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Bedřich Pondělíček

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MODULARITY AND DISTRIBUTIVITY OF TOLERANCE LATTICES OF COMMUTATIVE INVERSE SEMIGROUPS

BEDŘICH PONDĚLÍČEK, Praha

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By a tolerance on an algebra A we mean a reflexive and symmetric subalgebra of the direct product $A \times A$. The set $T(A)$ of all tolerances on A forms a complete algebraic lattice with respect to set inclusion (xee [1] and [2]). In the present paper we give a description of a commutative inverse semigroup S whose lattice $T(S)$ of tolerances is modular or distributive. Notice that S is found to be an algebra with a multiplication and a unary operation of inverse (see part III of [3]).

1. MODULARITY AND DISTRIBUTIVITY

By V we denote the variety of all commutative inverse semigroups of type $(., {}^{-1})$. Recall that every semigroup S of V is a semilattice of commutative groups (see [4]). For any integer m we denote by x^m the m-power of the element x of S in the maximal subgroup G_e of S containing an idempotent $e = x^0$. It is known that for each integer m and for all $x, y \in S$ we have

(1) $$(xy)^m = x^m y^m .$$

The set of all idempotents of S is denoted by $E(S)$ and is partially ordered by: $e \leqq f$ if $ef = e$. We write $e < f$ for $e \leqq f$ and $e \neq f$. Denote by $e \parallel f$ the fact that idempotents e, f are incomparable. The notation S^1 stands for S from V if S has an identity, otherwise it stands for S with an identity adjoined.

For any tolerance T on a semigroup S from V we have

(2) $$(au, bv) = (a, b)(u, v) \in T \quad \text{and}$$
$$(a, b)^{-1} = (a^{-1}, b^{-1}) \in T$$

whenver $(a, b) \in T$ and $(u, v) \in T$. This implies that for any integer m and all $(a, b) \in T$ we have

(3) $$(a, b)^m = (a^m, b^m) \in T .$$

We shall use the following notation: $(a, b)\, z = (az, bz)$ for all $a, b, z \in S$.

Let S be a semigroup of V and $\emptyset \neq A \subseteqq S \times S$. We denote by $I_S(A)$ (or simply $I(A)$) the least tolerance on S containing A.

Lemma 1.1. *For $x, y \in S$, $x \neq y$ we have $(x, y) \in I(A)$ if and only if $x = x_1 x_2 \dots$ $\dots x_n z$ and $y = y_1 y_2 \dots y_n z$, where $z \in S^1$ and either (x_i, y_i) or (y_i, x_i) or (x_i^{-1}, y_i^{-1}) or (y_i^{-1}, x_i^{-1}) lies in A $(i = 1, 2, \dots, n)$.*

Proof. Apply (1) and (2).

As a consequence we have

Lemma 1.2. *Let $a, b \in S$, $a \neq b$. For $x, y \in S$, $x \neq y$ we have $(x, y) \in I(a, b)$ if and only if there exist $z \in S^1$ and an integer m such that either $(x, y) = (a, b)^m z$ or $(x, y) = (b, a)^m z$.*

By $\vee$ or $\wedge$ we denote the join or meet in the lattice $T(S)$, respectively. It is easy to show that for $A, B \in T(S)$ we have $A \vee B = I(A \cup B)$ and $A \wedge B = A \cap B$.

It is clear that every commutative group G belongs to V. It is well known (see [5]) that the lattice $T(G)$ coincides with the lattice $C(G)$ of all congruences on a commutative group G. Hence, the lattice $T(G)$ is modular. According to Ore's theorem (see [6]), the lattice $T(G)$ is distributive if and only if the commutative group G is locally cyclic, i.e. every its subgroup generated by a finite set of generators is cyclic.

For any element x of a semigroup $S \in V$ we denote by $\langle x \rangle$ the subgroup of S generated by x. By the order of x we shall mean $\operatorname{ord} x = \operatorname{card} \langle x \rangle$, whenever $\langle x \rangle$ is finite.

Ore's theorem implies the following

Lemma 1.3. *For any pair of elements a, b of a commutative locally cyclic group we have*

$$ab \in (\langle a \rangle \cap \langle ab \rangle)(\langle b \rangle \cap \langle ab \rangle).$$

The main results of this paper are the following two theorems:

Theorem 1.1. *Let S be a semigroup from V. Then the lattice $T(S)$ is modular if and only if S satisfies the following conditions:*

(M1) *If e, f are two idempotents of S such that $e \parallel f$, then at least one of them is maximal with respect to the order in $E(S)$ and there exists no idempotent g of S such that $g \parallel ef$.*

(M2) *If e, f are two idempotents of S such that $e < f$, then $ze = e$ for every element z of the maximal subgroup G_f of S.*

(M3) *If e, f, g are three idempotents of S such that $e < f$ and $e \parallel g$, then the maximal subgroup G_g of S contains exactly one element.*

Theorem 1.2. *Let S be a semigroup from V. Then the lattice $T(S)$ is distributive if and only if S satisfies the following conditions:*

(M1), (M2), (M3) *and*

(D1) *Every maximal subgroup of S is locally cyclic.*

(D2) *Let G_e, G_f be two maximal subgroups of S such that $e \parallel f$, $e, f \in E(S)$. If $x \in G_e$, $y \in G_f$, $x \neq e$ and $y \neq f$, then the elements x, y are periodic and* ord x, ord y *are relatively prime.*

Corollary 1.1. *For a semilattice S, the following conditions are equivalent:*

(i) *$T(S)$ is modular;*

(ii) *$T(S)$ is distributive;*

(iii) *S satisfies the condition* (M1).

Note 1.1. Compare with Theorem 3 of [7].

Note 1.2. By $C(S)$ we denote the lattice of all congruences on a semilattice S. It is known (see [8], [9] and [10]) that the following conditions are equivalent:

(i) *$C(S)$ is modular;*

(ii) *$C(S)$ is distributive;*

(iii) *S is a tree.*

Recall that a semilattice S is called a *tree* if no two incomparable elements of S have an upper bound. It is easy to show that every semilattice satisfying (M1) is a tree. Hence we have the following

Corollary 1.2. *If the lattice $T(S)$ of a semilattice is modular, then the both lattices $T(S)$ and $C(S)$ are distributive.*

2. NECESSARY CONDITIONS

The following lemmas will be useful in obtaining necessary conditions for a semigroup S from V to have a modular or distributive lattice $T(S)$.

Lemma 2.1. *Let a semigroup $S \in V$ contain three idempotents e, f, g such that $e < g, f < g$ and $e \parallel g$. Then the lattice $T(S)$ is not modular.*

Proof. Put $A = I(e, g)$, $B = I(f, g)$ and $C = I((e, g), (ef, g))$. It is clear that $A \subseteqq C$. It follows from Lemma 1.1 that $(ef, g) \in (A \vee B) \wedge C$. We shall show that $(ef, g) \notin A \vee (B \wedge C)$.

Suppose that $(ef, g) \in A$. Then, by Lemma 1.2, for some $z \in S^1$ we have either $(ef, g) = (e, g)\, z$ or $(ef, g) = (g, e)\, z$. If $g = ez$, then $g \leqq e$, which is a contradiction. Then we have $(ef, g) = (e, g)\, z$. Thus we obtain that $e = eg = egz = efg \leqq f$, a contradiction.

Suppose that $(ef, g) \in B$. Then, by Lemma 1.2, for some $z \in S^1$ we have either $(ef, g) = (f, g)\, z$ or $(ef, g) = (g, f)\, z$. If $g = fz$, then $g \leqq f$, a contradiction. We can suppose that $(ef, g) = (f, g)\, z$. Then we have $f = fg = fgz = efg \leqq e$, a contradiction.

Now, we can assume that $(ef, g) \in A \vee (B \wedge C)$ and $(ef, g) \notin A \cup B$. According to Lemma 1.1, we have $(ef, g) = (uv, xy)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. It follows from (1) that $g \leqq y^0$ and so $y \notin Se$. Thus, by Lemma 1.2, we have $(v, y) = (ef, g)z$ for some $z \in S^1$. If $x \in Se$, then, by (1), we obtain $g \leqq x^0 \leqq e$, which is a contradiction. Hence $x \notin Se$. According to Lemma 1.2, we have $(u, x) = (e, g)w$ for some $w \in S^1$. Hence $(ef, g) = (u, x)(v, y) = (v, y)w \in B$, a contradiction.

Lemma 2.2. *Let a semigroup $S \in V$ contain four idempotents e, f, g, h such that $e < g$, $f < h$ and $e \parallel f$. Then the lattice $T(S)$ is not modular.*

Proof. Put $A = I(f, g)$, $B = I(e, h)$ and $C = I((f, g), (e, f))$. Evidently $A \subseteqq C$. Lemma 1.1 implies $(e, f) \in (A \vee B) \wedge C$. We shall show that either the lattice $T(S)$ is not modular or $(e, f) \notin A \vee (B \wedge C)$, which again means that $T(S)$ is not modular.

Assume that $(e, f) \in A$. If $(e, f) = (f, g)z$ for some $z \in S^1$, then $e = eg = fgz = f$, a contradiction. According to Lemma 1.2, we can suppose that $(e, f) = (g, f)z$ for some $z \in S^1$. It is clear that $z \in S$. It follows from (1) that $e \leqq z^0$ and $r \leqq z^0$. Hence, by Lemma 2.1, the lattice $T(S)$ is not modular.

Suppose that $(e, f) \in B$. Evidently $f \notin Se$. It follows from Lemma 1.2 that $(e, f) = (e, h)z$ for some $z \in S^1$. We have $z \in S$ and so, by (1), we obtain $e \leqq z^0$ and $f \leqq z^0$. Lemma 2.1 implies that the lattice $T(S)$ is not modular.

Now, we can suppose that $(e, f) \in A \vee (B \wedge C)$ and $(e, f) \notin A \cup B$. According to Lemma 1.1, we have $(e, f) = (uv, xy)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. It follows from (1) that $e \leqq v^0$ and so $v \notin Sf$. Thus, by Lemma 1.2, we have $(v, y) = (e, f)z$ for some $z \in S^1$. If $z \in S$, then $e \leqq v^0 \leqq z^0$ and $f \leqq y^0 \leqq z^0$ and so, by Lemma 2.1, the lattice $T(S)$ is not modular. We can suppose that $(v, y) = (e, f)$. Then $(e, f) \in B$, which is a contradiction.

Lemma 2.3. *Let a semigroup $S \in V$ contain three idempotents e, f, g such that $e \parallel f$ and $ef \parallel g$. Then the lattice $T(S)$ is not modular.*

Proof. Put $A = I(e, g)$, $B = I(f, g)$ and $C = I((e, g), (ef, g))$. Clearly $A \subseteqq C$. It follows from Lemma 1.1 that $(ef, g) \in (A \vee B) \wedge C$.

Suppose that $(ef, g) \in A$. If $ef \in Sg$, then $ef \leqq g$, a contradiction. We have $ef \notin Sg$. According to Lemma 1.2, we can assume that $(ef, g) = (e, g)z$ for some $z \in S^1$. Evidently $z \in S$ and so, by (1), we have $g \leqq z^0$. If $g = z^0$, then it follows from (1) that $ef \leqq g$, which is a contradiction. Thus we have $g < z^0$ and, by Lemma 2.2, the lattice $T(S)$ is not modular.

Analogously we can prove that $(ef, g) \in B$ implies that the lattice $T(S)$ is not modular.

Now, suppose that $(ef, g) \in A \vee (B \vee C)$ and $(ef, g) \notin A \cup B$. According to Lemma 1.1, we have $(ef, g) = (uv, xy)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. It follows from (1) that $ef \leqq v^0$ and so $v \notin Sg$. Hence, by Lemma 1.1, we have $(v, y) = (ef, g)z = (f, g)w$ for some $z, w \in S^1$. It is clear that $w \in S$, otherwise $f = efz \leqq e$, a contradiction. According to (1), we obtain $g \leqq y^0 \leqq w^0$. If $g = w^0$, then $ef \leqq v^0 \leqq w^0 = g$, which is a contradiction. Hence we have $g < w^0$ and, by Lemma 2.2, the lattice $T(S)$ is not modular.

Lemma 2.4. *Let a semigroup $S \in V$ contain an idempotent e and an element a such that $e < a^0 \neq a$ and $ae \neq e$. Then the lattice $T(S)$ is not modular.*

Proof. Put $A = I(e,f)$, $B = I(a,f)$ and $C = I((e,f),\ (ae,f))$, where $f = a^0$. Clearly $A \subseteq C$. It follows from Lemma 1.1 that $(ae,f) \in (A \vee B) \wedge C$.

Suppose that $(ae,f) \in A$. Evidently $f \notin Se$. It follows from Lemma 1.2 that $(ae,f) = (e,f)\, z$ for some $z \in S^1$. Then we have $e = ef = efz = fae = ae$, which is a contradiction.

Suppose that $(ae,f) \in B$. According to Lemma 1.2, we have either $(ae,f) = (a,f)^m z$ or $(ae,f) = (f,a)^m z$ for some $z \in S^1$ and some integer m. Hence we obtain either $ae = aef = a^m zf = a^m f = a^m$ or $a^{m+1}e = a^m fz = f$. By (1) this yields in both cases $fe = f$, which is a contradiction.

Now, assume that $(ae,f) \in A \vee (B \wedge C)$ and $(ae,f) \notin A \cup B$. According to Lemma 1.1, we have $(ae,f) = (uv, xy)$ for some $(u,x) \in A \setminus B$ and $(v,y) \in (B \cap C) \setminus A$. It follows from (1) that $f \leqq y^0$ and so $y \notin Se$. Thus, by Lemma 1.2, we have $(v,y) = (ae,f)^m z$ for some $z \in S^1$ and some integer m. It follows from (1) that $e = (ae)^0 \leqq v^0 \leqq e$ and so $e = v^0$. According to Lemma 1.2, we have either $(v,y) = (a,f)^k w$ or $(v,y) = (f,a)^k w$ for some $w \in S^1$ and some integer k. In both cases we obtain $w \in S$, $f \leqq y^0 \leqq w^0$ and so $fw^0 = f$. Thus, by (1) we get $e = v^0 = fw^0 = f$, a contradiction.

Lemma 2.5. *Let a semigroup $S \in V$ contain two idempotents e, f and an element a such that $e < f$ and $e \parallel a^0 \neq a$. Then the lattice $T(S)$ is not modular.*

Proof. Put $A = I(e,g)$, $B = I(a,f)$ and $C = I(e,a)$, where $g = a^0$. It follows from (3) that $(e,g) \in C$ and so $A \subseteq C$. By Lemma 1.1 we have $(e,a) \in (A \vee B) \wedge C$.

Suppose that $(e,a) \in A$. It is clear that $e \notin Sg$. According to Lemma 1.2, we have $(e,a) = (e,g)\, z$ for some $z \in S^1$. Evidently $z \in S$. Using (1) we get $e \leqq z^0$ and $g \leqq z^0$. Lemma 2.2 implies that the lattice $T(S)$ is not modular.

Assume that $(e,a) \in B$. If $e \in Sa$, then by (1) we have $e \leqq g$, which is a contradiction. Applying Lemma 1.2 we have $(e,a) = (f,a)^m z$ for some $z \in S^1$ and some integer m. It is clear that $z \in S$. According to (1), we have $e \leqq z^0$ and $g = a^0 \leqq z^0$. Lemma 2.2 shows that the lattice $T(S)$ is not modular.

Now suppose that $(e,a) \in A \vee (B \wedge C)$ and $(e,a) \notin A \cup B$. Lemma 1.1 implies that $(e,a) = (uv, xy)$ for some $(u,x) \in A \setminus B$ and $(v,y) \in (B \cap C) \setminus A$. If $y \in Se$, then by (1) we have $g = a^0 \leqq y^0 \leqq e$, a contradiction. We have $y \notin Se$ and according to Lemma 1.2, we can assume that $(v,y) = (e,a)^m z$ for some $z \in S^1$ and some integer m. If $z \in S$, then it follows from (1) that $e \leqq v^0 \leqq z^0$, $g = a^0 \leqq y^0 \leqq z^0$ and Lemma 2.2 implies that the lattice $T(S)$ is not modular. Therefore we can suppose that $(v,y) = (e,a)^m$. If $y \in Sf$, then using (1) we get $g = a^0 \leqq y^0 \leqq f$ and Lemma 2.2 implies that the lattice $T(S)$ is not modular. Let $y \notin Sf$. Since $(v,y) \in B$, we have by Lemma 1.2 $(e,a)^m = (v,y) = (f,a)^k w$ for some $w \in S^1$ and some integer k. Evidently $w \in S$. Using (1) we conclude $e \leqq w^0$ and $g = a^0 \leqq y^0 \leqq w^0$. According to Lemma 2.2, the lattice $T(S)$ is not modular.

Lemma 2.6. *Let G be a maximal subgroup of a semigroup S from V. Then the lattice $T(G)$ is distributive, whenever the lattice $T(S)$ is distributive.*

Proof. Suppose that the lattice $T(S)$ is distributive. By e we denote the idempotent of a maximal subgroup G of S. For any $X \in T(G)$ we put $\varphi(X) = I_S(X)$. We shall prove that φ is an isomorphism of $T(G)$ into $T(S)$.

First, we shall show that the mapping φ is isotone and injective. Let $X, Y \in T(G)$. If $X \subseteq Y$, then clearly $\varphi(X) = I_S(X) \subseteq I_S(Y) = \varphi(Y)$. Suppose that $\varphi(X) \subseteq \varphi(Y)$. Let $(x, y) \in X \setminus Y$. Then we have $(x, y) \in X \subseteq \varphi(X) \subseteq \varphi(Y) = I_S(Y)$ and $x \neq y$. According to Lemma 1.1, we obtain $(x, y) = (u, v)\, z$ for some $(u, v) \in Y$ and $z \in S^1$. Evidently $z \in S$. Using (1) we get $e = x^0 \leqq z^0$. If $e = z^0$, then $z \in G$ and so $(x, y) \in Y$, a contradiction. Thus we have $e < z^0$. If $z = z^0$, then $ez = e$. If $z \neq z^0$, then it follows from Lemma 2.4 that $ez = e$. In both cases we obtain $(x, y) = (u, v)\, z = (u, v) \in Y$, which is a contradiction. Consequently, we have $X \subseteq Y$.

Now, we shall show that φ is a lattice-isomorphism. Let $X, Y \in T(G)$. Assum that $(x, y) \in \varphi(X) \wedge \varphi(Y)$ and $x \neq y$. Then according to Lemma 1.1, we have $(x, y) = (u, v)\, z = (a, b)\, c$ for some $(u, v) \in X$, $(a, b) \in Y$ and $z, c \in S^1$. If $(u, v)\, z \notin X$, then $z \in S$ and it follows from (1) that $e = x^0 < z^0$. In this case, by Lemma 2.4, we have $ez = e$ and so $(u, v)\, z = (u, v) \in X$, which is a contradiction. Analogously we can obtain a contradiction if $(a, b)\, c \notin Y$. Therefore we have $(x, y) \in X \cap Y \subseteq I_S(X \cap Y) = \varphi(X \wedge Y)$. This implies $\varphi(X) \wedge \varphi(Y) \subseteq \varphi(X \wedge Y)$. Since φ is isotone, we have $\varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y)$.

It is clear that $X \subseteq \varphi(X)$ and $Y \subseteq \varphi(Y)$ for $X, Y \in T(G)$. Then we have $X \cup Y \subseteq \varphi(X) \vee \varphi(Y)$. It follows from Lemma 1.1 that $X \vee Y = I_G(X \cup Y) \subseteq \varphi(X) \vee \varphi(Y)$. Hence we get $\varphi(X \vee Y) = I_S(X \vee Y) \subseteq \varphi(X) \vee \varphi(Y)$. Since φ is isotone, we have $\varphi(X \vee Y) = \varphi(X) \vee \varphi(Y)$.

Consequently, the lattice $T(G)$ is isomorphic to a sublattice of the distributive lattice $T(S)$ and so it is distributive.

Lemma 2.7. *Let a semigroup $S \in V$ contain two elements a, b such that $a \neq a^0 \parallel b^0 \neq b$. If the lattice $T(S)$ is distributive, then the elements a, b are periodic and* ord a, ord b *are relatively prime.*

Proof. Suppose that the lattice $T(S)$ is distributive. Put $A = I(a^0, b)$, $B = I(a, b^0)$ and $C = I(a, b)$. By Lemma 1.1 we have $(a, b) \in (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$.

Assume that $(a, b) \in A$. If $b \in Sa^0$, then $b^0 \leqq a^0$, which is a contradiction. We have $b \notin Sa^0$ and so according to Lemma 1.2, we obtain $(a, b) = (a^0, b)^m z$ for some $z \in S^1$ and some integer m. It is clear that $z \in S$ and so, by (1), we have $a^0 \leqq z^0$, $b^0 \leqq z^0$. It follows from Lemma 2.1 that the lattice $T(S)$ is not modular, a contradiction. In an analogous manner it can be proved that $(a, b) \in B$ implies non-modularity of $T(S)$. Therefore we have $(a, b) \notin A \cup B$.

Using (1) we get $(a, b) = (uv, xy)$ for some $(u, x) \in (A \cap C) \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. If $u \in Sb^0$ or $v \in Sb^0$, then $a^0 \leqq b^0$, which is a contradiction. Thus we have $u \notin Sb^0$ and $v \notin Sb^0$. It follows from Lemma 1.2 that $(u, x) = (a^0, b)^m z$ and $(v, y) = (a, b^0)^n w$ for some $z, w \in S^1$ and some integers m, n. If $zw \in S$, then $a^0 = u^0 v^0 \leqq (zw)^0$ and $b^0 = x^0 y^0 \leqq (zw)^0$ (see (1)). Lemma 2.1 implies that the lattice $T(S)$ is not modular, a contradiction. We have $zw \notin S$ and this implies $a = a^n$ and $b = b^m$. Further we have $(a^0, b) = (u, x) \in C$ and $(a, b^0) = (v, y) \in C$. Using the same method of proof as above, we obtain that $(a^0, b) = (a, b)^i$ and $(a, b^0) = (a, b)^j$ for some integers i, j. It is clear that $i \neq 0 \neq j$ and so the elements a, b are periodic. Let k be a positive integer such that k divides ord a and ord b. Then k divides i and $i - 1$ and so k divides 1. Hence ord a and ord b are relatively prime.

3. SUFFICIENT CONDITIONS

In this section we shall present certain results concerning the properties of semigroups fulfilling the conditions (M) and (D) of Theorems 1.1 and 1.2.

By M we denote the class of all semigroups of V satisfying the conditions (M1), (M2) and (M3) of Theorem 1.1. Let D denote the subclass of M of all semigroups having the properties (D1) and (D2) of Theorem 1.2.

Lemma 3.1. *Let S be a semigroup of M and let $x, y, w, z \in S$.*

(i) *If $x^0 < y^0$, then $xy = x$.*

(ii) *If $x^0 \parallel y^0$, then $xy = x^0 y^0$.*

(iii) *If $x^0 \parallel y^0$ and $w^0 \parallel z^0$, then $xy = wz$.*

Proof. (i) According to (M2), we have $xy = xx^0y = xx^0 = x$.

(ii) It follows from (i) and (1) that $xy = xy(xy)^0 = x(xy)^0 = (xy)^0 = x^0 y^0$.

(iii) Suppose that $xy \neq wz$. By (ii) we have $xy = x^0 y^0 \in E(S)$ and $wz = w^0 z^0 \in E(S)$. It follows from (M1) that $xy < wz$ or $wz < xy$. Without loss of generality we can assume that $xy < wz$. It follows from (M1) that $x^0 < wz$ or $wz \leqq x^0$ and analogously we have $y^0 < wz$ or $wz \leqq y^0$. If $x^0 < wz$ and $y^0 < wz$, then this contradicts (M1). If $x^0 < wz \leqq y^0$, then $x^0 < y^0$, which is a contradiction. Similarly, $y^0 < wz \leqq x^0$ is not possible. If $wz \leqq x^0$ and $wz \leqq y^0$, then $wz \leqq x^0 y^0 = xy$, a contradiction. Hence we have $xy = wz$.

In Lemmas 3.2–3.9 we shall suppose that S is a semigroup of M, $P(a, u, b, v) = I(a, u) \vee (I(b, v) \wedge I((a, u), (x, y)))$ and $Q(a, u, b, v) = (I(a, u) \wedge I(x, y)) \vee (I(b, v) \wedge I(x, y))$, where $a, u, b, v \in S$ and $x = ab$, $y = uv$. It is easy to show that $Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Lemma 3.2. *If $b^0 \leqq a^0$ and $v^0 \leqq u^0$, then $(x, y) \in P(a, u, b, v)$.*

Proof. Suppose that $b^0 \leqq a^0$ and $v^0 \leqq u^0$. According to Lemma 3.1, we have $b = xa^{-1}$ and $v = yu^{-1}$. It follows from Lemma 1.1 that $(b, v) \in I((a, u), (x, y))$ and so we have $(x, y) = (a, u)(b, v) \in P(a, u, b, v)$.

Lemma 3.3. *If $S \in D$, $a^0 = b^0$ and $u^0 = v^0$, then $(x, y) \in Q(a, u, b, v)$.*

Proof. Suppose that $S \in D$, $a^0 = b^0$ and $u^0 = v^0$. We have the following possibilities:

Case 1. $a^0 = u^0$. Then there exists a maximal subgroup G of S such that $a, u, b, v \in G$. It follows from (D1) that G is locally cyclic and so by Ores's theorem, the lattice $T(G)$ is distributive. Put $U = I_G(a, u)$, $V = I_G(b, v)$ and $Y = I_G(x, y)$. It follows from Lemma 1.2 that $U \subseteq I_S(a, u)$, $V \subseteq I_S(b, v)$ and $Y \subseteq I_S(x, y)$. Using (2) and Lemma 1.1 we have $(x, y) \in (U \vee V) \wedge Y = (U \wedge Y) \vee (V \wedge Y) \subseteq Q(a, u, b, v)$.

Case 2. $a^0 < u^0$. It is clear that there is a maximal subgroup G of S such that $a, b \in G$. According to (D1), G is locally cyclic and so, by Lemma 1.3, there exist integers i, j, m, n such that $ab = a^m b^n$, $a^m = (ab)^i$ and $b^n = (ab)^j$. It follows from Lemma 1.2 and Lemma 3.1 that $(a^m, u) = (a, u)^m u^{1-m} \in I(a, u)$ and $(a^m, u) = (ab, uv)^i u^{1-i} v^{-i} \in I(x, y)$. Analogously we can show that $(b^n, v) \in I(b, v) \wedge I(x, y)$ and so, by Lemma 1.1, we have $(x, y) = (a^m, u)(b^n, v) \in Q(a, u, b, v)$.

Case 3. $u^0 < a^0$. Using the same method as in Case 2, we obtain $(x, y) \in Q(a, u, b, v)$.

Case 4. $a^0 \parallel u^0$.

Subcase 4a. First, we suppose that $u = u^0 = v$. This implies $y = u^0$. According to Lemma 1.3 and (D1), we have $ab = a^m b^n$, $a^m = (ab)^i$ and $b^n = (ab)^j$ for some integers i, j, m, n. It follows from Lemma 1.2 that $(a^m, u) = (a, u)^m \in I(a, u)$ and $(a^m, u) = (ab, u)^i \in I(x, y)$. Analogously it can be proved that $(b^n, v) \in I(b, v) \wedge I(x, y)$ and so, by Lemma 1.1, we have $(x, y) = (a^m, u)(b^n, v) \in Q(a, u, b, v)$.

Subcase 4b. By an analogous argument we can show that $a = a^0 = b$ implies $(x, y) \in Q(a, u, b, v)$.

Subcase 4c. Let $G_1(G_2)$ be a maximal subgroup of S containing a (u, respectively). Suppose that G_1 and G_2 are not trivial. It follows from (D2) that G_1 and G_2 are periodic. According to (D2), there exist integers s, t such that $s \operatorname{ord} a + t \operatorname{ord} u = 1$ and so Lemma 1.2 implies $(a, u^0) = (a, u)^{t \operatorname{ord} u} \in I(a, u)$. Then we have $I(a, u^0) \subseteq I(a, u)$. Analogously we obtain $I(b, v^0) \subseteq I(b, v)$ and $I(x, y^0) \subseteq I(x, y)$. This implies that $Q(a, u^0, b, v^0) \subseteq Q(a, u, b, v)$. Using the same method of proof as in Subcase 4a, we get $(x, y^0) \in Q(a, u^0, b, v^0)$ nad so $(x, y^0) \in Q(a, u, b, v)$. In an analogous manner it can be proved that $(x^0, y) \in Q(a, u, b, v)$. Consequently, by (2) we have $(x, y) = (x, y^0)(x^0, y) \in Q(a, u, b, v)$.

Lemma 3.4. *If $a^0 = b^0$ and $u^0 < v^0$, then $(x, y) \in P(a, u, b, v)$. Moreover, if $S \in D$, then $(x, y) \in Q(a, u, b, v)$.*

Proof. Suppose that $a^0 = b^0$ and $u^0 < v^0$. By Lemma 3.1 we have $y = uv = u$.

Case 1. $a^0 \leqq u^0$. It follows from Lemma 3.1 and Lemma 1.2 that $(b, u^0) = (b, v) u^0 \in I(b, v)$. Then we have $I(b, u^0) \subseteq I(b, v)$ and so $P(a, u, b, u^0) \subseteq P(a, u, b, v)$. It follows from Lemma 3.2 that $(x, y) \in P(a, u, b, u^0)$ and so $(x, y) \in P(a, u, b, v)$. If $S \in D$, then Lemma 3.3 implies $(x, y) \in Q(a, u, b, u^0) \subseteq Q(a, u, b, v)$.

Case 2. $u^0 < a^0$. According to Lemma 3.1 and Lemma 1.2, we have $(x, y) = (a, u)\, b \in I(a, u)$. Hence $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Case 3. $a^0 \parallel u^0$. By (M3) we have $a = a^0 < b$. It follows from Lemma 3.1 that $(x, y) = (a, u)$. Thus we have $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Lemma 3.5. *If $a^0 = b^0$ and $u^0 \parallel v^0$, then $(x, y) \in P(a, u, b, v)$. Moreover, if $S \in D$,* then $(x, y) \in Q(a, u, b, v)$.

Proof. Suppose that $a^0 = b^0$ and $u^0 \parallel v^0$. By Lemma 3.1 we have $y = uv = u^0 v^0 = y^0$. It follows from (M3) that $a^0 \leqq y$ or $y < a^0$.

Case 1. $a^0 \leqq y$. It follows from Lemma 3.1 and Lemma 1.2 that $(b, y) = (a^{-1}, u^{-1})\,(x, y) \in I((a, u), (x, y))$ and $(b, y) = (b, v)\, y \in I(b, v)$. Hence, by Lemma 1.1, we have $(x, y) = (a, u)\,(b, y) \in P(a, u, b, v)$.

Assume that $S \in D$. By (D1) and Lemma 1.3 there exist integers i, j, m, n such that $ab = a^m b^n$, $a^m = (ab)^i$ and $b^n = (ab)^j$. Lemma 1.2 and Lemma 3.1 imply that $(a^m, y) = (a, u)^m\, y \in I(a, u)$ and $(a^m, y) = (x, y)^i \in I(x, y)$. Analogously we can show that $(b^n, y) \in I(b, v) \wedge I(x, y)$. By virtue of Lemma 1.1 this implies $(x, y) = (a^m, y)\,(b^n, y) \in Q(a, u, b, v)$.

Case 2. $y < a^0$.

Subcase 2a. $a^0 < u^0$. Then, by Lemma 3.1 and Lemma 1.2, we have $(x, y) = (b, v)\, au \in I(b, v)$ and so $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Subcase 2b. $u^0 \leqq a^0$. If $a^0 \leqq v^0$, then $u^0 \leqq v^0$, which is a contradiction. If $v^0 < a^0$, then we obtain a contradiction by (M1). This implies that $a^0 \parallel v^0$. According to (M1), we have $u^0 \leqq a^0 v^0$ or $a^0 v^0 < u^0$. If $u^0 \leqq a^0 v^0$, then $u^0 \leqq v^0$, a contradiction. Therefore we have $a^0 v^0 < u^0$ and so $a^0 v^0 \leqq u^0 v^0 = y \leqq a^0 v^0$. Then, by Lemma 3.1, we obtain $y = a^0 v^0 = av$. This implies $(x, y) = (b, v)\, a \in I(b, v)$. Hence we have $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Subcase 2c. $a^0 \parallel u^0$. It follows from (M1) that $v^0 \leqq a^0 u^0$ or $a^0 u^0 < v^0$. If $v^0 \leqq a^0 u^0$, then $v^0 \leqq u^0$, a contradiction. We can suppose that $a^0 u^0 < v^0$ and so $a^0 u^0 \leqq u^0 v^0 = y$. Since $u^0 v^0 < a^0$, we have $u^0 v^0 \leqq a^0 u^0$. According to (M1) and Lemma 3.1, we obtain $y = a^0 u^0 = bu$. This and Lemma 1.2 imply $(x, y) = (a, u)\, b \in I(a, u)$. Therefore $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Lemma 3.6. *If $a^0 < b^0$ and $u^0 < v^0$, then $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.*

Proof. If $a^0 < b^0$ and $u^0 < v^0$, then it follows from Lemma 3.1 that $(x, y) = (a, u) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Lemma 3.7. *If $a^0 < b^0$ and $v^0 < u^0$, then $(x, y) \in P(a, u, b, v)$. Moreover, if $S \in D$, then $(x, y) \in Q(a, u, b, v)$.*

Proof. Suppose that $a^0 < b^0$ and $v^0 < u^0$. It follows from (M1) that $a^0 < v^0$ or $v^0 \leqq a^0$.

Case 1. $a^0 < v^0$. According to Lemma 3.1, we have $(x, y) = (a, v) = (a, u) v \in \in I(a, u)$. This implies that $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Case 2. $v^0 < a^0$. It follows from Lemma 3.1 that $(x, y) = (a, v) = (b, v) a \in I(b, v)$ and so $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.

Case 3. $a^0 = v^0$. By Lemma 3.1 and Lemma 1.2 we have $(a^0, v) = (b, v) a^0 \in \in I(b, v)$ and using Lemma 1.1 we get $(a^0, v) = (a^{-1}, u^{-1}) (a, v) \in I((a, u), (a, v))$. Furthermore, we have $(a, a^0) = (a, u) a^0 \in I(a, u)$. This implies that $(x, y) = (a, v) = = (a^0, v) (a, a^0) \in P(a, u, b, v)$.

Now, we shall suppose that $S \in D$. According to (D1) and Lemma 1.3, there exist integers i, j, m, n such that $av^{-1} = a^m v^{-n}$, $a^m = (av^{-1})^i$ and $v^{-n} = (av^{-1})^j$. By Lemma 3.1 and Lemma 1.2 we have $(a^m, a^0) = (a, u)^m a^0 \in I(a, u)$ and $(a^m, a^0) = = (a, v)^i v^{-i} \in I(a, v)$. In an analogous manner it can be shown that $(v^{-n}, a^0) = = (v, b)^{-n} a^0 = (a, v)^j v^{-j} \in I(b, v) \wedge I(a, v)$. Hence we have $(x, y) = (a, v) = = (a^m, a^0) (v^{-n}, a^0) v \in Q(a, u, b, v)$.

Lemma 3.8. *If $a^0 < b^0$ and $u^0 \parallel v^0$, then $(x, y) \in Q(a, u, b, v) \subseteqq P(a, u, b, v)$.*

Proof. Suppose that $a^0 < b^0$ and $u^0 \parallel v^0$. According to Lemma 3.1, we have $y = uv = u^0 v^0 = y^0$. It follows from (M1) that $a^0 \leqq y$ or $y < a^0$.

Case 1. $a^0 \leqq y$. Hence $a^0 < v^0$. By Lemma 3.1 and Lemma 1.2 we have $(x, y) = = (a, uv) = (a, u) v \in I(a, u)$ and so $(x, y) \in Q(a, u, b, v)$.

Case 2. $y < a^0$.

Subcase 2a. $a^0 < u^0$. It follows from Lemma 3.1 and Lemma 1.2 that $(x, y) = = (a, uv) = (b, v) au \in I(b, v)$ and so $(x, y) \in Q(a, u, b, v)$.

Subcase 2b. $u^0 \leqq a^0$. Using the same method of proof as in Lemma 3.5 (subcase 2b), we obtain $(x, y) \in I(b, v)$ and so $(x, y) \in Q(a, u, b, v)$.

Subcase 2c. $a^0 \parallel u^0$. By a simple adaptation of the proof of Lemma 3.5 (subcase 2c) we obtain $y = a^0 u^0 = a^0 u$. This implies that $(x, y) = (a, y) = (a, u) a^0 \in I(a, u)$. Thus we have $(x, y) \in Q(a, u, b, v)$.

Lemma 3.9. *If $b^0 < a^0$ and $u^0 \parallel v^0$, then $(x, y) \in P(a, u, b, v)$.*

Proof. Suppose that $b^0 < a^0$ and $u^0 \parallel v^0$. According to Lemma 3.1, we have $ab = b$.

Case 1. $b^0 < u^0$. It follows from Lemma 3.1 that $bu = b$ and so $(x, y) = (b, v) u \in \in I(b, v)$. This implies $(x, y) \in P(a, u, b, v)$.

Case 2. $u^0 \leqq b^0$. Then we have $u^0 < a^0$. According to (M1), we have $a^0 \leqq v^0$ or $a^0 \parallel v^0$. If $a^0 \leqq v^0$, then $u^0 < v^0$, which is a contradiction. We can suppose that $a^0 \parallel v^0$. It follows from (M1) that $u^0 \leqq a^0 v^0$ or $a^0 v^0 < u^0$. If $u^0 \leqq a^0 v^0$, then $u^0 \leqq v^0$, a contradiction. Hence we have $a^0 v^0 < u^0$. This implies that $a^0 v^0 \leqq u^0 v^0 \leqq a^0 v^0$ and so, by Lemma 3.1, we obtain $av = a^0 v^0 = u^0 v^0 = uv$. Therefore $(x, y) = = (b, v) a \in I(b, v)$ and so $(x, y) \in P(a, u, b, v)$.

Case 3. $b^0 \parallel u^0$. According to Lemma 3.1, we have $v^0 \leqq b^0u^0$ or $b^0u^0 < v^0$. If $v^0 \leqq b^0u^0$, then $v^0 \leqq u^0$, a contradiction. We can suppose that $b^0u^0 < v^0$. In an analogous manner it can be proved that $u^0v^0 < b^0$. Thus we have $u^0v^0 \leqq b^0u^0 \leqq \leqq u^0v^0$ and so, by Lemma 3.1, we obtain $uv = u^0v^0 = b^0u^0 = bu$. Consequently, $(x, y) = (a, u)\, b \in I(a, u)$ and so $(x, y) \in P(a, u, b, v)$.

4. PROOFS OF THEOREMS

In this section we complete the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let S be a semigroup of V. If the lattice $T(S)$ is modular, then it follows from Lemmas 2.2, 2.3, 2.4 and 2.5 that $S \in M$.

Suppose that $S \in M$. To prove that the lattice $T(S)$ is modular it suffices to show that $(A \vee B) \wedge C \subseteqq A \vee (B \wedge C)$ for all $A, B, C \in T(S)$ with $A \subseteqq C$. Let $(x, y) \in \in (A \vee B) \wedge C$. If $(x, y) \in A$ or $(x, y) \in B$, then we have $(x, y) \in A \vee (B \wedge C)$. We can assume that neither $(x, y) \in A$ nor $(x, y) \in B$. Then, by Lemma 1.1, we have $(x, y) = (x_1x_2, y_1y_2)$, where $(x_1, y_1) \in A \setminus B$ and $(x_2, y_2) \in B \setminus A$. Using the notation of Section 3 and (2) we obtain $P(y_1, x_1, y_2, x_2) = P(x_1, y_1, x_2, y_2) \subseteqq A \vee (B \wedge C)$.

If $x_1^0 = x_2^0$, then it follows from Lemmas 3.2, 3.4 and 3.5 that $(x, y) \in A \vee (B \wedge C)$. If $x_1^0 < x_2^0$, then according to Lemmas 3.4, 3.6, 3.7 and 3.8, we have $(x, y) \in A \vee \vee (B \wedge C)$. If $x_2^0 < x_1^0$, then Lemmas 3.2, 3.7 and 3.9 imply that $(x, y) \in A \vee \vee (B \wedge C)$. Finally, if $x_1^0 \parallel x_2^0$, then it follows from Lemmas 3.5, 3.8, 3.9 and 3.1 that $(x, y) \in A \vee (B \wedge C)$. Therefore $(A \vee B) \wedge C = A \vee (B \wedge C)$. Hence the lattice $T(S)$ is modular.

Proof of Theorem 1.2. Let S be a semigroup of V. If the lattice $T(S)$ is distributive, then it follows from Theorem 1.1 and Lemmas 2.6 and 2.7 that $S \in D$.

Suppose that $S \in D$. Let $A, B, C \in T(S)$. We shall show that $(A \vee B) \wedge C \subseteqq \subseteqq (A \wedge C) \vee (B \wedge C)$. Let $(x, y) \in (A \vee B) \wedge C$. We can suppose that neither $(x, y) \in A$ nor $(x, y) \in B$. Then we have $(x, y) = (x_1x_2, y_1y_2)$, where $(x_1, y_1) \in A \setminus B$ and $(x_2, y_2) \in B \setminus A$. Thus we get $Q(x_1, y_1, x_2, y_2) = Q(y_1, x_1, y_2, x_2) = = Q(x_2, y_2, x_1, y_1) = Q(y_2, x_2, y_1, x_1) \subseteqq (A \wedge C) \vee (B \wedge C)$ (see Section 3).

If $x_1^0 = x_2^0$, then Lemmas 3.3, 3.4 and 3.5 imply that $(x, y) \in (A \wedge C) \vee (B \wedge C)$. If $x_1^0 < x_2^0$ or $x_2^0 < x_1^0$, then it follows from Lemmas 3.4, 3.6, 3.7 and 3.8 that $(x, y) \in \in (A \wedge C) \vee (B \wedge C)$. Finally, if $x_1^0 \parallel x_2^0$, then according to Lemmas 3.5, 3.8 and 3.1 we obtain $(x, y) \in (A \wedge C) \vee (B \wedge C)$. Hence $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$. Consequently, the lattice $T(S)$ is distributive.

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Author's address: 166 27 Praha 6, Suchbátarova 2, Czechoslovakia (Elektrotechnická fakulta ČVUT).