Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 1, 162–166

Persistent URL: http://dml.cz/dmlcz/102005

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

ON LOCALLY QUASICONNECTED GRAPHS AND THEIR UPPER EMBEDDABILITY

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(Received March 29, 1984)

0. It was proved in [5] that if G is a connected, locally connected graph with $p \geqq 3$ vertices, then G contains a spanning tree T with the property that exactly one of the components of the graph $G - E(T)$ is nontrivial (i.e. that exactly one of the components of $G - E(T)$ is different from an isolated vertex). This result together with a certain characterization of upper embeddable graphs (see below) led to the theorem saying that if G is a connected, locally connected graph, then G is upper embeddable (see [5]). In the present paper the notion of a locally quasiconnected graph will be introduced and the above mentioned results on locally connected graphs will be generalized.

1. By a graph we mean a graph in the sense of [1]; if G is a graph, then the symbols $V(G)$, $E(G)$, and $c(G)$ denote the vertex set of G, the edge set of G, and the number of components of G, respectively. Let G be a graph without isolated vertices. If $v \in V(G)$, then we denote by $G_{(v)}$ the subgraph of G induced by the vertices adjacent to v in G. We shall say that G is *locally quasiconnected* if for each pair of adjacent vertices u and w of G at least one of the graphs $G_{(u)}$ and $G_{(w)}$ is connected. It can be easily shown that if G is locally quasiconnected then it contains no pair of adjacent cut-vertices. Obviously, if G is a star, then it is locally quasiconnected. We say that G is locally connected if for each $v \in V(G)$, $G_{(v)}$ is connected. If G is locally connected, then it contains no cut-vertex (see [2], where locally connected graphs were studied).

The following two theorems represent two distinct generalizations of the result mentioned at the very beginning of the present paper:

Theorem 1. *Let G be a nontrivial connected, locally quasiconnected graph. If G is different from a star, then there exists a spanning tree T of G with the property that exactly one of the components of the graph $G - E(T)$ is nontrivial.*

Theorem 2. *Let G be a connected, locally connected graph with $p \geqq 3$ vertices. Then there exists a spanning tree T of G with the properties that exactly one of the components of the graph $G - E(T)$ is nontrivial, and at most one of the components of $G - E(T)$ is trivial.*

Corollary (Zelinka [11]). *If G is a connected, locally connected graph with $p \geqq 2$ vertices and q edges then $q \geqq 2p - 3$.*

Before proving Theorems 1 and 2 we state two lemmas. The first of them follows from the fact that a tree contains no cycle.

Lemma 1. *Let G be a connected graph, and let T be a spanning tree of G. Assume that there exist distinct $u, v, w \in V(G)$ such that $uv, vw \in E(T)$ and v is an isolated vertex of $G - E(T)$. If u and w belong to distinct components of $G - E(T)$ then $G_{(v)}$ is not connected.*

Let G be a connected graph with $p \geqq 3$ vertices, let T be a spanning tree of G, and let $v \in V(G)$. We denote by $T(v, G)$ the subgraph of T induced by $V(G_{(v)}) \cup \{v\}$, and by $T[v, G]$ the component of $T(v, G)$ which contains v. Finally, we denote by $T^{(v,G)}$ the spanning subgraph of G induced by the set of edges

$$(E(T) - E(T[v, G])) \cup \{vw;\ w \in V(T[v, G] - v)\}\ .$$

Clearly, $T^{(v,G)}$ is a spanning tree of G. For any adjacent vertices u and w of $G_{(v)}$, if $uw \in E(T^{(v,G)})$, then $uv, vw \in E(G) - E(T^{(v,G)})$. The proof of the following lemma is easy.

Lemma 2. *Let G be a connected graph with $p \geqq 3$ vertices, let T be a spanning tree of G, let $v \in V(G)$, and let $u, w \in V(G - v)$. Assume that $G_{(v)}$ is connected, and that either $u, w \in V(G_{(v)})$ or there exists a component F of $G - E(T)$ such that $u, w \in V(F)$. Then there exists a component F' of $G - E(T^{(v,G)})$ such that $u, w \in V(F')$.*

If H is a graph, then we denote by $c^*(H)$ the number of nontrivial components of H. Let G be a connected graph. For every spanning tree T_0 of G, we define $h_G(T_0) = c^*(G - E(T_0))$. We denote by h_G the minimum integer m with the property that there exists a spanning tree T of G such that $h_G(T) = m$.

Proof of Theorem 1. Assume that G is different from a star. Since G is locally quasiconnected, it is obvious that $h_G \geqq 1$. We wish to prove that $h_G = 1$. On the contrary, let $h_G \geqq 2$.

For every spanning tree T_0 of G, we denote by $i(T_0)$ the minimum integer n_0 with the property that there exist vertices u and w of G which belong to distinct nontrivial components of $G - E(T_0)$ and the distance between u and w in T_0 equals n_0. Moreover, we denote by i the minimum integer n' such that there exists a spanning tree T' of G with the properties that $h_G(T') = h_G$ and $i(T') = n'$. Obviously, $i \geqq 1$.

Consider a spanning tree T of G such that $h_G(T) = h_G$ and $i(T) = i$. There exist distinct nontrivial components F and F' of $G - E(T)$ and vertices $u \in V(F)$ and $u' \in V(F')$ such that the distance between u and u' in T equals i. Clearly, there exists exactly one vertex v of G with the properties that $uv \in E(T)$ and v belongs to $u - u'$ path in T. Since G is locally quasiconnected, it follows from Lemma 1 that $i \leqq 2$. Let first $i = 1$. Then $v = u'$. Since G is locally quasiconnected, at least one of the graphs $G_{(u)}$ and $G_{(v)}$ is connected. Without loss of generality we assume that $G_{(u)}$

is connected. Lemma 2 implies that there exists a component F'' of $G - E(T^{(u,G)})$ such that $V(F - u) \cup V(F') \subseteqq V(F'')$. We get that $h_G(T^{(u,G)}) < h_G(T)$, which is a contradiction. Let now $i = 2$. Then v is an isolated vertex of $G - E(T)$. As follows from Lemma 1, $G_{(u)}$ is connected. Since $h_G(T^{(u,G)}) \geqq h_G$, Lemma 2 implies that $h_G(T^{(u,G)}) = h_G$ and $i(T^{(u,G)}) < i$, which is a contradiction.

Therefore, $h_G = 1$, which completes the proof.

Proof of Theorem 2. For every spanning tree T_0 of G, the number of isolated vertices of $G - E(T_0)$ will be denoted by $j(T_0)$. We denote by j the minimum integer m with the property that there exists a spanning tree T of G such that $h_G(T) = 1$ and $j(T) = m$. According to Theorem 1, the number j is well-defined. We wish to prove that $j \leqq 1$. On the contrary, let $j \geqq 2$.

For every spanning tree T_0 of G with $j(T_0) \geqq 2$, we denote by $k(T_0)$ the minimum integer n_0 such that there exist distinct isolated vertices u and w of $G - E(T_0)$ with the property that the distance between u and w in T_0 equals n_0. Finally, we denote by k the minimum integer n' such that there exists a spanning tree T' of G with the properties that $h_G(T') = 1$, $j(T') = j$ and $k(T') = n'$.

Consider a spanning tree T of G with the properties that $h_G(T) = 1$, $j(T) = j$ and $k(T) = k$. There exist isolated vertices u and w of $G - E(T)$ with the property that the distance between u and w in T equals k. As follows from Lemma 1, $k \geqq 2$. There exists exactly one vertex v of G with the properties that $uv \in E(T)$ and v belongs to the $u - w$ path in T. Since $k \geqq 2$, $v \neq w$. Lemma 2 implies that $h_G(T^{(v,G)}) = 1$. Since $j(T^{(v,G)}) \geqq j$, Lemma 2 implies that $j(T^{(v,G)}) = j$, v is an isolated vertex of $G - E(T^{(v,G)})$, and $vw \notin E(G)$. Therefore, w is an isolated vertex of $G - E(T^{(v,G)})$. Since the distance between v and w in $T^{(v,G)}$ does not exceed that in T, $k(T^{(v,G)}) < k$, which is a contradiction.

Therefore, $j \leqq 1$, which completes the proof.

2. We shall now derive further properties of connected, locally quasiconnected graphs. If G is a graph and U_1, U_2 are disjoint subsets of $V(G)$, then we denote by $E(G; U_1, U_2)$ the set of edges e with the property that e is incident both with a vertex in U_1 and with a vertex in U_2.

Lemma 3. *Let G be a connected, locally quasiconnected graph with $p \geqq 4$ vertices. Consider a partition P of $V(G)$ such that $|P| \geqq 2$, and that for every $U \in P$, the subgraph of G induced by U is nontrivial and connected. There exist distinct $U_1, U_2 \in P$ such that $|E(G; U_1, U_2)| \geqq 2$.*

Proof. Since G is connected, there exist distinct $U_1, U \in P$ such that $E(G; U_1, U) \neq \emptyset$. This implies that there exist $u' \in U_1$ and $u \in U$ such that $u'u \in E(G)$. Since G is locally quasiconnected, at least one of the graphs $G_{(u')}$ and $G_{(u)}$ is connected. Without loss of generality, let $G_{(u')}$ be connected. Since $|U_1| \geqq 2$, $|U_1 \cap V(G_{(u')})| \geqq 1$. Since $G_{(u')}$ is connected, there exist $v_1, v_2 \in V(G_{(u')})$ with the properties that $v_1 \in U_1$, $v_2 \notin U_1$, and $v_1 v_2 \in E(G)$. Obviously, there exists $U_2 \in P - \{U_1\}$ such that

$v_2 \in U_2$. Since $v_2 \in V(G_{(u')})$, $u'v_2 \in E(G)$. We get that $|E(G; U_1, U_2)| \geqq 2$, and the lemma is proved.

Theorem 3. *Let G be a nontrivial connected, locally quasiconnected graph. Then*

$$c(G - A) + c^*(G - A) - 2 \leqq |A| \quad \textit{for every} \quad A \subseteq E(G)\,.$$

Proof. There exists $A_0 \subseteq E(G)$ with the properties that

$$c(G - A_0) + c^*(G - A_0) - 2 - |A_0| \geqq c(G - A) + c^*(G - A) - 2 - |A|$$
$$\text{for every} \quad A \subseteq E(G)$$

and

$$c(G - A_0) + c^*(G - A_0) - 2 - |A_0| \geqq c(G - A_1) + c^*(G - A_1) - 2 - |A_1|$$
$$\text{for every proper subset } A_1 \text{ of } A_0\,.$$

It is easy to see that each component of $G - A_0$ is a nontrivial induced subgraph of G.

We now wish to show that $c(G - A_0) = 1$. On the contrary, let $c(G - A_0) \geqq 2$. It follows from Lemma 3 that there exist distinct components F' and F'' of $G - A_0$ such that $|E(G; V(F'), V(F''))| \geqq 2$. Denote $A' = A_0 - E(G; V(F'), V(F''))$. Since F' and F'' are nontrivial, $c(G - A_0) + c^*(G - A_0) - 2 - |A_0| \leqq c(G - A') + c^*(G - A') - 2 - |A'|$. Since A' is a proper subset of A_0, we get a contradiction. Thus, $c(G - A_0) = 1$.

Clearly, $A_0 = \emptyset$. We have

$$0 = c(G - A_0) + c^*(G - A_0) - 2 - |A_0|\,.$$

Hence the theorem follows.

3. The theory of 2-cell embeddings of graphs in closed surfaces is a very fruitful branch of graph theory; cf. [8], [9] or Chapter 5 in [1]. A connected graph G is said to be upper embeddable if there exists a 2-cell embedding of G in the orientable closed surface of genus $[(|E(G)| - |V(G)| + 1)/2]$. Note that the concept of an upper embeddable graph is closely related to the concept of the maximum genus of a graph (see [7], for example).

If H is a graph, then we denote by $b(H)$ the number of components F of H with the property that $|E(F)| - |V(F)| + 1$ is odd.

The next theorem gives two characterizations of upper embeddable graphs:

Theorem A. *If G is a connected graph, then the following three statements are equivalent:*

(I) *G is upper embeddable;*

(II) *there exists a spanning tree T of G with the property that for at most one component F_0 of $G - E(T)$, $|E(F_0)|$ is odd;*

(III) $c(G - A) + b(G - A) - 2 \leqq |A|$, *for every* $A \subseteq E(G)$.

The equivalence (I) ⇔ (II) was proved independently in [3], [4] and [10] (note that this equivalence was also applied in [5]). The equivalence (II) ⇔ (III) was proved in [6].

The following theorem, which is a generalization of the theorem in [5], can be obtained in two distinct ways: as a consequence of Theorem 1 and the implication (II) ⇒ (I), and as a consequence of Theorem 3 and the implication (III) ⇒ (I):

Theorem 4. *Let G be a nontrivial connected graph. If G is locally quasiconnected, then it is upper embeddable.*

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