Czechoslovak Mathematical Journal

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Fixed point theorems for multivalued mappings

Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 4, 533–542

Persistent URL: http://dml.cz/dmlcz/102047

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Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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(Received May 16, 1984)

In [7] M. Švec has proved versions of Schauder's Fixed Point Theorem for more general Banach spaces (spaces of n times differentiable functions with bounded derivatives of order $\leqq n$ defined on a finite or infinite interval). The aim of the present paper is to extend these theorems to the case of multifunctions. In the sequel we shall follow the method of M. Švec and, instead of Schauder's theorem, we shall use a modification of the fixed point theorem by Ky Fan.

Now we shall introduce the notation and give preliminary results which will be needed in the paper.

Let X and Y be topological spaces. Let us denote by 2^Y the family of all nonempty subsets of the space Y. A map F is called a *multifunction* if it assigns to each element of the space X exactly one set belonging to 2^Y. In a symbolic notation $F: X \to 2^Y$. From now on we shall use capital leters to denote multifunctions.

Definition 1. The mapping $F: X \to 2^Y$ is *upper semicontinuous at a point* $x \in X$, iff for an arbitrary neighbourhood $O_{F(x)}$ of the set-image $F(x)$ there exists such a neighbourhood O_x of the point x that $F(O_x) \subset O_{F(x)}$, where $F(O_x) = \bigcup_{z \in O_x} F(z)$.

Definition 2. The mapping $F: X \to 2^Y$ is *upper semicompact at a point* $x \in X$ iff the assumptions $x_n \in X$, $x_n \to x$, $y_n \in F(x_n)$ imply that there exists a subsequence of the sequence $\{y_n\}$, convergent to some $y \in F(x)$.

Definition 3. The map F is *upper semicompact at a point* $x \in X$ iff F is i) upper semicontinuous at the point x and ii) the set $F(x)$ is compact.

Lemma 1. (W. Sobieczek, P. Kowalski [6].) *Let X fulfil the first axiom of countability and let Y fulfil the second axiom of countability. Then Definitions* 2, 3 *of upper semicompactness are equivalent.*

Let A be a subset of E_n. Then $|A| = \sup \{|a| : a \in A\}$, and co$A$ will denote the convex hull of A. $\bar{A}$ will denote the closure of A. Further, cf (Y) will denote the set of all nonempty closed convex subsets of the topological vector space Y. Let $I \subset E$ be an arbitrary interval (bounded or not). By $B_m(I)$ $(m \geqq 0)$ we shall denote the Banach space of all continuous and bounded real functions having bounded derivatives to the

m-th order on I with the norm

$$|f|_{B_m} = \max_{0 \leq i \leq m} \{\sup_{x \in I} |f^{(i)}(x)|\} .$$

Definition 4. The sequence $f_k \in B_m(I)$ *quasi-converges* (q-*converges*) to $f \in B_m(I)$ iff $\lim_{k \to \infty} f_k^{(i)}(x) = f^{(i)}(x)$ for every $x \in I$ and $i = 1, 2, \ldots, m$. This will be denoted by $f_k \to^q f$.

Definition 5. A set $M \subset B_m(I)$ is said to be *q-compact in $B_m(I)$* iff every infinite subset of M contains a sequence q-convergent in $B_m(I)$.

Definition 6. A set $M \subset B_m(I)$ is said to be *uniformly bounded* iff there is $K > 0$ such that for every $f \in M$, $x \in I$ and $i = 0, 1, \ldots, m$ we have $|f^{(i)}(x)| \leqq K$.

Definition 7. A set $M \subset B_m(I)$ is said to be *equicontinuous* iff for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for $|x - x'| < \delta(\varepsilon)$, $x, x' \in I$ and each $f \in M$ we have

$$|f^{(i)}(x) - f^{(i)}(x')| < \varepsilon , \quad i = 0, 1, 2, \ldots, m .$$

Lemma 2. (M. Švec [7].) *Let $M \subset B_m(I)$ be a uniformly bounded and equicontinuous set of functions. Then* co M *and* $\overline{\text{co}\, M}$ *are also uniformly bounded and equicontinuous sets of functions.*

Lemma 3. (S. Mazur's theorem, L. Collatz [1], p. 351.) *If $M \subset B_m(I)$ is a relatively compact set then* co M *is a relatively compact set.*

Theorem 1. (Ky Fan [2].) *If B is a locally convex topological vector space and B_0 is a compact convex subset of B, then for every semicontinuous mapping T from B_0 into* cf (B_0), *there exists a point $x \in B_0$ such that $x \in T(x)$.*

Corollary 1. *Let M be a closed convex subset of $B_m(I)$. If $T: M \to$* cf (M) *is upper semicontinuous and $\overline{TM}$ is compact, then there is $x \in M$ such that $x \in Tx$.*

Proof. Let $M_0 = \overline{\text{co}\,(TM)}$. Then M_0 is convex and by Mazur's theorem it is also compact. On the other hand $TM \subset M$ and since M is closed and convex we have $\overline{\text{co}\,(TM)} \subset M$. Hence we have

$$TM_0 = T(\overline{\text{co}\,(TM)}) \subset TM \subset \overline{\text{co}\,(TM)} = M_0 \subset M .$$

The assertion of the corollary now follows from Ky Fan's theorem.

Lemma 4. *Let $M \subset B_m(I)$ be a uniformly bounded set of functions which are equicontinuous on every compact subinterval of I. Then M is q-compact.*

Proof. Let $\langle a_j, b_j \rangle$ be such a sequence of intervals that $a_{j+1} \leqq a_j$, $b_j \leqq b_{j+1}$, $\lim_{j \to \infty} a_j$ is equal to the left end point of I, $\lim_{j \to \infty} b_j$ is equal to the right end point of I. By the Ascoli Theorem, M is q-compact on every compact subinterval of I. Thus M contains a sequence $\{f_{jk}\}$, $k = 1, 2, \ldots$, which is q-convergent on $\langle a_j, b_j \rangle$ and

and $\{f_{j+1,k}\}$, $k = 1, 2, \ldots$, is a subsequence of $\{f_{jk}\}$, $k = 1, 2, \ldots$. Define $f_j = f_{jj}$. Then $\{f_n\}$ is the required sequence which is q-convergent on I.

Definition 8. The operator $T: B_m(I) \to 2^{B_m(I)}$ is *upper q-continuous* iff the assumptions

$$f_k(x) \to^q f(x)\,, \quad f_k(x)\,, \quad f(x) \in B_m(I)\,, \quad y_n \in T(f_n)$$

imply that there exists a subsequence of the sequence $\{y_n\}$ convergent to some $y \in T(f)$ (in the norm).

Corollary 2. *If T is an upper q-continuous operator, then T is upper semicompact.*

Lemma 5. *If the operator $T: B_m(I) \to 2^{B_m(I)}$ is upper q-continuous, then T is upper semicontinuous.*

Proof. Lemma 5 is a consequence of Corollary 2 and Lemma 1.

Theorem 2. *Assume that*

i) $g(t) \geqq 0$ *for* $t \geqq t_0$ *and* $\int_{t_0}^{\infty} g(t)\,\mathrm{d}t = \varrho < +\infty$,

ii) $B_{0,\varrho} = \{x(t) : x(t) \in C(\langle t_0, +\infty)) \text{ and } |x(t)| \leqq \varrho\}$,
$\tilde{B}_\varrho = \{x(t) : x(t) \text{ measurable on } \langle t_0, +\infty) \text{ and } |x(t)| \leqq \varrho\}$,

iii) $F: \tilde{B}_\varrho \to \mathrm{cf}(\tilde{B}_\varrho)$, $|F(x(t))| \leqq g(t)$ *and F is upper semicompact.*

Then the operator $TF: B_{0,\varrho} \to \mathrm{cf}(B_{0,\varrho})$ *defined by*

$$TF\,x(t) = \left\{z \in B_{0,\varrho} : z = \int_t^{\infty} y(s)\,\mathrm{d}s,\ y \in F(x) \text{ and } x \in B_{0,\varrho}\right\}$$

is upper q-continuous.

Proof. Let $x_k \to^q x$, $x_k, x \in B_{0,\varrho}$ and $z_k \in TFx_k$. We have to prove that there is a subsequence of $\{z_k\}$ converging to some $z \in TFx$ (in the norm of B_0).

Let $z_k = \int_t^{\infty} y_k(s)\,\mathrm{d}s$, $y_k \in F(x_k)$. Since F is upper semicompact and $\tilde{B}_\varrho$ is equipped with a.e. pointwise convergence, there is a subsequence $\{y_{1k}\}$ of $\{y_k\}$ which converges to some q $\in F(x)$ a.e. on $\langle t_0, +\infty)$.

Further,

$$|z_{1k}(t_1) - z_{1k}(t_2)| = \left|\int_{t_1}^{t_2} y_{1k}(s)\,\mathrm{d}s\right| \leqq \int_{t_1}^{t_2} g(s)\,\mathrm{d}s\,, \quad t_1 \leqq t_2\,.$$

Thus the functions z_{1k} are equicontinuous and uniformly bounded. By Lemma 4, there is a subsequence $\{z_{2k}\}$ of the sequence $\{z_{1k}\}$ which q-converges to some z on $\langle t_0, +\infty)$.

On the other hand,

$$z_{2k}(t) = \int_t^{\infty} y_{2k}(s)\,\mathrm{d}s\,.$$

This yields (by the Lebesgue Theorem)

$$z(t) = \int_t^\infty y(s)\, ds\,,$$

i.e. $z \in TFx$.

In virtue of

$$\sup |z_{2k}(t) - z(t)| \leqq \int_{t_0}^\infty |y_{2k}(s) - y(s)|\, ds\,,$$

the Lebesgue Theorem implies that $\{z_{2k}\}$ converges to z in the norm of B_0.

Theorem 3. *Let $J = \langle 0, +\infty)$ and let a mapping $F: J \times E_n \to \mathrm{cf}(E_n)$ satisfy the following conditions:*

(c_0) *$F(t, x)$ is a nonempty, compact and convex subset of E_n for each $(t, x) \in J \times E_n$,*

(c_1) *for every fixed $t \in J$ the function $F(t, x)$ is upper semicontinuous,*

(c_2) *for each measurable function $x: J \to E_n$, there exists a measurable function $f_x: J \to E_n$ such that*

$$f_x(t) \in F(t, x(t)) \quad a.e.\ on \quad J\,.$$

Further suppose that there exists $g: J \times J \to J$ such that

i) *$g(t, u)$ is monotone nondecreasing in u for each fixed $t \in J$,*

ii) *$\int_0^\infty g^{p'}(s, c)\, ds < +\infty$ for any constant $c > 0$; $p' > 1$,*

iii) *for each $x \in E_n$,*

$$|F(t, x)| \leqq g(t, |x|) \quad a.e.\ on\ J\,.$$

Given a function $x \in C^n(J)$, denote by $M(x)$ the set of all measurable functions $y: J \to E_n$ such that $y(t) \in F(t, x(t))$ a.e. on J.

Then the correspondence $x \to M(x)$ defines a bounded mapping of

$$B^n_{0,\varrho} = \{x(t) \in C^n(J) : |x(t)| \leqq \varrho\}$$

into $\mathrm{cf}(L^n_{p'}(J))$.

Proof. We have to show that for every $x \in B^n_{0,\varrho}$,

a) $M(x)$ is nonempty,

b) $M(x)$ is convex,

c) $M(x)$ is closed,

d) $M(x) \subset L^n_{p'}(J)$,

e) for every $\delta > 0$ there is a constant $K > 0$ such that $\sup_{t \in J} |x(t)| \leqq \delta$ implies $|y|_{p'} \leqq K$ for every $y \in M(x)$.

a), b) are trivial. e) follows from assumptions ii) and iii) and obviously implies d). Thus we have to prove c) only. Let $\{y_n\}$, $y_n \in M(x)$ be a sequence such that $|y_n - y|_{p'} \to 0$. By the Riesz theorem there is a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ such that $\{y_{1n}(t)\}$ converges a.e. on J to $y(t)$ as $n \to +\infty$.

On the other hand,

$$y_{1n}(t) \in F(t, x(t)) \quad \text{a.e. on} \quad J\,.$$

Because of (c_0),

$$y(t) \in F(t, x(t)) \quad \text{a.e. on} \quad J\,.$$

Thus $y \in M(x)$.

Definition 9. Let X and Y be normed linear spaces. The mapping $F: X \to 2^Y$ is *weakly upper q-continuous at a point* $x \in X$ iff the assumptions

$$x_n, x \in X\,, \quad x_n \to^q x\,, \quad y_n \in F(x_n)$$

imply that there is a subsequence of the sequence $\{y_n\}$ which weakly converges to some $y \in F(x)$.

Theorem 4. *Let the hypotheses of Theorem 3 be satisfied. Then the mapping* $M: B^n_{0,\varrho} \to \operatorname{cf}(L^n_{p'}(J))$ *is weakly upper q-continuous.*

Proof. Let $x_n \to^q x$, x_n, $x \in B^n_{0,\varrho}$ and $y_n \in M(x_n)$. Then

$$|y_n|_{p'} \leqq c = \left(\int_J g^{p'}(t, \varrho)\, \mathrm{d}s\right)^{1/p'}$$

and thus there is a subsequence $\{y_{1n}\}$ which weakly converges to some $y \in L^{n'}_p(J)$. We only have to prove that $y \in M(x)$. By the Banach-Saks Theorem, there is a subsequence $\{y_{2n}\}$ of the sequence $\{y_{1n}\}$ such that

$$\left|\frac{1}{n}\sum_{k=1}^{n} y_{2k} - y\right|_{p'} \to 0 \quad \text{for} \quad n \to +\infty\,.$$

Now, by the Riesz theorem, there is a sequence $\{\sigma_n\}$, $\sigma_n \in N$, $\sigma_n \geqq n$ such that

$$\frac{1}{\sigma_n}\sum_{k=1}^{\sigma_n} y_{2k}(t) \to y(t) \quad \text{a.e. on} \quad J \quad \text{for} \quad n \to +\infty\,.$$

On the other hand, by assumption (c_1), for almost every fixed $t \in J$ and any $\varepsilon > 0$ there is an integer $N(\varepsilon, t)$ such that $F(t, x_i(t)) \subset F(t, x(t)) + K_\varepsilon = \{u + v: u \in F(t, x(t)), |v| \leqq \varepsilon\}$ for $i \geqq N(\varepsilon, t)$.

Thus

$$y_{2k}(t) \in F(t, x(t)) + K_\varepsilon\,, \quad 2k \geqq N(\varepsilon, t)$$

and by convexity of $F(t, x(t))$,

$$\frac{1}{\sigma_n}\sum_{k=1}^{\sigma_n} y_{2k}(t) \in F(t, x(t)) + K_\varepsilon\,, \quad 2\sigma_n \geqq N(\varepsilon, t)$$

so that

$$y(t) \in F(t, x(t)) \quad \text{a.e. on} \quad J\,.$$

The proof is complete.

Theorem 5. *Let the hypotheses of Theorem 3 be satisfied and let* $h(t) \in L^n_p(J)$ *where* $1/p + 1/p' = 1$.

Then the operator $TM: B^n_{0,\varrho} \to \mathrm{cf}(B^n_0)$ *defined by*

$$TM\,x(t) = \left\{ z \in B^n_0 : z = \int_t^\infty h(s)\, y(s)\, \mathrm{d}s,\ y \in M(x) \right\}$$

is upper q-continuous.

Proof. Let $x_n \to^q x$, $x_n, x \in B^n_{0,\varrho}$ and $z_n \in TM\, x_n$. We shall show that there exists a subsequence of $\{z_n\}$ that uniformly converges to some $z \in TM\, x$.

Let

$$z_i(t) = \int_t^\infty h(s)\, y_i(s)\, \mathrm{d}s\,, \quad y_i \in M(x_i)\,.$$

Then, since M is weakly upper q-continuous, there is a subsequence $\{y_{1i}\}$ of $\{y_i\}$ which weakly converges to some $y \in M(x)$, i.e.

$$z_{1i}(t) \to z(t) = \int_t^\infty y(s)\, h(s)\, \mathrm{d}s \in TM\, x(t) \quad \text{a.e. on } J \text{ as } i \to +\infty\,.$$

Further, the functions $z_{1i}(t)$, $i = 1, 2, \ldots$, are uniformly bounded and by virtue of

$$|z_{1i}(t_1) - z_{1i}(t_2)| \leqq \int_{t_1}^{t_2} |h(s)\, y_{1i}(s)|\, \mathrm{d}s \leqq \int_{t_1}^{t_2} |h(s)|\, g(s, \varrho)\, \mathrm{d}s\,, \quad t_1 \leqq t_2$$

they are also equicontinuous on J. By the Ascoli theorem, as well as Cantor's diagonalization process, the sequence $\{z_{1i}\}$ contains a subsequence $\{z_{2i}\}$, which is uniformly convergent on every compact subinterval of J. This fact together with the inequality

$$|z_{2i}(t)| \leqq \int_t^\infty |h(s)|\, g(s, \varrho)\, \mathrm{d}s\,, \quad i = 1, 2, \ldots,$$

guarantees the uniform convergence of $\{z_{2i}\}$ on J.

Corollary 3. *Let the hypotheses of Theorem* 3 *be satisfied and let* $h(t) \in L^n_{p'}(J)$ *where* $1/p + 1/p' = 1$. *Then the operator* $TM: B^n_{0,\varrho} \to \mathrm{cf}(B^n_0)$, *defined by*

$$TM\,x(t) = \left\{ z \in B^n_0 : z = \int_0^t h(s)\, y(s)\, \mathrm{d}s,\ y \in M(x) \right\},$$

is upper q-continuous.

Proof. The proof proceeds analogously as that of Theorem 5.

Corollary 4. *Let mappings* $F_i : X \to 2^Y$, $i = 1, 2$ *be (weakly) upper q-continuous at a point* $x \in X$. *Then the mappings* $-F_i$ $(i = 1, 2)$, $F_1 + F_2$ *are (weakly) upper q-continuous.*

Theorem 6. *Let the hypotheses of Theorem 3 be satisfied and let* $h_1(t), h_2(t) \in L^n_p(J)$, *where* $1/p + 1/p' = 1$. *Then the operator* $TM: B^n_{0,\varrho} \to \mathrm{cf}(B^n_0)$ *defined by*

$$TM\, x(t) = \left\{ z \in B^n_0 \colon z = \int_0^t h_1(s)\, y(s)\, \mathrm{d}s - \int_t^\infty h_2(s)\, y(s)\, \mathrm{d}s,\ y \in M(x) \right\}$$

is upper q-continuous.

Theorem 7. *Let the hypotheses of Theorem 3 be satisfied and let D be a Banach space. Suppose that* $T: L^n_{p'}(J) \to D$ *is a compact linear operator.*

Then the operator TM defined by

$$TM\, x = \{ z \in D \colon z = Ty \ \text{and}\ y \in Mx \}$$

maps $B^n_{0,\varrho}$ *into cf (D) and is upper q-continuous.*

Proof. First we shall prove that the operator TM is upper q-continuous.

Let $x_n \to^q x$, $x_n, x \in B^n_{0,\varrho}$ and $z_n \in TM\, x_n$. We have to show that there is a subsequence of the sequence $\{z_n\}$ that converges (in the norm of D) to some $z \in TM\, x$.

Let $z_i = Ty_i$, $y_i \in Mx_i$. Since M is weakly upper q-continuous, there is a subsequence $\{y_{1i}\}$ of the sequence $\{y_i\}$ which weakly converges to some $y \in Mx$. Since $\{y_{1i}\}$ is bounded and T is a compact linear operator there is a subsequence $\{y_{2i}\}$ of the sequence $\{y_{1i}\}$ such that $Ty_{2i} \to z \in D$ as $i \to +\infty$. We shall show that $z = Ty \in TM\, x$.

Because $\{y_{1i}\}$ weakly converges to y we have that also $\{y_{2i}\}$ weakly converges to y. By the Banach-Saks theorem there is a subsequence $\{y_{3i}\}$ of the sequence $\{y_{2i}\}$ such that

$$\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i} \to y$$

as $i \to +\infty$ in the norm of $L^n_{p'}(J)$.

Since T is compact and linear (hence T is continuous), we have

$$(*) \qquad T\left(\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i}\right) \to Ty \quad \text{as} \quad i \to +\infty\,.$$

On the other hand, since $Ty_{3i} \to z \in D$ and T is linear we have

$$(**) \qquad z = \lim_{i \to +\infty} Ty_{3i} = \lim_{i \to +\infty} \frac{Ty_{31} + Ty_{32} + \ldots + Ty_{3i}}{i} = \lim_{i \to +\infty} T\left(\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i}\right).$$

By $(*)$ and $(**)$ we have that $z = Ty \in TM\, x$. Thus the operator TM is upper q-continuous. From this we conclude that $TM\, x$ is closed. Further, Mx is a convex set and T is a linear operator. Thus $TM\, x$ is also a convex set.

Lemma 6. *Suppose that $M \subset B_m$ is a q-compact set and T is a upper q-continuous operator. Then $\overline{TM}$ is compact.*

Proof. Let $\{g_k\}$ be an infinite sequence from TM and let $\{f_k\}$ be such a sequence that $f_k \in M$ and $g_k \in Tf_k$, $k = 1, 2, \ldots$. Since M is a q-compact set, there is a subsequence $\{f_{1k}\}$ of the sequence $\{f_k\}$ and a function $f \in B_m$ such that $f_{1k} \to^q f$. Further, T is an upper q-continuous operator. Hence there is a subsequence $\{g_{2k}\}$ of the sequence $\{g_{1k}\}$ such that

$$\lim_{k \to +\infty} g_{2k}(t) = g(t) \in Tf(t) \subset TM \quad \text{in the norm.}$$

Theorem 8. *Suppose that $M \subset B_m$ is a nonempty, convex, closed and q-compact set and $T: M \to \mathrm{cf}(M)$ is an upper q-continuous operator. Then there is a point $x \in M$ such that $x \in Tx$.*

Proof. By the hypotheses all assumptions of Lemma 6 are fulfilled. Thus $\overline{TM}$ is compact. Further, by Lemma 5, T is an upper semicontinuous operator. Now the existence of at least one point $x \in M$ such that $x \in Tx$ follows from Corollary 1.

Theorem 9. *Suppose that $M \subset B_m$ is a nonempty, convex and closed set and $T: M \to \mathrm{cf}(M)$ is an upper q-continuous operator. Further, let TM be a uniformly bounded set of functions which are equicontinuous on every compact subinterval of J. Then there is a point $x \in M$ such that $x \in Tx$.*

Proof. Let $M_0 = \overline{\mathrm{co}(TM)}$. Then M_0 is convex and closed. By Lemma 2, M_0 is a uniformly bounded set of functions which are equicontinuous on every compact subinterval of J. Thus, by Lemma 4, M_0 is a q-compact set. Furthermore

$$TM_0 \subset TM \subset \overline{\mathrm{co}(TM)} = M_0 \subset M .$$

Thus, by Theorem 8, there is a point $x \in M_0 \subset M$ such that $x \in Tx$.

Definition 10. An operator $T: B_m(I) \to 2^{B_m(I)}$ fulfils the *condition* (a) iff

$$f_k \to^q f\,, \quad f_k, f \in B_m(I)\,, \quad \{|f_k|\} \ \text{is bounded}\,, \quad g_k \in Tf_k$$

implies that there is a subsequence $\{g_{1k}\}$ of the sequence $\{g_k\}$ such that $g_{1k} \to g \in Tf$ in the norm.

The condition that $\{|f_k|\}$ is bounded does not follow from the q-convergence of $\{f_k\}$ to f as the following example shows.

Example. Define a sequence $\{f_k\}$ of functions on $\langle 0, +\infty)$ by

$$f_k(x) = \begin{cases} 0\,, & x \in \langle 0, k-1) \cup \langle k, +\infty) \\ k(x - k + 1)\,, & x \in \langle k-1, k - 1/2) \\ -k(x - k)\,, & x \in \langle k - 1/2, k)\,. \end{cases}$$

Then $f_k \to^q 0$ as $k \to +\infty$ and $f_k = \frac{1}{2}k \to +\infty$ as $k \to +\infty$.

Corollary 5. *If the operator* $T\colon B_m(I) \to 2^{B_m(I)}$ *fulfils the condition* (a), *then it is upper semicompact and upper semicontinuous. If the operator* T *is upper* q-*continuous, then it fulfils the condition* (a).

Theorem 10. *Suppose that* $M \subset B_m(I)$ *is a nonempty, convex and closed set and the operator* $T\colon M \to \operatorname{cf}(M)$ *fulfils the condition* (a).

Further, let TM *be a uniformly bounded set of functions which are equicontinuous on every compact subinterval of* I. *Then there is a point* $x \in M$ *such that* $x \in Tx$.

Proof. Let $M_0 = \overline{\operatorname{co}(TM)}$. Then, as in the proof of Theorem 9, we have that M_0 is a nonempty, convex, closed, uniformly bounded and q-compact set such that $TM_0 \subset M_0 \subset M$. By Corollary 5 the operator T is upper semicontinuous. Thus we only have to prove that TM_0 is compact.

Let $\{g_k\}$ be an infinite sequence from TM_0 and let $\{f_k\}$ be such a sequence that $g_k \in Tf_k$ and $f_k \in M_0$. Since M_0 is a q-compact set there is a subsequence $\{f_{1k}\}$ of the sequence $\{f_k\}$ and a function $f \in B_m$ such that $f_{1k} \to^q f$. Further, because M_0 is uniformly bounded, so is $\{f_{1k}\}$ and $g_{1k} \in Tf_{1k}$. Now, by the condition (a), there is a subsequence $\{g_{2k}\}$ of the sequence $\{g_{1k}\}$ such that $g_{2k} \to g \in Tf$ in the norm. The proof of the Theorem 10 is complete.

Definition 11. A set $M \subset B_m(I)$ is said to be *q-closed* iff $f_\kappa \in M$, $f_k \to^q f$ implies $f \in M$.

Theorem 11. *Suppose that* $M \subset B_m(I)$ *is a nonempty, convex and* q-*closed set and* $T\colon M \to \operatorname{cf}(M)$ *is an upper* q-*continuous operator such that* TM *is a uniformly bounded set of functions which are equicontinuous on every compact subinterval of* I. *Then there is a point* $x \in M$ *such that* $x \in Tx$.

Proof. It is easy to see that each q-closed set is closed. Thus we have

$$TM_0 \subset M_0 = \overline{\operatorname{co}(TM)} \subset M\,,$$

M_0 is convex, closed and, as in the proof of Theorem 9, we have that TM_0 is a q-compact set. By Lemma 5, operator T is upper semicontinuous. In order to apply Corollary 1, we have to prove that $\overline{TM_0}$ is compact.

Let $\{g_k\}$ be an infinite sequence of functions $g_k \in TM_0$ and let $\{f_k\}$ be a sequence of functions such that $g_k \in Tf_k$ and $f_k \in M_0$. Since M_0 is a q-compact set, there is a subsequence $\{f_{1k}\}$ of the sequence $\{f_k\}$ and a function $f \in B_m$ such that $f_{1k} \to^q f$. Because the operator T is upper q-continuous, there is a subsequence $\{g_{2k}\}$ of the sequence $\{g_{1k}\}$ such that $g_{2k} \to g \in Tf$ as $k \to +\infty$ in the norm. This completes the proof of Theorem 11.

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