]$ is a subinterval of both the intervals $[x \wedge r, x \vee r]$ and $[x \wedge s, x \vee s]$, we get that $f([(x \wedge r) \vee (x \wedge s), x]) \subseteq [f'(r), f'(x)] \cap [f'(x), f'(s)] = [f'(x), f'(x)]$. Thus $(x \wedge r) \vee (x \wedge s) = x$. Dually, $(x \vee r) \wedge (x \vee s) = x$.

Lemma 5. *Let x, y be elements of A. If $x \wedge r = y \wedge r$ and $x \wedge s = y \wedge s$, then $x = y$. If $x \vee r = y \vee r$ and $x \vee s = y \vee s$, then $x = y$.*

Proof. Let $x \wedge r = y \wedge r$ and $x \wedge s = y \wedge s$. Then, by Lemma 4, $x = (x \wedge r) \vee (x \wedge s) = (y \wedge r) \vee (y \wedge s) = y$. Dually we get the rest.

Lemma 6. *Let x, y be elements of A. The following equalities hold:*

$$r \wedge (x \vee y) = (r \wedge x) \vee (r \wedge y), \quad r \vee (x \wedge y) = (r \vee x) \wedge (r \vee y),$$

$$s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y), \quad s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y).$$

Proof. Denote $a = r \wedge (x \vee y)$ and $b = (r \wedge x) \vee (r \wedge y)$. It is clear that $r \vee a = r \vee b = r$. By Lemma 4, $s \vee a = s \vee (s \wedge (x \vee y)) \vee (r \wedge (x \vee y)) = s \vee x \vee y$ and $s \vee b = s \vee (s \wedge x) \vee (s \wedge y) \vee (r \wedge x) \vee (r \wedge y) = s \vee x \vee y$. Thus, by Lemma 5, $a = b$. The rest can be shown similarly.

2. CONGRUENCE RELATIONS

In this section let A and B be lattices and let f be an isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$. Suppose that the induced mapping f' is neither an isomorphism nor a dual isomorphism. By Lemma 2, there exist elements $a, b, c, d \in A$ such that $a < b$, $c < d$ and $f'(a) < f'(b)$, $f'(c) > f'(d)$. Let M be the set of all intervals of A containing elements a, b, c, d. For any interval $I \in M$, the mapping $f/\mathrm{Int}(I)$ is an isomorphism of $\mathrm{Int}(I)$ onto $\mathrm{Int}(f(I))$ such that the induced mapping f'/I is neither an isomorphism nor a dual isomorphism. Let r_I and s_I be elements of I such that $f(I) = [f'(r_I), f'(s_I)]$.

By Lemma 3, $I = [r_I \wedge s_I, r_I \vee s_I]$. It is evident that any interval of A is a subinterval of an interval from M.

Lemma 7. *Let $I, J \in M$ and $I \subseteq J$. Let x be an element of I. The following equalities hold*:

$$x \wedge r_J = x \wedge r_I \wedge r_J\,, \quad x \vee r_J = x \vee r_I \vee r_J\,,$$

$$x \wedge s_J = x \wedge s_I \wedge s_J\,, \quad x \vee s_J = x \vee s_I \vee s_J\,.$$

Proof. By Lemma 1, $f([r_I \wedge x, r_I \vee x]) = [f'(r_I), f'(x)]$ and $f([r_J \wedge x, r_J \vee x]) = [f'(r_J), f'(x)]$. If $I \subseteq J$, then $f'(r_J) \leqq f'(r_I)$ and so $[r_I \wedge x, r_I \vee x] \subseteq \subseteq [r_J \wedge x, r_J \vee x]$. Thus we have $r_J \wedge x \leqq r_I \wedge x$ and $r_I \vee x \leqq r_J \vee x$. Hence $x \wedge r_J = x \wedge r_I \wedge r_J$ and $x \vee r_J = x \vee r_I \vee r_J$. The rest can be proved in the same way.

Lemma 8. *Let x, y be elements of an interval $I \in M$ such that $x \wedge r_I = y \wedge r_I$* (or $x \vee r_I = y \vee r_I$, $x \wedge s_I = y \wedge s_I$, $x \vee s_I = y \vee s_I$). *Then $x \wedge r_J = y \wedge r_J$ ($x \vee r_J = y \vee r_J, x \wedge s_J = y \wedge s_J, x \vee s_J = y \vee s_J$, respectively) for any interval $J \in M$ such that $I \subseteq J$.*

Proof follows from Lemma 7.

Now define relations α, β on A by the following rules:

$$x\alpha y \quad \text{iff} \quad x \wedge r_I = y \wedge r_I \quad \text{for some} \quad I \in M\,,$$

$$x\beta y \quad \text{iff} \quad x \wedge s_I = y \wedge s_I \quad \text{for some} \quad I \in M\,.$$

Lemma 9. *The relations α, β are congruence relations on A and $\alpha \cap \beta = \mathrm{id}_A$ (the identity relation on A).*

Proof. The relation α is clearly reflexive and symmetric. Let $x\alpha y$ and $y\alpha z$, i.e. $x \wedge r_I = y \wedge r_I$ and $y \wedge r_J = z \wedge r_J$ for some $I, J \in M$. Let $K \in M$ be an interval such that $I \subseteq K$ and $J \subseteq K$. Then, by Lemma 8, $x \wedge r_K = y \wedge r_K = z \wedge r_K$. One can easily show that $x\alpha y$ implies $(x \wedge c)\,\alpha(y \wedge c)$ and, by Lemmas 6 and 8, $(x \vee c)\,\alpha(y \vee c)$ for all $c \in A$. Thus α is a congruence relation on A. In the same way we can prove that β is a congruence relation on A. It follows from Lemma 5 that $\alpha \cap \beta = \mathrm{id}_A$.

Lemma 10. *Let u, v be elements of A and let $I \in M$ be an interval containing u, v. Then $((u \wedge r_I) \vee (v \wedge s_I))\,\alpha u$ and $((u \wedge r_I) \vee (v \wedge s_I))\,\beta v$.*

Proof. Using Lemma 6 we get $((u \wedge r_I) \vee (v \wedge s_I)) \wedge r_I = u \wedge r_I$ and $((u \wedge r_I) \vee (v \wedge s_I)) \wedge s_I = v \wedge s_I$.

For $x \in A$, denote by $\alpha(x)$ and $\beta(x)$ the congruence classes of the congruence relations α and β containing x.

Lemma 11. *Let x, y be elements of A. Then $f'(x) \leqq f'(y)$ if and only if $\alpha(x) \geqq \alpha(y)$ in the lattice A/α and $\beta(x) \leqq \beta(y)$ in the lattice A/β.*

Proof. Let $I \in M$ be an interval containing x, y. Using Lemma 1 we get that

$f'(x) \leqq f'(y)$ iff $[x \wedge r_I,\ x \vee r_I] \subseteq [y \wedge r_I,\ y \vee r_I]$, i.e. $x \wedge r_I \geqq y \wedge r_I$ and $x \vee r_I \leqq y \vee r_I$. Evidently, $\alpha(x) = \alpha(x \wedge r_I)$ and $\alpha(y) = \alpha(y \wedge r_I)$. By Lemma 6, $\beta(x) = \beta(x \vee r_I)$ and $\beta(y) = \beta(y \vee r_I)$.

Lemma 12. *The lattice A is isomorphic to $A/\alpha \times A/\beta$ and the lattice B is isomorphic to $(A/\alpha)^* \times A/\beta$, where $(A/\alpha)^*$ denotes the lattice dual to A/α.*

Proof follows from Lemmas 9, 10 and 11.

Lemma 13. *Let L, M be lattices. The lattice* $\mathrm{Int}(L \times M)$ *is isomorphic to* $\mathrm{Int}(L \times M^*)$, *where M^* is the dual of M.*

Proof is easy.

3. THE RESULTS

Let A be a lattice. Denote by $\mathrm{CSub}(A)$ the lattice of all convex sublattices of A (including the empty set). The lattice $\mathrm{Int}(A)$ is a sublattice of $\mathrm{CSub}(A)$ and any nonempty interval of A is either an atom or a join of two atoms in $\mathrm{CSub}(A)$. Thus whenever $\mathrm{CSub}(A)$ is isomorphic to $\mathrm{CSub}(B)$, then $\mathrm{Int}(A)$ is isomorphic to $\mathrm{Int}(B)$, too. On the other hand, any isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$ can be extended to an isomorphism of $\mathrm{CSub}(A)$ onto $\mathrm{CSub}(B)$ in a natural way (any convex sublattice of A is the join of all intervals of this sublattice).

Using Lemmas 12 and 13 one can easily prove the following theorems and corollaries.

Theorem 1. *Let A and B be lattices. The following assertions are equivalent:*

(i) $\mathrm{Int}(A)$ *is isomorphic to* $\mathrm{Int}(B)$.

(ii) $\mathrm{CSub}(A)$ *is isomorphic to* $\mathrm{CSub}(B)$.

(iii) *There exist lattices A_1, A_2, B_1, B_2 such that $A = A_1 \times A_2$, $B = B_1 \times B_2$, A_1 is isomorphic to B_1 and A_2 is dually isomorphic to B_2.*

Theorem 2. *Let A be a lattice. The following two assertions are equivalent:*

(i) *Whenever B is a lattice and f an isomorphism of* $\mathrm{Int}(A)$ *onto* $\mathrm{Int}(B)$, *then the induced mapping f' is either an isomorphism or a dual isomorphism.*

(ii) *A is directly irreducible.*

Theorem 3. *For a lattice A, the following two conditions are equivalent:*

(i) *Whenever B is a lattice such that* $\mathrm{Int}(A)$ *is isomorphic to* $\mathrm{Int}(B)$, *then A is either isomorphic or dual isomorphic to B.*

(ii) *Either A is directly irreducible or whenever $A = A_1 \times A_2$ (both A_1 and A_2 have more than one element), then both A_1 and A_2 are self dual.*

Corollary 1. *The lattice* $\mathrm{Int}(A)$ *determines A up to isomorphism if and only if whenever $A = A_1 \times A_2$, then both A_1 and A_2 are self dual.*

Corollary 2 ([2]). *Let V be a variety of lattices. With any lattice A, the variety V contains all lattices B such that* Int (A) *is isomorphic to* Int (B) *if and only if V is self dual.*

Remark. V. I. Igošin proved in [3] that any finite lattice A having just one atom is determined by Int (A) up to isomorphism or dual isomorphism. This result follows immediately from Theorem 3.

References

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