Czechoslovak Mathematical Journal

Katherine Johnston

Non-modular congruence lattices of Rees matrix semigroups

Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 3, 429–443

Persistent URL: http://dml.cz/dmlcz/102033

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

NON-MODULAR CONGRUENCE LATTICES OF REES MATRIX SEMIGROUPS

KATHERINE JOHNSTON, Atlanta

(Received April 18, 1984)

Congruence lattices of semigroups have been studied extensively (see the recent survey article [7]). For the class of completely (0 –) simple semigroups, Lallement [6] has given a characterization of the congruences in terms of the Rees matrix representation. In [4], the author used this characterization to determine necessary and sufficient conditions for a modular lattice to be the congruence lattice of a Rees matrix semigroup. In addition, it was shown there that these conditions were not sufficient in the non-modular case.

As was pointed out by Jones in [5], even a characterization of arbitrary congruence lattices of completely *simple* semigroups is unknown. One hindrance to their study is lack of knowledge of congruence lattices of groups. In this paper we will present some necessary conditions on a (non-modular) lattice of congruences of a Rees matrix semigroup which do not follow from those given in [4], and give some sufficient conditions which apply to certain classes of Rees matrix semigroups.

1. INTRODUCTION

The terminology and notation will mainly be that of [4]. Additional information on semigroups may be found in [1, 3], and on lattice theory in [2].

For any set A, we will denote by $\Pi(A)$ the lattice of equivalence relations on A. Also, ε_A and ω_A will denote the least and greatest elements of $\Pi(A)$, respectively. Where there is no chance of confusion, we will omit the subscript. If G is a group, $\mathcal{N}(G)$ will denote the lattice of normal subgroups of G. For any lattice L, if $x, y \in L$ and $x \geqq y$, let $x/y = \{z \in L: x \geqq z \geqq y\}$.

Let $S = \mathcal{M}^0(I, G, M; P)$ be a regular Rees matrix semigroup, and define the equivalence relation r on I and the equivalence relation π on M by:

(1.1) $i\ r\ j$ if and only if for every $\mu \in M$, $p_{\mu i} = 0$ if and only if $p_{\mu j} = 0$,

(1.2) $\mu\ \pi\ \nu$ if and only if for every $i \in I$, $p_{\mu i} = 0$ if and only if $p_{\nu i} = 0$.

The sandwich matrix P is said to be *normalized* if for each r-class K there is a $\mu \in M$ such that $p_{\mu i} = e$ for all $i \in K$; and for each π-class Δ there is a $j \in I$ such that $p_{\nu j} = e$ for all $\nu \in \Delta$. The following theorem of Tamura [9] will be useful later.

Theorem 1.1. *A semigroup is completely* 0*-simple (completely simple) if and only if it is isomorphic to a Rees matrix semigroup with normalized sandwich matrix.*

Theorem 1.1 allows us to simplify somewhat Lallement's characterization of congruences on a Rees matrix semigroup. In what follows, we will assume that the sandwich matrix of any Rees matrix semigroup is normalized.

Definition. Let s be an equivalence relation on I, N a normal subgroup of G, and ϱ an equivalence relation on M. Then (s, N, ϱ) is an *admissible triple* on S if the following conditions are satisfied:

A. If $i\, s\, j$, then for all $\mu \in M$,
   1. $p_{\mu i} \neq 0$ if and only if $p_{\mu j} \neq 0$;
   2. If $p_{\mu i} \neq 0$, then $p_{\mu i} p_{\mu j}^{-1} \in N$.

B. If $\mu\, \varrho\, \nu$, then for all $i \in I$,
   1. $p_{\mu i} \neq 0$ if and only if $p_{\nu i} \neq 0$;
   2. if $p_{\mu i} \neq 0$, then $p_{\mu i} p_{\nu i}^{-1} \in N$.

We will call a congruence on S *proper* if it is not the universal congruence. Let $C(S)$ denote the lattice of congruences of S, and $C'(S) = C(S) \setminus \{w_s\}$. The following result due to Lallement [6] shows the relationship between admissible triples and proper congruences on S.

Theorem 1.2. *Let $S = \mathscr{M}^0(I, G, M; P)$ with P normalized. If (s, N, ϱ) is an admissible triple, then the relation $\theta = \theta(s, N, \varrho)$ on S defined by $0\,\theta\,0$,*

$$(i, a, \mu)\, \theta\, (j, b, \nu) \quad \text{if} \quad a \neq 0, \quad b \neq 0$$
$$i\, s\, j, \quad \mu\, \varrho\, \nu, \quad \textit{and} \quad ab^{-1} \in N$$

is a proper congruence on S. Conversely, every proper congruence on S can be written in the form $\theta(s, N, \varrho)$ for some admissible triple (s, N, ϱ).

The following results will be needed in the sequel.

Lemma 1.3. [4; Lemma 4] *Let I and M be sets, and G a group. Let L be a subset of $\Pi(I) \times \mathscr{N}(G) \times \Pi(M)$ satisfying the following:*

1. *If $\{(r_\alpha, N_\alpha, \pi_\alpha)\}_{\alpha \in A} \subseteqq L$, then $\bigwedge(r_\alpha, N_\alpha, \pi_\alpha) = (\bigwedge r_\alpha, \bigwedge N_\alpha, \bigwedge \pi_\alpha) \in L$, and dually.*
2. *If $(s, K\ \varrho) \in L\ \ s' \leqq s$, $K' \geqq K$, $\varrho' \leqq \varrho$, then $(s', K', \varrho') \in L$.*
3. *$(\varepsilon, \{e\}, \varepsilon) \in L$.*

Then L is a subdirect product of $r/\varepsilon \times \mathscr{N}(G) \times \pi/\varepsilon$ for some $r \in \Pi(I)$, $\pi \in \Pi(M)$.

Theorem 1.4. [4; Theorem 5] *If $S = \mathscr{M}^0(I, G, M; P)$, then $C'(S)$ is a subdirect product of $r/\varepsilon \times \mathscr{N}(G) \times \pi/\varepsilon$ for some $r \in \Pi(I)$, $\pi \in \Pi(M)$.*

If L is a lattice satisfying the hypothesis of Lemma 1.3, we will say that L satisfies $(*)$. For the remainder of this section, we will assume that L is a lattice satisfying $(*)$.

Lemma 1.5. [4; Lemma 6] *If $s \in r/\varepsilon$, $\varrho \in \pi/\varepsilon$, then there is a unique minimal normal subgroup N of G such that $(s, N, \varrho) \in L$.*

We shall denote by $N(s, \varrho)$ the minimal normal subgroup of G associated with $s \in r/\varepsilon$ and $\varrho \in \pi/\varepsilon$. Where there is no danger of confusion, we will use $N_s = N(s, \varepsilon)$, and $N_\varrho = N(\varepsilon, \varrho)$.

Lemma 1.6. [4; Lemma 7] *If $s_i \leqq r$ and $\varrho_i \leqq \pi$ for all $i \in I$, then $N(\bigvee_{i \in I} s_i, \bigvee_{i \in I} \varrho_i) = \bigvee_{i \in I} N(s_i, \varrho_i)$.*

If A is any set and $S \subseteq A$, we will define $[S] \in \Pi(A)$ by

$$i[S]j \quad \text{iff} \quad i, j \in S\,, \quad \text{or} \quad i = j\,.$$

By $[i_1, i_2, \ldots, i_n]$ we will mean $[\{i_1, i_2, \ldots, i_n\}]$.

Lemma 1.7. [4; Lemma 10] *If i, j and k are all in the same r-class, then $N_{[i,j]} \leqq N_{[i,k]} N_{[j,k]}$.*

2. SUFFICIENT CONDITIONS

Before exhibiting sufficient conditions for a lattice to be the proper congruence lattice of a Rees matrix semigroup, we present here some technical results which will make subsequent proofs easier.

Theorem 2.1. *Let $S = \mathscr{M}^0(I, G, M; P)$. Then there is a Rees matrix semigroup $T = \mathscr{M}^0(I', G, M'; Q)$ such that $C'(S) \cong C'(T)$, and $C'(T) \leqq r'/\varepsilon \times \mathscr{N}(G) \times \varepsilon/\varepsilon$, where $r \in \Pi(I')$.*

Proof. Assume I and M are disjoint, and without loss of generality $|I| > 1$, since $C'(\mathscr{M}^0(I, G, M; P)) \cong C'(\mathscr{M}^0(M, G, I; P^t))$ by theorem 1.2. Let B be a set disjoint from both I and M having cardinality $I \cup M$, and let $f\colon B \to I \cup M$ be a one-to-one correspondence. Let $I' = I \cup M \cup B$, $M' = I \cup M$. Set $T = \mathscr{M}^0(I', G, M'; Q)$ with the entries of $Q\colon M' \times I' \to G^0$ as follows:

$$q_{\beta\alpha} = \begin{cases} 0 & \beta \in I\,, \quad \alpha \in I \\ P_{\alpha\beta} & \beta \in I\,, \quad \alpha \in M \\ P_{\beta\alpha} & \beta \in M\,, \quad \alpha \in I \\ 0 & \beta \in M\,, \quad \alpha \in M \\ 0 & \alpha \in B\,, \quad f(\beta) = \alpha \\ e & \beta \in B\,, \quad f(\beta) \neq \alpha \end{cases}$$

Suppose $C'(S) \leqq r/\varepsilon \times \mathscr{N}(G) \times \pi/\varepsilon$ for $r \in \Pi(I)$, $\pi \in \Pi(M)$. For $s \in r/\varepsilon$, define $\bar{s}$ on I' by

$$i\,\bar{s}\,j \quad \text{iff} \quad i, j \in I \quad \text{and} \quad i\,s\,j\,, \quad \text{or} \quad i = j\,.$$

Similarly, for $\varrho \in \pi/\varepsilon$, define $\bar{\varrho}$ on I' by

$$\mu\, \bar{\varrho}\, \nu \quad \text{iff} \quad \mu, \nu \in M \quad \text{and} \quad \mu\, \varrho\, \nu\,, \quad \text{or} \quad \mu = \nu\,.$$

Clearly, $\bar{s}$ and $\bar{\varrho}$ are equivalence relations on I'.

Let $(s, K, \varrho) \in C'(S)$. Then $(\bar{s} \vee \bar{\varrho}, K, \varepsilon) \in \Pi(I') \times \mathcal{N}(G) \times \Pi(M')$. The proof that $(\bar{s} \vee \bar{\varrho}, K, \varepsilon)$ is an admissible triple for T is straightforward and will be omitted. It is routine to verify that the matrix Q is normalized.

Now define Φ: $C'(S) \to C'(T)$ by $(s, K, \varrho)\, \Phi = (\bar{s} \vee \bar{\varrho}, K, \varepsilon)$.

First we will show that if $(t, K, \sigma) \in C'(T)$, then $\sigma = \varepsilon$. For if $\sigma \neq \varepsilon$, there are μ, $\nu \in M'$ so that $\mu\, \sigma\, \nu$ and $\mu \neq \nu$. Pick $i \in B$ such that $f(i) = \nu$. Then $q_{\mu i} = 0$, and since $\nu \neq \mu$ and f is one-to-one, $f(i) \neq \nu$, yielding $q_{\nu i} = e$. This contradicts the admissibility conditions for (t, K, σ), hence we must have $\sigma = \varepsilon$.

Next we show that $t = \bar{s} \vee \bar{\varrho}$ for some $s \in r/\varepsilon$, $\varrho \in \varrho/\varepsilon$. Define $s \in \Pi(I)$, $\varrho \in \Pi(M)$ by $s = t_{|I}$, $\varrho = t_{|M}$. Clearly, $\bar{s} \leqq t$ and $\bar{\varrho} \leqq t$, so $\bar{s} \vee \bar{\varrho} \leqq t$. Suppose $\alpha\, t\, \beta$, for $\alpha, \beta \in I'$. If $\alpha = \beta$, then $\alpha(\bar{s} \vee \bar{\varrho})\, \beta$, so assume $\alpha \neq \beta$. We have several cases.

Case 1. One of α or β is in B, say $\alpha \in B$. Then $f(\alpha) = \sigma \in I \cup M$, so $q_{\sigma\alpha} = 0$. By the admissibility conditions, $q_{\sigma\beta} = 0$, and hence we must have $\beta, \sigma \in I$, or $\beta, \sigma \in M$. If $\beta, \sigma \in I$, pick $\gamma \in I$ such that $\gamma \neq \sigma$. Then $q_{\gamma\alpha} = e$, $q_{\gamma\beta} = 0$, which contradicts the admissibility conditions. In a similar manner, we can eliminate the case $\beta, \sigma \in M$. Hence neither of α, β is in B, and thus $\alpha, \beta \in I \cup M$.

Case 2. $\alpha \in I$, $\beta \in M$. In this case there exists a $\sigma \in I$ such that $p_{\beta\sigma} \neq 0$. Thus $q_{\sigma\alpha} = 0$, and $q_{\sigma\beta} = p_{\beta\sigma} \neq 0$, which contradicts the admissibility conditions.

Case 3. $\alpha, \beta \in I$. Then $\alpha\,(t_{|I})\, \beta$, or $\alpha\, s\, \beta$, which implies $\alpha\, \bar{s}\, \beta$, and therefore $\alpha\,(\bar{s} \vee \bar{\varrho})\, \beta$, as desired.

Case 4. $\alpha, \beta \in M$. Then $\alpha\,(t_{|M})\, \beta$, hence $\alpha\, \varrho\, \beta$ and so $\alpha\,(\bar{s} \vee \bar{\varrho})\, \beta$.

In all cases, $\alpha\, t\, \beta$ implies $\alpha\,(\bar{s} \vee \bar{\varrho})\, \beta$, which shows $t \leqq \bar{s} \vee \bar{\varrho}$, and thus $t = \bar{s} \vee \bar{\varrho}$. With this result we now have $(t, K, \sigma) = (\bar{s} \vee \bar{\varrho}, K, \varepsilon) = (s, K, \varrho)\, \phi$, proving that ϕ is onto.

It is easy to see that ϕ is one-to-one, and that ϕ is order-preserving. We wish to shown that ϕ^{-1} is order-preserving, so assume that $(\bar{s} \vee \bar{\varrho}, K, \varepsilon) \leqq (\bar{t} \vee \bar{\eta}, H, \varepsilon)$. It follows that $K \leqq H$ and $\bar{s} \vee \bar{\varrho} \leqq \bar{t} \vee \bar{\eta}$.

Suppose that $i, j \in I$ and $i\, s\, j$. Then $i\,(\bar{s} \vee \bar{\varrho})\, j$, and hence $i\,(\bar{t} \vee \bar{\varrho})\, j$. There must be a sequence $i = i_0, i_1, \ldots, i_n = j$ of elements of I' such that $i_0\, \bar{t}\, i_1\, \bar{\eta}\, i_2 \ldots \bar{t}\, i_n$. Since $i_0 = i \in I$ and $i_0\, \bar{t}\, i_1$, we must have $i_0\, t\, i_1$, and $i_1 \in I$. From $i_1\, \bar{\eta}\, i_2$ and $i_1 \in I$, we get $i_1 = i_2$. Continuing in this manner, we see that $i_0\, t\, i_1\, t\, i_2 \ldots t\, i_n$, and hence $i\, t\, j$. Thus $s \leqq t$, and in a similar manner, $\varrho \leqq \eta$, giving us $(s, K, \varrho) \leqq (t, H, \eta)$ and ϕ^{-1} is order-preserving. It is now evident that ϕ is an isomorphism.

Let I be any set, G a group, and $r \in \Pi(I)$. We will say the function α: $r/\varepsilon \to \mathcal{N}(G)$ *represents* the lattice $L \leqq r/\varepsilon \times \mathcal{N}(G)$ provided that $(s, K) \in L$ if and only if $s\alpha \leqq K$.

Theorem 2.2. *A lattice satisfies* $(*)$ *if and only if it is isomorphic to a lattice L represented by a function* $\alpha: r/\varepsilon \to \mathcal{N}(G)$ *having the properties*

(2.1) α *is a complete join homomorphism*

(2.2) $\varepsilon\alpha = \{e\}$.

Proof. Let $L' \leqq \Pi(I) \times \mathcal{N}(G) \times \Pi(M)$ be a lattice satisfying $(*)$. By Lemma 1.3, L is a subdirect product of $s/\varepsilon \times \mathcal{N}(G) \times \sigma/\varepsilon$ for some $s \in \Pi(I)$, $\sigma \in \Pi(M)$.

For any $t \in s/\varepsilon$, define the relation $\bar{t}$ on $I \cup M$ by

$$i \, \bar{t} \, j \quad \text{iff} \quad i, j \in I \quad \text{and} \quad i \, t \, j \,, \quad \text{or} \quad i = j$$

In a similar manner, for $\tau \in \sigma/\varepsilon$, define $\bar{\tau}$ on $I \cup M$ by

$$\mu \, \bar{\tau} \, v \quad \text{iff} \quad \mu, v \in M \quad \text{and} \quad \mu \, \tau \, v \,, \quad \text{or} \quad \mu = v \,.$$

Let $r \in \Pi(I \cup M)$ be the relation $r = \bar{s} \vee \bar{\sigma}$. By the same argument as in the proof of theorem 2.1, we have $L' \cong L \leqq r/\varepsilon \times \mathcal{N}(G)$ under the mapping $(t, K, \eta) \to \to (\bar{t} \vee \bar{\eta}, K)$.

Now define $\alpha: r/\varepsilon \to \mathcal{N}(G)$ as follows: $t\alpha = N(v, \varrho)$, where $\bar{v} \vee \bar{\varrho} = t$ for $v \in s/\varepsilon$, $\varrho \in \sigma/\varepsilon$. That α is a complete join homomorphism follows directly from lemma 1.6. Condition (3) of $(*)$ insures that $\varepsilon\alpha = e$. That α represents L is an immediate consequence of $(*)$ and the definition of $N(v, \varrho)$.

To prove the inverse implication, let L be a lattice represented by $\alpha: r/\varepsilon \to \mathcal{N}(G)$, $r \in \Pi(I)$, with α satisfying (2.1) and (2.2). Let $L' = L \times \{e\}$. We will show L' satisfies $(*)$.

Note first that since α is a join homomorphism, α is order-preserving. Suppose that $\{(r_\beta, N_\beta, \varepsilon)\}_{\beta \in B} \subseteq L'$. Since L' is a sub/lattice of a complete lattice, it suffices to show that $(\bigwedge r_\beta, \bigwedge N_\beta, \varepsilon)$ and $(\bigvee r_\beta, \bigvee N_\beta, \varepsilon)$ are in L'. Since $\bigwedge r_\beta \leqq r_\beta$ for each $\beta \in B$, and α is order-preserving, $(\bigwedge r_\beta)\,\alpha \leqq r_\beta\alpha$ for all $\beta \in B$. But $r_\beta\alpha \leqq N_\beta$ because α represents L, whence $(\bigwedge r_\beta)\,\alpha \leqq N_\beta$ for all β, and therefore $(\bigwedge r_\beta)\,\alpha \leqq \bigwedge N_\beta$. By the definition of α representing L, this implies $(\bigwedge r_\beta, \bigwedge N_\beta) \in L$, and $(\bigwedge r_\beta, \bigwedge N_\beta, \varepsilon) \in L'$ as desired.

Now consider $(\bigvee r_\beta, \bigvee N_\beta, \varepsilon)$. Since α represents L, we know $r_\beta\alpha \leqq N$ for each $\beta \in B$, and hence $(r_\beta\alpha) \leqq \bigvee N_\beta$. The fact that is a complete join homomorphism implies $\bigvee(r_\beta\alpha) = (\bigvee r_\beta)\,\alpha \leqq \bigvee N_\beta$, which gives us $(\bigvee r_\beta, \bigvee N_\beta) \in L$, and $(\bigvee r_\beta, \bigvee N_\beta, \varepsilon) \in \in L'$. Thus part (1) of $(*)$ is satisfied.

Suppose $(s, K, \varepsilon) \in L'$, $t \leqq s$, and $H \geqq K$. As was noted earlier, α is order-preserving, so $t\alpha \leqq s\alpha \leqq K \leqq H$, and since α represents L, $(t, H) \in L$. It follows that $(t, H, \varepsilon) \in \in L'$, and part (2) of $(*)$ is proved. Part (3) of $(*)$ is immediate from condition (2.2).

The next theorem shows it suffices to know $[i, j]\,\alpha$ for each $[i, j] \leqq r$ to get a complete join homomorphism.

Theorem 2.3. *Suppose the function* $\alpha_0: \{[i, j]: i \, r \, j\} \to \mathcal{N}(G)$, $r \in \Pi(I)$, *has the property that if* $[i, k] \leqq r$ *and* $[j, k] \leqq r$,

$$[i, j]\,\alpha_0 \leqq [i, k]\,\alpha_0 \vee [j, k]\,\alpha_0 \,. \tag{2.3}$$

Then α_0 *can be extended to a complete join-homomorphism on* r/ε.

Proof. For $s \in r/\varepsilon$, suppose $s = \bigvee_{\beta \in B} [i_\beta, j_\beta]$. Define $s\alpha$ to be $\bigvee_{\beta \in B} [i_\beta, j_\beta] \alpha_0$. We must show α is well-defined, so assume also $s = \bigvee_{\gamma \in C} [i_\gamma, j_\gamma]$. For each $\beta \in B$, $[i_\beta, j_\beta] \leqq \leqq \bigvee_{\gamma \in C} [i_\gamma, j_\gamma]$. There must be elements $i_\beta = i_0, i_1, \ldots, i_n = j_\beta$ of I such that $i[i_{\gamma_k}, j_{\gamma_k}]\, i_{k+1}$ for some $\gamma_k \in C$ and each $k \in n$. Without loss of generality we may assume $i_k \neq i_{k+1}$. Thus $[i_k, i_{k+1}] = [i_{\gamma_k}, j_{\gamma_k}]$.

Now we may calculate using (2.3),

$$[i_\beta, j_\beta]\, \alpha_0 = [i_0, i_n]\, \alpha_0 \leqq [i_0, i_1]\, \alpha_0 \vee [i_1, i_n]\, \alpha_0 \leqq$$
$$\leqq [i_0, i_1]\, \alpha_0 \vee ([i_1, i_2]\, \alpha_0 \vee [i_2, i_n]\, \alpha_0 \leqq \ldots \leqq [i_0, i_1]\, \alpha_0 \vee$$
$$\vee [i_1, i_2]\, \alpha_0 \vee \ldots \vee [i_{n-1}, i_n]\, \alpha_0 \leqq \bigvee_{\gamma \in C} [i_\gamma, j_\gamma]\, \alpha_0 \,.$$

Since this inequality holds for each $\beta \in B$, it follows that $\bigvee_{\beta \in B} [i_\beta, j_\beta]\, \alpha_0 \leqq \bigvee_{\gamma \in C} [i_\gamma, j_\gamma]\, \alpha_0$, and by symmetry we get equality. Hence α is well defined.

That α is a complete join-homomorphism now follows immediately from definition of α.

The next theorem shows that if a lattice is a certain type of sublattice of a proper congruence lattice in a class of Rees matrix semigroups, then it is itself a proper congruence lattice. Recall that a group is Hamiltonian if all of its subgroups are normal. In particular, abelian groups are Hamiltonian.

Theorem 2.4. *If L is the lattice of proper congruences of a Rees matrix semigroup over a Hamiltonian group, then any principal ideal of L is also the lattice of proper congruences of a Rees matrix semigroup.*

Proof. From Theorem 2.1, we may assume $L = C'(S)$ where $S = \mathscr{M}^0(I, G, M; P)$, and L is a complete sublattice of $r/\varepsilon \times \mathscr{N}(G) \times \varepsilon/\varepsilon$ for some $r \in \Pi(I)$. Let $(s, K, \varepsilon) \in \in L$, and we will show $(s, K, \varepsilon)/(\varepsilon, \{e\}, \varepsilon)$ is isomorphic to a proper congruence lattice of a Rees matrix semigroup.

Index the set of s-classes by A where $M \cap A = \emptyset$ and $I \cap A = \emptyset$. Pick a representative from each s-class, and for $i \in I$, let $f(i)$ be the representative from the s-class containing i. Define a matrix $A\colon M \cup A \times I \cup A \to G^0$ as follows:

$$q_{\mu i} = \begin{cases} p_{\mu i} p_{\mu f(i)}^{-1} & \mu \in M\,, \quad i \in I \\ e & \mu \in A\,, \quad i \in s_\mu \\ 0 & \mu \in A\,, \quad i \notin s_\mu \\ e & \mu \in A\,, \quad i \in A\,, \quad \mu \neq i \\ 0 & \mu \in A\,, \quad i \in A\,, \quad \mu = i \\ e & \mu \in M\,, \quad i \in A\,. \end{cases}$$

Note that Q is normalized.

We claim that all the non-zero entries of Q are in K. The only case we need to check is $\mu \in M$ and $i \in I$. If $q_{\mu i} \neq 0$, then $p_{\mu i} \neq 0$ and $p_{\mu f(i)} \neq 0$. Since $i\, s\, f(i)$, the admissibility of (s, K, ε) gives us $p_{\mu i} p_{\mu f(i)}^{-1} = q_{\mu i} \in K$.

Define the Rees matrix semigroup $T = \mathcal{M}^0(I \cup A, K, M \cup A; Q)$. If $t \in \Pi(I)$, define the relation $\bar{t}$ on $I \cup A$ by $i\, \bar{t}\, j$ iff $i, j \in I$ and $i\, t\, j$, or $i = j$. We claim that if $(t, F, \varepsilon) \in (s, k, \varepsilon)/(\varepsilon, \{e\}, \varepsilon)$, then $(\bar{t}, F, \varepsilon) \in C'(T)$. Assume $i\, \bar{t}\, j$ and let $\mu \in M \cup A$. The admissibility conditions hold if $i = j$, so assume $i \neq j$. In this case, $i\, \bar{t}\, j$ implies $i, j \in I$ and $i\, t\, j$. If $\mu \in A$ and $q_{\mu i} \neq 0$, then $i \in s_\mu$. Since $i\, t\, j$ and $t \leqq s$, we have also $j \in s_\mu$, and thus $q_{\mu j} = e \neq 0$.

If $\mu \in M$ and $q_{\mu i} \neq 0$, then $p_{\mu i} \neq 0$. The admissibility of (t, F, ε) gives us $P_{\mu j} \neq 0$, and because $j\, s\, f(i)$ we also know $P_{\mu f(j)} \neq 0$. Hence $q_{\mu j} = p_{\mu j} p_{\mu f(j)}^{-1} \neq 0$.

For the second part of the admissibility conditions, notice that since $i\, t\, j$, $f(i) = f(j)$. It follows that if $\mu \in M$, $q_{\mu i} q_{\mu j}^{-1} = (p_{\mu i} p_{\mu f(i)}^{-1})(p_{\mu j} p_{\mu f(j)}^{-1})^{-1} = p_{\mu i} p_{\mu j}^{-1} \in F$. If $\mu \in A$ and we assume $q_{\mu i} \neq 0$, then $i \in s_\mu$, and so $q_{\mu i} = e$. Since also $j \in s_\mu$, $q_{\mu j} = e$, and therefore $q_{\mu i} q_{\mu j}^{-1} = e \in F$.

Now define the mapping $\phi: (s, K, \varrho)/(\varepsilon, \{e\}, \varepsilon) \to C'(T)$ by $(t, F, \varepsilon)\phi = (\bar{t}, F, \varepsilon)$. That ϕ is one to one is obvious. To prove ϕ is onto, let $(v, F, \sigma) \in C'(T)$. First we must show $\sigma = \varepsilon$, so suppose $\mu\, \sigma\, \nu$ for some $\mu, \nu \in M \cup A$, and $\mu \neq \nu$.

Case 1. Both $\mu, \nu \in M$. Since $C'(S) \leqq r/\varepsilon \times \mathcal{N}(G) \times \varepsilon/\varepsilon$, there must be an $i \in I$ such that one of $p_{\mu i}$, $p_{\nu i}$ is zero and the other is not. For otherwise we would have $(r, G, [\mu, \nu])$ an admissible triple for S, a contradiction.

Case 2. At least one of $\mu, \nu \in A$, say $\nu \in A$. Then $q_{\mu\nu} = e$ and $q_{\nu\nu} = 0$, contradicting the admissibility of (v, F, σ).

In both cases we get a contradiction, so we must have $\mu = \nu$, and therefore $\sigma = \varepsilon$.

Define $t \in \Pi(I)$ by $t = v_{|I}$. Clearly, $\bar{t} \leqq v$, so to prove the reverse conclusion, suppose $i, j \in I \cup A$ and $(i, j) \notin \bar{t}$. It must be that $i \neq j$, and that one of i or j is not in I, say $i \notin I$. Then $i \in A$, and we have $q_{ii} = 0$ and $q_{ij} = e \neq 0$, and thus $(i, j) \notin v$ because (v, F, ε) is admissible. We now have $\bar{t} = v$.

We need to show that $t \leqq s$. Suppose $(i, j) \notin s$. Let s_μ, $\mu \in A$, be the s-class containing i. Then $j \notin s_\mu$, whence $q_{\mu j} = 0$ and $q_{\mu i} = e \neq 0$. From this it follows that $(i, j) \notin v = \bar{t}$ since (v, F, ε) is admissible, and thus $(i, j) \notin t$, and we have $t \leqq s$.

Since $(t, F, \varepsilon) \leqq (s, K, \varepsilon)$, G is Hamiltonian and $\bar{t} = v$, $(v, F, \varepsilon) = (t, F, \varepsilon)\phi$, and ϕ is onto. Clearly, ϕ and ϕ^{-1} are order-preserving, and it follows that ϕ is an isomorhism. The proof that $(s, K, \varepsilon)/(\varepsilon, \{e\}, \varepsilon) \cong C'(T)$ is now complete.

Aplying the Third Isomorphism Theorem to Theorem 2.4 we get the following corollary.

Corollary 2.5. *If L is the lattice of proper congruences of a Rees matrix semigroup over a Hamiltonian group, then any interval sublattice of L is also the lattice of proper congruences of a Rees matrix semigroup.*

We will need the following lemma.

Lemma 2.6. [4, Corollary 9] *Suppose* $\varepsilon \neq s \in r/\varepsilon$ *and* $K \in \mathcal{N}(G)$. *If* $N_s \nleqq K$, *then there are elements* i, j *of* I *such that* $(i, j) \in s$ *and* $N_{[i,j]} \leqq K$.

Suppose that $s \in \Pi(A)$ for some set A, and let $a \in A$. We will denote by $s(a)$ the equivalence class of s containing a. Thus $[s(a)]$ identifies all the members of the s-class containing a, but no other elements.

Theorem 2.7. *Suppose that* L *is a lattice satisfying* $(*)$, *and for each* $i \in I$ *and* $\mu \in M$, $N_{[r(i)]}$ *and* $N_{[\pi(\mu)]}$ *are either minimal normal subgroups of* G, *or* $\{e\}$. *Then* L *is isomorphic to the proper congruence lattice of a Rees matrix semigroup.*

Proof. If $|I/r| \neq |M/\pi|$, say $|I/r| < |M/\pi|$, then we can add $|M/\pi| - |I/r|$ elements to I and make them singleton r-classes without disturbing the hypotheses. So we may assume that $|I/r| = |M/\pi|$. Denote by I_λ, M_λ, $\lambda \in \Delta$ the elements of I/r and M/π respectively.

Let B be the set of symbols $\{i_{\varrho H}\}$ where $\varepsilon \neq \varrho \in \pi/\varepsilon$ and $H \ngeqq N_\varrho$. Let C be the set $\{\mu_{sK}\}$ where $\varepsilon \neq s \in r/\varepsilon$ and $K \ngeqq N_s$. Assume $|C| \leqq |B|$, and let T be a set of cardinality $|B| - |C|$. If $|B| = |C|$, set $T = \emptyset$. Let $I = I \cup B$, $M' = M \cup C \cup T$.

For each non-trivial $s \in r/\varepsilon$ and each $K \ngeqq N_s$, by lemma 2.6 we may choose an i and j in I such that $i\,s\,j$ and $N_{[i,j]} \nleqq K$. Let $\alpha_{sK} \in N_{[i,j]} \setminus K$. For $k \in I$, define

$$p_{\mu_{sK}k} = \begin{cases} \alpha_{sK} & \text{if } N_{[i,k]} \neq \{e\} \text{ and } k\,s\,i\,, \\ e & \text{otherwise}\,. \end{cases}$$

Similarly, for each non-trivial $\varrho \in \pi/\varepsilon$ and each $H \ngeqq N$, we may choose $\mu, \nu \in M$ such that $\mu\,\varrho\,\nu$ and $N_{[\mu,\nu]} \nleqq H$. Let $\gamma_{\varrho H} \in N_{[\mu,\nu]} \setminus H$. For $\sigma \in M$, define

$$p_{\sigma i_{\varrho H}} = \begin{cases} \gamma_{\varrho H} & \text{if } N_{[\mu,\sigma]} \neq \{e\} \text{ and } \mu\,\varrho\,\nu\,, \\ e & \text{otherwise}\,. \end{cases}$$

Now set $S = \mathcal{M}^0(I', G, M'; P)$ with the entries of P defined as follows:

(1) If $i \in I$, $\mu \in M$, then $i \in I_{\lambda_0}$, $\mu \in M_{\lambda_1}$ for some $\lambda_0, \lambda_1 \in \Delta$. If $\lambda_0 = \lambda_1$, set $p_{\mu i} = e$; otherwise, set $p_{\mu i} = 0$.

(2) If $i \in B$, $\mu \in M$, then $i = i_{\varrho H}$ for some $\varrho \in \pi/\varepsilon$ $(\varrho \neq \varepsilon)$ and some $H \ngeqq N$. Define $p_{\mu i} = p_{\mu i_{\varrho H}}$.

(3) If $i \in I$, $\mu \in C$, then $\mu = \mu_{sK}$ for some $s \in r/\varepsilon$ $(s \neq \varepsilon)$ and some $K \ngeqq N_s$. Define $p_{\mu i} = p_{\mu_{sK} i}$.

(4) If $i \in I$, $\mu \in T$, let $p_{\mu i} = e$.

(5) Since $|B| = |C \cup T|$, there is a one-to-one correspondence $\psi: B \leftrightarrow C \cup T$. If $i \in B$, $\mu \in C \cup T$, and $\psi(i) = \mu$, set $p_{\mu i} = 0$; otherwise set $p_{\mu i} = e$.

We wish to show that if $(s, H, \varrho) \in L$, then it corresponds to an admissible triple on S. If $(s, H, \varrho) \in L$, then $s \in \Pi(I)$, $\varrho \in \Pi(M)$.

We extend s to an equivalence relation on I' by making the elements of $I' \setminus I$ singleton s-classes. Similarly, we extend ϱ to M'. We will show that (A) of the admissibility conditions holds; that (B) holds will follow by a similar argument. If $s = \varepsilon$, then clearly (A) holds, so we may assume $s \neq \varepsilon$.

Suppose for $i, j \in I'$ that $i\, s\, j$, and let $\mu \in M'$. Then we must have $i, j \in I$, so only parts (1), (3) and (4) of the construction apply. Only in (1) is it possible to have any zeros, so it is sufficient to consider this case. Then if $p_{\mu i} = 0$, $p_{\mu j}$ must be 0 since $i\, s\, j$ and $s \leqq r$. Hence condition A.1 holds.

Now assume that $p_{\mu i} \neq 0$.

Case 1. If $\mu \in M \cup T$, then $p_{\mu i} = p_{\mu j} = e$, hence $p_{\mu i} p_{\mu j}^{-1} = e \in H$.

Case 2. If $\mu \in C$, then $\mu = \mu_{tK}$ for some $t \in r/\varepsilon$ $(t \neq \varepsilon)$ and some $K \ngeq N_t$. If $p_{\mu i} = p_{\mu j}$ we are done, so assume they are not equal. By construction we must have one of $p_{\mu i}, p_{\mu j}$ equal to α_{tK}, and the other to e, say $p_{\mu i} = \alpha_{tK}$ and $p_{\mu j} = e$. We know $\alpha_{tK} \in N_{[h,k]} \setminus K$ where $h\, t\, k$, and since $p_{\mu i} = \alpha_{tK}$, we know $N_{[h,i]} \neq \{e\}$ and $h\, t\, i$. Since $t \leqq r$, we must have $h\, r\, k$ and $h\, r\, i$, and by hypothesis we see that $\{e\} < N_{[h,k]} = N_{[h,i]} = N_{[r(i)]}$ since $N_{[r(i)]}$ is minimal. That $p_{\mu j} = e$ implies $N_{[h,j]} = \{e\}$. From lemma 1.7, $\{e\} < N_{[h,i]} \leqq N_{[h,j]} N_{[i,j]} = N_{[i,j]}$. Because $i\, r\, j$, $N_{[i,j]} \leqq N_{[r(i)]}$, and the minimality of $N_{[r(i)]}$ yields $N_{[i,j]} = N_{[r(i)]}$. It now follows that $p_{\mu i} p_{\mu j}^{-1} = \alpha_{tK} \in N_{[h,k]} = N_{[i,j]} \leqq N_s \leqq H$.

Hence in both cases, $p_{\mu i} p_{\mu j}^{-1} \in H$, so condition A.2. holds.

We wish to show that only triples corresponding to elements of L are admissible. Suppose that $(s, H, \varrho) \notin L$. Then either $s \nleqq r$, $\varrho \nleqq \pi$ or $H \ngeq N(s, \varrho)$. If $s \nleqq r$, then there exist $i, j \in I'$ such that $i\, s\, j$ but $(i, j) \notin r$. If i and j are both elements of I, then by the construction in (1), we may pick $\mu \in M$ such that $p_{\mu i} = 0$ and $p_{\mu j} \neq 0$. In the case that $i, j \in B$, for some $\mu \in C \cup T$ we will have $p_{\mu i} = 0$ and $p_{\mu j} \neq 0$ from the construction in (5). If $i \in T$, $j \in B$, then we may pick $\mu \in C \cup T$ such that $p_{\mu j} = 0$ by (5), and from (3) and (4) it follows that $p_{\mu i} \neq 0$. Hence there is always a $\mu \in M$ so that exactly one of $p_{\mu i}, p_{\mu j}$ is zero, and hence (s, H, ϱ) is not admissible. A similar argument holds if $\varrho \nleqq \pi$.

So assume that $s \leqq r$ and $\varrho \leqq \pi$. Then $H \ngeq N(s, \varrho) = N_s \vee N_\varrho$, so etiher $H \ngeq N_s$ or $H \ngeq N$, say $H \ngeq N_s$. By construction there are elements i, j in I and elements $p_{\mu_{sH} i}, p_{\mu_{sH} j}$ such that $p_{\mu_{sH} i} p_{\mu_{sH} j}^{-1} \in N_{[i,j]} \setminus H$. Let $\mu = \mu_{sH}$. Then $p_{\mu i} p_{\mu j}^{-1} \notin H$, and we see that (s, H, ϱ) is not admissible, completing the proof.

Corollary 2.8. *If L is a lattice satisfying $(*)$ and G is simple, then L is isomorphic to the lattice of proper congruences of a Rees matrix semigroup.*

Theorem 2.9. *Let L be a lattice satisfying $(*)$. Suppose that for each $[i, j] \in r/\varepsilon$ and each $[\mu, \nu] \in \pi/\varepsilon$, $N_{[i,j]} \leqq \bigwedge\{N_{[i,x]}: x \neq 1 \text{ and } x \neq j\} \vee \bigwedge\{N_{[j,y]}: y \neq j \text{ and } y \neq i$, and $N_{[\mu,\nu]} \leqq \bigwedge\{N_{[\mu,\sigma]}: \sigma \neq \nu \text{ and } \sigma \neq \mu\} \vee \bigwedge\{N_{[\nu,\eta]}: \eta \neq \mu \text{ and } \eta \neq \nu\}$. Then L is isomorphic to the congruence lattice of a Rees matrix semigroup.*

Proof. As in the proof of theorem 2.7, assume $|I/r| = |M/\pi|$, and denote by $I_\lambda, M_\lambda, \lambda \in \Delta$ the elements of I/r and M/π respectively. Define the sets B, C, T, I' and M' as in the proof of theorem 2.7.

For each non-trivial $s \in r/\varepsilon$ and each $K \in (G)$ such that $K \ngeq N_s$, choose an i and j in I such that $i\, s\, j$ and $N_{[i,j]} \ngeq K$. Pick $\alpha_{sK} \in N_{[i,j]} \setminus K$. By hypothesis, $\alpha_{sK} = \beta_{sK} \gamma_{sK}$

where $\beta_{sK} \in \bigwedge\{N_{[i,x]}: x \neq i, \quad x \neq j\}$ and $\gamma_{sK} \in \bigwedge\{N_{[j,y]}: y \neq j, \quad y \neq i\}$. Define $p_{\mu_{sK}i} = \beta_{sK}$, $p_{\mu_{sK}j} = \gamma_{sK}^{-1}$, $p_{\mu_{sK}l} = e$ if $l \in I \setminus \{i, j\}$.

Note that by this construction, if $t \in r/\varepsilon$ and $D \geqq N_t$, and $h, l \in I$ such that $h\ t\ l$, then $p_{\mu_{tD}h}p_{\mu_{tD}l}^{-1} \in N_{[h,l]}$. Similarly we can define the entries of P restricted to $M \times B$ so that if $C \ngeq N_\sigma$, and $\delta, \xi \in M$ such that $\delta\ \sigma\ \xi$, then $p_{\delta i_{\sigma C}}p_{\xi i_{\sigma C}}^{-1} \in N_{[\delta,\xi]}$.

The remainder of the proof is exactly like the proof of theorem 2.7.

We will now construct a class of examples of lattices satisfying the hypotheses of theorem 2.9. Let n be a finite ordinal or ω and $\mathscr{P}(n)$ the power set Boolean algebra on n. Let $\{p_i\}_{i\in n}$ be a set of distinct primes, and set $G = \sum_{i\in n} p_i$. Then $\mathscr{P}(n) \cong \mathscr{N}(G)$. Define the function α_0 from the atoms of $\Pi(n + 1)$ to $\mathscr{N}(G)$ by $[i, j]\,\alpha_0 = Z_{p_i} \times Z_{p_j}$ if $i, j \in n$, and $[i, n]\,\alpha_0 = Z_{p_i}$. Also let $\varepsilon\alpha_0 = \{e\}$.

We claim that α_0 satisfies (2.3). Let $i, j, k \in n$.

Case 1. $k = n; j \in n$.

Then $[i, k]\,\alpha_0 \vee [j, k]\,\alpha_0 = [i, n]\,\alpha_0 \vee [j, n]\,\alpha_0 = Z_{p_i} \vee Z_{p_j} = Z_{p_i} \times Z_{p_j} = [i, j]\,\alpha_0$.

Case 2. $j = n, \ k = n, \ i \in n$.

In this case, $[j, k]\,\alpha_0 = \varepsilon\alpha_0 = \{e\}$, and $[i, k]\,\alpha_0 = [i, n]\,\alpha_0 = Z_{p_i}$. Thus $[i, j]\,\alpha_0 = [i, n]\,\alpha_0 = Z_{p_i} = Z_{p_i} \times \{e\} = [i, k]\,\alpha_0 \vee [j, k]\,\alpha_0$.

Case 3. $i, j, k \in n$.

Then we have $[i, k]\,\alpha_0 \vee [j, k]\,\alpha_0 = Z_{p_i} \times Z_{p_k} \vee Z_{p_j} \times Z_{p_k} = Z_{p_i} \times Z_{p_j} \times Z_{p_k} \geqq Z_{p_i} \times Z_{p_j} = [i, j]\,\alpha_0$.

Thus we see that α_0 satisfies (2.3), so by Theorem 2.3, α_0 can be extended to a complete join-homomorphism $\alpha: \Pi(n + 1) \to \mathscr{N}(G)$. By definition of α_0, we also have $\varepsilon\alpha = \{e\}$, so that (2.2) is satisfied. Hence by Theorem 2.2, if we let L' be the lattice represented by α, and $L = L' \times \varepsilon/\varepsilon$, then $L \leqq \Pi(n + 1) \times \mathscr{N}(G) \times \Pi(1)$ and L satisfies $(*)$. Also, $[i, j]\,\alpha = N_{[i,j]}$.

To show that L satisfies the hypotheses of theorem 2.9, let $i, j \in n + 1$, $i \neq j$.

Case 1. $i, j \in n$.

We have $\bigwedge\{N_{[i,x]}: x \neq j, \ x \neq i\} = N_{[i,n]} \wedge (\bigwedge\{N_{[i,x]}: x \neq n, \ x \neq j, \ x \neq i\}) = Z_{p_i} \wedge (\bigwedge\{Z_{p_i} \times Z_{p_x}: x \neq j\}) = Z_{p_i} = N_{[i,n]}$. in a similar manner we can show $\bigwedge\{N_{[j,y]}: y \neq 1, \ y \neq j\} = N_{[j,n]}$. Since α_0 satisfies (2.3), we know $N_{[i,j]} \leqq N_{[i,n]} \vee N_{[j,n]} = \bigwedge\{N_{[i,x]}: x \neq j, \ x \neq i\} \vee \bigwedge\{N_{[j,y]}: y \neq j, \ y \neq i\}$, as desired.

Case 2. $i \in n, \ j = n$.

For every $x \in n$, $x \neq i$, $N_{[i,x]} = Z_{p_i} \times Z_{p_x} \geqq Z_{p_i} = N_{[i,n]}$. Thus $N_{[i,j]} \leqq \bigwedge\{N_{[i,x]}: x \neq j, \ x \neq i\}$, and we conclude that $N_{[i,j]} \leqq \bigwedge\{N_{[i,x]}: x \neq j, \ x \neq i\} \vee \bigwedge\{N_{[j,y]}: y \neq i, \ y \neq j\}$.

Hence, L satisfies the hypotheses of theorem 2.9, and is therefore the lattice of proper congruences of a Rees matrix semigroup.

Let L be a lattice satisfying $(*)$. By lemma 1.3, $\{(s, \{e\}, \varrho) \in L\} = (r_e, \{e\}, \pi_e) \in L$, where $r_e \in \Pi(I)$ and $\pi_e \in \Pi(M)$.

Theorem 2.10. *If L is a lattice satisfying $(*)$, and the lattice interval $(r, G, \pi)/(r_e, \{e\}, \pi_e)$ is the lattice of proper congruences of $\mathcal{M}^0(I/r_e, G, M/\pi_e; P)$ for some $P: M/\pi_e \times I/r_e \to G^0$, then L is the lattice of proper congruences of a Rees matrix semigroup.*

Proof. For $i \in I$ and $\mu \in M$, let $\bar{i} = r_e(i)$ and $\bar{\mu} = \pi_e(\mu)$. Define $S = \mathcal{M}^0(I, G, M; Q)$ where the entries of $Q: M \times I \to G^0$ are defined by $q_{\mu i} = p_{\bar{\mu}\bar{i}}$. Observe that $(s, K, \varrho) \in L$ implies $(s, K, \varrho) \vee (r_e, \{e\}, \pi_e) = (s \vee r_e, K, \varrho \vee \pi_e) \in (r, G, \pi)/(r_e, \{e\}, \pi_e)$.

We wish to show $L = C'(S)$. First assume $(s, K, \varrho) \in L$, and suppose $i\, s\, j$ and $\mu \in M$. If $q_{\mu i} = 0$ then $p_{\bar{\mu}\bar{i}} = 0$. Since $i\, s\, j$, $i\,(s \vee r_e/r_e)\, j$, and thus $p_{\bar{\mu}\bar{j}} = 0$, yielding $q_{\mu j} = 0$, and A.1 of the admissibility conditions holds. Since $(s \vee r_e, K, \varrho \vee \pi_e) \in (r, G, \pi)/(r_e, \{e\}, \pi_e)$, $(s \vee r_e/r_e, K, \varrho \vee \pi_e/\pi_e)$ is admissible for $\mathcal{M}^0(I/r_e, G, M/\pi_e; P)$ and this along with $i\,(s \vee r_e/r_e)\, j$ implies $p_{\mu i} p_{\bar{\mu}\bar{j}}^{-1} = q_{\mu i} q_{\mu j}^{-1} \in K$, and condition A.2. is satisfied. The proof that B holds proceeds in the same manner. Thus $(s, K, \varrho) \in C'(S)$, and we have $L \leqq C'(S)$.

Now suppose $(s, K, \varrho) \notin L$. Then either $s \nleqq r$, $\varrho \nleqq \pi$, or $K \ngeqq N(s, \varrho)$. If $s \nleqq r$, then $s \vee r_e \nleqq r$, so $s \vee r_e/r_e \nleqq r/r_e$. Thus there must exist $(i, j) \in s \vee r_e/r_e$ and $\mu \in M/\pi_e$ such that one of $p_{\bar{\mu}\bar{i}}, p_{\bar{\mu}\bar{j}}$ is zero, and the other is not. Hence one of $q_{\mu i}, q_{\mu j}$ is zero, and the other is not, and so (s, K, ϱ) is not admissible for S, and $(s, K, \varrho) \notin C'(S)$. A similar argument holds if $\varrho \nleqq \pi$.

So suppose $s \leqq r$, $\varrho \leqq \pi$, and $K \ngeqq N(s, \varrho) = N_s \vee N_\varrho$. Then either $K \ngeqq N_s$ or $K \ngeqq N$, say $K \ngeqq N_s$. Note that $N_s = N_{s \vee r_e}$ since $(s, N_s, \varepsilon) \vee (r_e, \{e\}, \varepsilon) = (s \vee r_e, N_s, \varepsilon) \in L$, and thus $K \ngeqq N_s \vee r_e/r_e$. It follows from Lemma 2.6 that there exist $\bar{i}, \bar{j} \in I/r_e$ such that $K \ngeqq N_{[\bar{i},\bar{j}]}$, and so $([i, j], K, \varepsilon) \in L'$. Therefore there is a $\bar{\mu} \in M/\pi_e$ such that $p_{\bar{\mu}\bar{i}} p_{\bar{\mu}\bar{j}}^{-1} \notin K$, and we have $q_{\bar{\mu}i} q_{\mu j}^{-1} = p_{\bar{\mu}i} p_{\mu j}^{-1} \notin K$, showing that (s, K, ϱ) is not admissible. We have shown $C'(S) \leqq L$, and finally $C'(S) = L$, as desired.

Corollary 2.11. *If L is a lattice satisfying $(*)$, and each r-class (respectively ϱ-class) contains at most three r_e-classes (respectively π_e-classes), then L is the proper congruence lattice of a Rees matrix semigroup.*

3. NECESSARY CONDITIONS

In this section we will examine some necessary conditions which do not follow from $(*)$ which must be satisfied by a non-modular lattice of proper congruences of a Rees matrix semigroup. We will make use of the following lemma, whose proof is an easy application of the admissibility conditions.

Lemma 3.1. *Let $N_{[i,j]} \in C'(S)$ for a Rees matrix semigroup $S = \mathcal{M}^0(I, G, M; P)$. Then $N_{[i,j]}$ is the normal subgroup of G generated by $\{p_{\mu i} p_{\mu j}^{-1} : \mu \in M,\ P_{\mu i} \neq 0\}$.*

We will also make use of the following theorem, which is due to Remark [8].

Theorem 3.2. *Let H be a group having distinct normal subgroups A, B, and C such that $A \wedge B = B \wedge C = C \wedge A = \{e\}$ and $A \vee B = B \vee C = C \vee A = H$. Then H is abelian.*

If $L \leqq r/\varepsilon \times \mathcal{N}(G) \times \pi/\varepsilon$ is a non-modular congruence lattice of a Ress matrix semigroup, then one of r and π must have an equivalence class containing at least four elements.

Theorem 3.3. *Let $S = \mathcal{M}^0(I, G, M; P)$ with P normalized. Suppose $C'(S) \leqq \leqq r/\varepsilon \times \mathcal{N}(G) \times \pi/\varepsilon$, $r \in \Pi(I)$, $\pi \in \Pi(M)$, is non-modular with an r-class containing four elements 0, 1, 2 and 3. If $N_{[i,j]}N_{[k,l]} \wedge N_{[i,k]}N_{[j,l]} = \{e\}$ whenever $\{i, j, k, l\} = 4$, then either $N_{[4]} = \{e\}$ or $N_{[4]}$ is a direct sum of copies of Z_2.*

Proof. Assume $N_{[4]} \neq e$. First note that if $\{i, j, k, l\} = 4$, by Lemma 1.6, $N_{[i,k]} \vee N_{[i,j]}N_{[k,l]} = N_{[4]}$.

We claim that if $\{i, j, k, l\} = 4$, then $N_{[i,j]} = N_{[k,l]}$. Let $a \in N_{[i,j]} \leqq N_{[k,l]} \vee \vee N_{[i,k]}N_{[j,l]} = N_{[4]}$. Then $a = cd$ where $c \in N_{[k,l]}$ and $d \in N_{[i,k]}N_{[j,l]}$. Now $c^{-1}a \in \in N_{[k,l]}N_{[i,j]}$, and $c^{-1}a = d$, hence $c^{-1}a \in N_{[k,l]}N_{[i,j]} \wedge N_{[i,k]}N_{[j,l]} = \{e\}$ by hypothesis. Thus $a = c \in N_{[k,l]}$, and we have shown $N_{[i,j]} \leqq N_{[k,l]}$. By symmetry we conclude $N_{[k,j]} = N_{[k,l]}$.

From this it follows that if $\{i, j, k, l\} = 4$, then $\{e\} = N_{[i,j]}N_{[k,l]} \wedge N_{[i,k]}N_{[j,l]} = = N_{[i,j]} \wedge N_{[i,k]}$, and similarly, $\{e\} = N_{[i,j]} \wedge N_{[i,l]} = N_{[i,j]} \wedge N_{[i,l]}$. We also have $N_{[4]} = N_{[i,j]} \vee N_{[i,k]}N_{[j,l]} = N_{[i,j]} \vee N_{[i,k]}$, and similarly $N_{[4]} = N_{[i,j]} \vee N_{[i,l]} = = N_{[i,k]} \vee N_{[i,l]}$. Thus by theorem 3.2, we know that $N_{[4]}$ is abelian.

Let $\mu \in M$ be such that $p_{\mu i} \neq 0$ for all $i \in 4$. Let $\{i, j, k, l\} = 4$ and define the following elements:

$$a = p_{\mu i}p_{\mu j}^{-1} \in N_{[i,j]},$$

$$b = p_{\mu i}p_{\mu k}^{-1} \in N_{[i,k]}, \quad \text{and}$$

$$c = p_{\mu i}p_{\mu l}^{-1} \in N_{[i,l]}.$$

Then we have

$$ab^{-1} = p_{\mu k}p_{\mu j}^{-1} \in N_{[j,k]} = N_{[i,l]},$$

$$bc^{-1} = p_{\mu l}p_{\mu k}^{-1} \in N_{[k,l]} = N_{[i,j]}, \quad \text{and}$$

$$ac^{-1} = p_{\mu l}p_{\mu j}^{-1} \in N_{[j,l]} = N_{[i,k]}.$$

Thus we can calculate,

$$(ab^{-1})\,c \in N_{[i,l]},$$

$$a(b^{-1}c) \in N_{[i,j]}, \quad \text{implying}$$

$$ab^{-1}c \in N_{[i,l]} \wedge N_{[i,j]} = \{e\}, \quad \text{and hence}$$

$$ab^{-1} = c^{-1}. \tag{3.1}$$

In a like manner we compute

$$
\begin{aligned}
&(ab^{-1})\,c^{-1} \in N_{[i,l]}\,,\\
&(ac^{-1})\,b^{-1} \in N_{[i,k]}\,, \quad \text{whence}\\
&ab^{-1}c^{-1} \in N_{[i,l]} \wedge N_{[i,k]} = \{e\}\,, \quad \text{and}\\
(3.2)\qquad &ab^{-1} = c\,.
\end{aligned}
$$

Combining (3.1) and (3.2), we see that $c = c^{-1}$, and thus $c^2 = e$. Likewise we can obtain $a^2 = b^2 = e$. Thus if $\mu \in M$ such that $p_{\mu i} \neq 0$, $(p_{\mu i}p_{\mu j}^{-1})^2 = e$. From Lemma 3.1 we conclude that the generators of $N_{[i,j]}$, which are all conjugates of the elements $p_{\mu i}p_{\mu j}^{-1}$, all have order two. Since $N_{[i,j]} \leqq N_{[4]}$ which is abelian, every element of $N_{[i,j]}$ has order two. By a similar argument, so does every element of $N_{[i,k]}$. Since $N_{[4]} = N_{[i,j]} \times N_{[i,k]}$, every element of $N_{[4]}$ has order two, and it follows that $N_{[4]}$ is a direct sum of copies of Z_2.

Denote by M_k the two-dimensional modular lattice with k atoms.

Lemma 3.4. *Let L be a subdirect product of $\Pi(n) \times M_{p+1} \times \Pi(1)$ which satisfies $(*)$, with $n \geqq 4$ and $p \geqq 3$. Suppose $L = C'(\mathscr{M}^0(I, G, M; P))$. Then L is a subdirect product of $r/\varepsilon \times \mathscr{N}(G) \times \pi/\varepsilon$ for some $r \in \Pi(I)$, $\pi \in \Pi(M)$, and $r/\varepsilon \cong \Pi(n)$, $G \cong \cong Z_p \times Z_p$, and $\pi/\varepsilon \cong \Pi(1)$.*

The proof of this lemma is essentially the content of section 4 of [4]. It is easy to verify that the lattice L constructed there satisfied the hypothesies of theorem 3.3. By lemma 3.4, if L is the proper congruence lattice of a Rees matrix semigroup $\mathscr{M}^0(I, G, M; P)$, then $G = N_{[4]} \cong Z_3 \times Z_3$, contradicting the conclusion of theorem 3.3. Note that this also shows that theorem 3.3 does not follow from the conditions $(*)$.

Theorem 3.5 *Suppose $L = C(\mathscr{M}^0(n, Z_p \times Z_p, M; P))$ where L is a subdirect product of $\Pi(n) \times M_{p+1} \times \pi/\varepsilon$ where $\pi \in \Pi(M)$, and $n \geqq 4$. Let i, j, k and l be distinct elements of 4. Then if $N_{[i,l]}$, $N_{[k,l]}$ and $N_{[i,k]}$ are distinct atoms of M_{p+1}, and $N_{[i,k]} = N_{[j,l]}$ and $N_{[i,j]} = Z_p \times Z_p$, then $N_{[j,k]} = Z_p \times Z_p$.*

Proof. Since $N_{[i,j]}$ is abelian, by lemma 3.1 $N_{[i,j]} = \langle p_{\mu i}p_{\mu j}^{-1} : \mu \in M,\ p_{\mu i} \neq 0\rangle$. We know $N_{[i,j]}$ is not cyclic since $N_{[i,j]} = Z_p \times Z_p$, so there must be distinct $\mu, \nu \in M$ such that $\langle p_{\mu i}p_{\mu j}^{-1}\rangle \times \langle p_{\mu i}p_{\mu j}^{-1}\rangle = N_{[i,j]}$. Since $N_{[j,l]} < N_{[i,j]}$, we cannot have both $p_{\mu i}p_{\mu j}^{-1}$ and $p_{\mu i}p_{\mu j}^{-1}$ in $N_{[j,l]}$, so assume $p_{\mu i}p_{\mu j}^{-1} \notin N_{[j,l]}$.

Define the following elements:

$$
\begin{aligned}
a &= p_{\mu i}p_{\mu j}^{-1} \in N_{[i,j]}\,,\\
b &= p_{\mu i}p_{\mu k}^{-1} \in N_{[i,k]}\,,\\
c &= p_{\mu i}p_{\mu l}^{-1} \in N_{[i,l]}\,,\\
d &= p_{\nu i}p_{\nu j}^{-1} \in N_{[i,j]}\,.
\end{aligned}
$$

We claim that $b \neq e$ and $c \neq e$. For if $c = e$, then $a = p_{\mu j}^{-1}p_{\mu l} \in N_{[j,l]}$, contradicting the assumption that $p_{\mu i}p_{\mu j}^{-1} \notin N_{[j,l]}$. If $b = e$ then $c = p_{\mu l}^{-1}p_{\mu k} \in N_{[k,l]}$, whence $c \in N_{[i,l]} \wedge N_{[k,l]} = \{e\}$, a contradiction.

Since $N_{[i,k]}$ and $N_{[i,l]}$ are cyclic of order p, we have $N_{[i,k]} = \langle b \rangle$ and $N_{[i,l]} = \langle c \rangle$. Thus $p_{\nu i}p_{\nu k}^{-1} = b^n$ and $p_{\nu i}p_{\nu l}^{-1} = c^m$ for some integers $0 \leqq n, m < p$. We claim that $n = m$, for we have $(bc^{-1})^n = b^n c^{-n} \in N_{[k,l]}$, and $b^n c^{-m} = p_{\nu k}^{-1}p_{\nu l} \in N_{[k,l]}$, and it follows that $c^{n-m} \in N_{[k,l]} \wedge N_{[i,l]} = \{e\}$.

We now calculate

$$(3.3) \qquad ab^{-1} \in N_{[j,k]}, \quad \text{so} \quad a^n b^{-n} \in N_{[j,k]}.$$

Also

$$(3.4) \qquad db^{-n} \in N_{[j,k]},$$

and combining (3.3) and (3.4) we get $a^n d^{-1} \in N_{[j,k]}$.

Similarly, $ac^{-1} \in N_{[j,l]}$ and $dc^{-n} \in N_{[j,l]}$ imply

$$(3.5) \qquad a^n d^{-1} \in N_{[j,l]}.$$

Thus we have $a^n d^{-1} \in N_{[j,k]} \wedge N_{[j,l]}$. We know that $\langle a \rangle \times \langle d \rangle = Z_p \times Z_p$, and so we cannot have $a^n = d$, whence $a^n d^{-1} \neq e$. Therefore $N_{[j,k]} \wedge N_{[j,l]} \neq \{e\}$, and it must be that $N_{[j,k]} = N_{[j,l]}$ or $N_{[j,k]} = Z_p \times Z_p$.

However, by lemma 1.6 we must have $Z_p \times Z_p = N_{[i,j]} \leqq N_{[i,k]} \vee N_{[j,k]}$, whence $N_{[j,k]} \neq N_{[i,k]} = N_{[j,l]}$. Thus $N_{[j,k]} = Z_p \times Z_p$, as desired.

Let $A_0, \ldots, A_3$ denote the atoms of M_4. Define the mapping α_0: $\{[i, j]: i, j \in 4\} \to \mathcal{N}(Z_3 \times Z_3)$ by

$$\begin{aligned}
&[0, 3]\,\alpha_0 = A_0, \\
&[2, 3]\,\alpha_0 = A_1, \\
&[0, 1]\,\alpha_0 = Z_3 \times Z_3, \\
&[1, 2]\,\alpha_0 = A_2, \\
&[0, 2]\,\alpha_0 = [1, 3]\,\alpha_0 = A_3, \quad \text{and} \\
&\varepsilon\alpha_0 = \{e\}.
\end{aligned}$$

Let L be the lattice represented by α, the extension of α_0 to $\Pi(4)$. Suppose L is the lattice of proper congruences of a Rees matrix semigroup $S = \mathcal{M}^0(I, G, M; P)$, so that $L \leqq r/\varepsilon \times \mathcal{N}(G) \times \pi/\varepsilon$ for some $r \in \Pi(I)$, $\pi \in \Pi(M)$. By lemma 3.4 we know $r/\varepsilon \cong \Pi(4)$, $G \cong Z_3 \times Z_3$, and $\pi/\varepsilon \cong \Pi(1)$. We will show that the representation of L is unique up to a permutation of the atoms of M_4 or a permutation of 4.

L has four atoms. Since L is a subdirect product of $\Pi(4) \times M_4 \times \Pi(1)$, there must be a $\sigma_i \in \Pi(4)$ such that $(\sigma_i, A_i, \varepsilon) \in L$ for each $i \in 4$. In order for L to satisfy (*), $(\varepsilon, A_i, \varepsilon)$, $i \in 4$ must be in L, and these elements must then be the four atoms.

Three of the atoms are covered by two elements each, and one atom, say $(\varepsilon, A_k, \varepsilon)$, is covered by three elements. By (*), $(\varepsilon, 1, \varepsilon)$ is in L, and it is a cover of all the atoms.

If $i \neq k$, the other atom covering $(\varepsilon, A_i, \varepsilon)$ must be of the form (s, A_i, ε) where $s \to\to \varepsilon$. The two other atoms covering $(\varepsilon, A_k, \varepsilon)$ are of the form (r, A_k, ε), and (t, A_k, ε) where $r \to\to \varepsilon$ and $t \to\to \varepsilon$, say $r = [a, b]$, $t = [c, d]$, where $a, b, c, d \in 4$. If $\{a, b\} \cap \{c, d\} \neq 0$, say $b = c$, then $(r, A_k, \varepsilon) \vee (t, A_k, \varepsilon) = ([a, b, d], A_k, \varepsilon)$. The elements (r, A_k, ε), $([a, d], A_k, \varepsilon)$, (t, A_k, ε) are then three covering elements of $(\varepsilon, A_k, \varepsilon)$ other than $(\varepsilon, 1, \varepsilon)$, a contradiction. Hence $(r, A_k, \varepsilon) \vee (t, A_k, \varepsilon)$ must be of the form $([a, b] \vee [c, d], A_k, \varepsilon)$, $a, b, c, d = 4$. Now we see that L is unique up to any permutation of 4 or any permutation of the atoms of M_4.

It is now easy to see that L satisfies the hypothesis of Theorem 3.5, with $A_0 = N_{[i,l]}$, $A_1 = N_{[k,l]}$, $A_3 = N_{[i,k]} = N_{[j,l]}$, but $N_{[j,k]} = A_2 \neq Z_3 \times Z_3$. Hence L is not the proper congruence lattice of a Rees matrix semigroup. It is also evident that theorem 3.5 does not follow from the conditions $(*)$, nor from Theorem 3.3.

References

[1] *A. H. Clifford* and *G. B. Preston:* The Algebraic Theory of Semigroups, Math. Surveys, No. 7, Vol. *I*, (1961), Vol. *II*, (1967), Amer. Math. Soc., Providence, R.I.

[2] *P. Crawley* and *R. P. Dilworth:* Algebraic Theory of Lattices, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[3] *J. M. Howie:* An Introduction to Semigroup Theory, Academic Press, London, 1976.

[4] *K. G. Johnston:* Congruence lattices of Rees matrix semigroups, Semigroup Forum *15* (1978), 247—261.

[5] *P. R. Jones:* Universal aspects of completely simple semigroups, Semigroups, Proc. Conf. Monash Univ. 1979, pp. 27—46, Academic Press, NY, 1980.

[6] *G. Lallement:* A note on congruences on Rees matrix semigroups, Semigroup Forum *8* (1974), 89—92.

[7] *H. Mitsch:* Semigroups and their lattice of congruences, Semigroup Forum *26* (1983), 1—63.

[8] *R. Remak:* Über minimale invariante Untergruppen in der Theorie der endlichen Gruppen, Journ. reine u. angew. Math. *162* (1930), 1—16.

[9] *T. Tamura:* Decompositions of a completely simple semigroup, Osaka Math. J. *12* (1960), 269—275.

Author's address: Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322 U.S.A.