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# MODULARITY AND DISTRIBUTIVITY OF TOLERANCE LATTICES OF COMMUTATIVE SEPARATIVE SEMIGROUPS

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By a tolerance on a semigroup $S$ we mean a reflexive and symmetric subsemigroup of the direct product $S \times S$. The set $\mathscr{L}(S)$ of all tolerances on $S$ forms a complete algebraic lattice with respect to set inclusion (see [1] and [2]). The aim of this paper is to consider modularity and distributivity of $\mathscr{L}(S)$ when $S$ is a commutative separative semigroup.

For any tolerance $T$ on a semigroup $S$ we have

$$(1) \qquad (au, bv) = (a, b)(u, v) \in T$$

whenever $(a, b) \in T$ and $(u, v) \in T$. This implies that for every positive integer $m$ we have

$$(2) \qquad (a, b)^m = (a^m, b^m) \in T$$

whenever $(a, b) \in T$. For all $a, b, z \in S$ we shall use the following notation: $(a, b) z = (az, bz)$.

Let $\emptyset \neq A \subseteq S \times S$. By $T(A)$ we denote the least tolerance on $S$ containing $A$. The symbol $S^1$ stands for $S$ if $S$ has an identity, otherwise it stands for $S$ with an identity adjoined.

**Lemma 1.** *Let $S$ be a commutative semigroup. For $x, y \in S$, $x \neq y$, we have $(x, y) \in T(A)$ if and only if $x = x_1 x_2 \dots x_n z$ and $y = y_1 y_2 \dots y_n z$, where $z \in S^1$ and either $(x_i, y_i) \in A$ or $(y_i, x_i) \in A$.*

Proof. Apply (1).

This implies the following:

**Lemma 2.** *Let $S$ be a commutative semigroup and $a, b \in S$, $a \neq b$. For $x, y \in S$, $x \neq y$, we have $(x, y) \in T(a, b)$ if and only if there exist $z \in S^1$ and a positive integer $m$ such that either $(x, y) = (a, b)^m z$ or $(x, y) = (b, a)^m z$.*

By $\vee$ and $\wedge$ we denote the join or meet in the lattice $\mathscr{L}(S)$. Clearly we have $A \vee B = T(A \cup B)$ and $A \wedge B = A \cap B$ for all $A, B \in \mathscr{L}(S)$.

First, we shall consider commutative regular semigroups. Recall that every commutative regular semigroup $S$ is a semilattice of commutative groups (see [3]). The set of all idempotents of $S$ is denoted by $E(S)$ and is partially ordeted by: $e \leqq f$ if $ef = e$. We write $e < f$ for $e \leqq f$ and $e \neq f$. By $e \parallel f$ we denote the fact that idempotents $e, f$ are incomparable. For any integer $k$, by $x^k$ we denote the $k$-power of an element $x$ of $S$ in the maximal subgroup $G_e$ of $S$ containing an idempotent $e = x^0$. It is known that for all $x, y \in S$ and all integers $k$ we have

$$(xy)^k = x^k y^k . \tag{3}$$

By $J(S)$ we denote the set of all tolerances $I$ on a commutative regular semigroup $S$ satisfying the following condition:

$$\text{If} \quad (a, b) \in I, \quad \text{then} \quad (a^{-1}, b^{-1}) \in I . \tag{4}$$

Using (3) and Lemma 1 we can show that $J(S)$ is a sublattice of $\mathscr{L}(S)$.

**Theorem 1.** *Let $S$ be a commutative regular semigroup. Then the lattice $\mathscr{L}(S)$ is modular if and only if $S$ satisfies the following conditions*:

(M1) *If $e, f$ are two idempotents of $S$ such that $e \parallel f$, then at least one of them is maximal with respect to the order in $E(S)$ and there exists no idempotent $g$ of $S$ such that $g \parallel ef$.*

(M2) *If $e, f$ are two idempotents of $S$ such that $e < f$, then $ze = e$ for every element $z$ of the maximal subgroup $G_f$ of $S$.*

(M3) *If $e, f, g$ are three idempotents of $S$ such that $e < f$ and $e \parallel g$, then the maximal subgroup $G_g$ of $S$ contains exactly one element.*

(M4) *$S$ is either periodic or $E(S)$ contains the greatest element $i$ and the maximal subgroup $G_e$ of $S$ is periodic for each $e < i$.*

Proof. I. Suppose that the lattice $\mathscr{L}(S)$ is modular. Then the sublattice $J(S)$ of $\mathscr{L}(S)$ is modular and so according to Theorem 1.1 of [4], $S$ satisfies the conditions (M1), (M2) and (M3). Now, we shall show that $S$ has the property (M4).

Let $a$ be a non periodic element of $S$. By way of contradiction, assume that there exists an idempotent $e$ of $S$ such that either $a^0 < e$ or $a^0 \parallel e$. Put $A = T(a^2, e)$, $B = T(a^3, e)$ and $C = T((a^2, e), (a^5, e))$. It is clear that $A \subseteq C$. Since the lattice $\mathscr{L}(S)$ is modular, it follows from Lemma 1 that $(a^5, e) \in (A \vee B) \wedge C = A \vee \vee (B \wedge C)$.

Suppose that $(a^5, e) \in A$. Then, by Lemma 2 and (3), we have $(a^5, e) = (a^2, e)^m z$ for a positive integer $m$ and $z \in S^1$. Assume that $z \in S$, then $e \leqq z^0$. If $a^0 < e$, then $a^0 < z^0$ and so, by (M2), we have $a^0 = a^0 z$. This implies that $a^5 = a^{2m} z = a^{2m}$, which is a contradiction. If $a^0 \parallel e$, then according to (M3), we obtain that $e = z^0$. Then, by (3), we have $a^0 = (a^5)^0 = (a^{2m} z)^0 = a^0 z^0 \leqq z^0 = e$, a contradiction. If $z \notin S$, then we have $a^5 = a^{2m}$, a contradiction.

In an analogous manner it can be shown that $(a^5, e) \notin B$. According to Lemma 1 and (1), we have $(a^5, e) = (u, x)(v, y)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$.

From (3) it follows that $e \leqq x^0$ and so $x \notin Sa^0$. Thus, by Lemma 2, we have $(u, x) = (a^2, e)^m z$ for some $z \in S^1$ and a positive integer $m$. Analogously it can be shown that $(v, y) = (a^3, e)^n w$ for some $w \in S^1$ and a positive integer $n$. We shall prove that $v = a^3$. We have $(a^5, e) = (a^{2m+3n}, e) zw$. Assume that $zw \in S$, then $e \leqq (zw)^0$. If $a^0 < e$, then $a^0 < (zw)^0$. It follows from (M2) that $a^0 = a^0 zw = a^0 w$ and so $a^5 = a^{2m+3n}$. This implies that $n = 1$ and so $v = a^3 w = a^3$. If $a^0 \parallel e$, then, by (M3), we have $e = (zw)^0$. Using (3) we get $a^0 = (a^5)^0 = a^0(zw)^0 \leqq e$, which is a contradiction. If $zw \notin S$, then we have $a^5 = a^{2m+3n}$ and so $v = a^3$. Since $(v, y) \in C \setminus A$ and $y \notin Sa^0$, we have according to Lemma 1, $(v, y) = (a^5, e)^r t$ or $(v, y) = (a^5, e)^r (a^2, e)^s t$ for some $t \in S^1$ and some positive integers $r, s$. This implies $a^3 = v = a^{5r} t$ or $a^3 = a^{5r+2s} t$ and so $t \in S$. It follows from (3) that $e \leqq y^0 \leqq t^0$. If $a^0 < e$, then $a^0 < t^0$ and so, by (M2), we have $a^0 = a^0 t$. Therefore $a^3 = a^{5r}$ or $a^3 = a^{5r+2s}$, which is a contradiction. If $a^0 \parallel e$, then according to (M3) we have $e = t^0$. Using (3) we get $a^0 \leqq v^0 \leqq t^0 = e$, a contradiction. Therefore $S$ satisfies the condition (M4).

II. Suppose that $S$ satisfies the conditions (M1), (M2), (M3) and (M4). According to Theorem 1.1 of [4], the lattice $J(S)$ is modular. It follows from (M4) and Theorem 7 of [5] that $\mathscr{L}(S) = J(S)$ and so the lattice $\mathscr{L}(S)$ is modular.

**Corollary 1.** *Let $S$ be a commutative regular semigroup. If the lattice $\mathscr{L}(S)$ is modular, then $\mathscr{L}(S) = J(S)$.*

The following result is a generalization of the well known Ores' theorem (see [6]) that for every commutative group $G$ the lattice $\mathscr{L}(G)$ (which coincides with the lattice of all congruences on $G$, see [7]) is distributive if and only if $G$ is locally cyclic, i.e. every its subgroup generated by a finite set of generators is cyclic. Let $x$ be a periodic element of a commutative regular semigroup $S$. By ord $x$ we denote the order of $x$ in the maximal subgroup $G_{x^0}$ of $S$.

**Theorem 2.** *Let $S$ be a commutative regular semigroup. Then the lattice $\mathscr{L}(S)$ is distributive if and only if $S$ satisfies:*

(M1), (M2), (M3) *and* (M4).

(D1) *Every maximal subgroup of $S$ is locally cyclic.*

(D2) *Let $G_e, G_f$ be two maximal subgroups of $S$ such that $e \parallel f$, $e, f \in E(S)$. If $x \in G_e$ and $y \in G_f$, then* ord $x$, ord $y$ *are relatively prime.*

The proof follows from Theorem 1, Theorem 7 of [5] and Theorem 1.1 of [4].

A semigroup $S$ is said to be *separative* if $a^2 = ab = b^2$ imply $a = b$ $(a, b \in S)$.

**Lemma 3.** *Let $S$ be a commutative separative semigroup, $a \in S$. If $a^2 \in a^3 S^1$, then $a \in a^2 S^1$.*

Proof. Suppose that $a^2 \in a^3 S^1$. Then $a^2 = a^2(ab)$ for some $b \in S^1$ and so $a^2(ab) = a^2(ab)^2$. Hence we have $a = a(ab) \in a^2 S^1$.

**Theorem 3.** *Let $S$ be a commutative semigroup whose lattice $\mathscr{L}(S)$ is modular. Then $S$ is regular if and only if it is separative.*

Proof. Suppose that the lattice $\mathscr{L}(S)$ of a commutative semigroup is modular. It follows from (3) that every commutative regular semigroup is separative. Now, we shall assume that $S$ is separative and not regular. Then there exists $a \in S$ such that $a \notin a^2S^1$.

We shall distinguish two cases.

Case 1: $a$ is periodic. Then there exist positive integers $k$ and $m$ such that $a^k = e = e^2$, $a^me \neq a^m$ and $a^{2m}e = a^{2m}$. We have $(a^m)^2 = a^m(a^me) = (a^me)^2$ and so $a^m = a^me$, which is a contradiction.

Case 2: $a$ is not periodic. Put $A = T(a^2, a)$, $B = T(a^3, a)$ and $C = T((a^2, a), (a^5, a^2))$. It is clear that $A \subseteqq C$ and so, by Lemma 1, $(a^5, a^2) \in (A \vee B) \wedge C = A \vee (B \wedge C)$.

Suppose that $(a^5, a^2) \in A$. Then, by Lemma 2, we have $(a^5, a^2) = (a^2, a)^m z$ or $(a^5, a^2) = (a, a^2)^m z$ for a positive integer $m$ and $z \in S^1$. Assume that $(a^5, a^2) = (a^2, a)^m z$. Then $a^5 = a^{2m}z = a^m(a^mz) = a^{m+2}$ and so $m = 3$. Therefore $a^2 \in a^3S^1$. Lemma 3 implies that $a \in a^2S^1$, which is a contradiction. Assume that $(a^5, a^2) = (a, a^2)^m z$. Then $a^2 = a^{2m}z = a^m(a^mz) = a^{m+5}$, a contradiction.

Analogously we can show that $(a^5, a^2) \notin B$. According to Lemma 1 and (1), we have $(a^5, a^2) = (u, x)(v, y)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. Then, by Lemma 2, we obtain that $(u, x) = (a^2, a)^m z$ or $(u, x) = (a, a^2)^m z$ for a positive integer $m$ and $z \in S^1$. Further, there exist a positive integer $n$ and $w \in S^1$ such that $(v, y) = (a^3, a)^n w$ or $(v, y) = (a, a^3)^n w$. We have the following two possibilities:

Case a: $(u, x) = (a^2, a)^m z$ and $(v, y) = (a^3, a)^n w$. Then $a^5 = uv = a^{2m+3n}zw = a^{m+2n}(a^mz)(a^nw) = a^{m+2n}xy = a^{m+2n+2}$ and so $m = n = 1$. Hence we have $a^2 = xy = a^2zw = a^2(zw)^2$ and so $a = azw$. Since $(v, y) \in C \setminus A$, it follows from Lemma 1 that $y = a^2c$ for some $c \in S^1$. Thus we have $a = awz = yz = a^2cz$, which is a contradiction.

Case b: $(u, x) = (a, a^2)^m z$ or $(v, y) = (a, a^3)^n w$. In both cases we have $a^2 = xy \in a^3S^1$. Lemma 3 implies $a \in a^2S^1$, which is a contradiction.

**Corollary 2.** *Let $S$ be a commutative semigroup. Then $S$ is separative with the modular (or distributive) lattice $\mathscr{L}(S)$ if and only if $S$ is regular and satisfies the conditions* (Mi) (*respectively* (Mi), (D1) *and* (D2)) *for* i = 1, 2, 3 *and* 4.

The proof follows from Theorems 1, 2 and 3.

Compare with [8].

It is easy to show that every commutative cancellative semigroup is separative and every commutative regular cancellative semigroup is a group.

**Corollary 3.** *Let $S$ be a commutative semigroup. Then $S$ is cancellative with the modular (or distributive) lattice $\mathscr{L}(S)$ if and only if $S$ is a group (respectively a locally cyclic group).*

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