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THE UNFOLDINGS OF A GERM OF VECTOR FIELDS IN THE PLANE WITH A SINGULARITY OF CODIMENSION 3

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1. INTRODUCTION

There are many papers which deal with bifurcation problems concerning families of vector fields depending on a single parameter. A relatively extensive bibliography on bifurcations of one-parameter families of vector fields can be found in [16, 17]. However, there are only several results concerning generic bifurcations of vector fields depending on more-dimensional parameters (see e.g. [3, 4, 5, 7, 8, 9, 11, 13, 14, 19, 20]). There are two basic reasons for this. The first is that the dimension of the bifurcation equation is growing along with the number of parameters in the generic cases. This causes difficulties concerning the computation of critical points. The second is that even for a three-dimensional central manifold very complicated topological structures of trajectories can occur also in generic cases. In such cases the so called "strange" or "chaotic attractor" may appear, on which trajectories oscillate chaotically for long periods of time (see e.g. [18]). Despite of these difficulties a considerable progress, mainly in the theory of two-parameter bifurcations, has been made. Let us mention some articles devoted to these problems.

V. I. Arnold [3] analyses generic bifurcations of two-parameter families of vector fields which are unfoldings of codimension two degenerate singularities. One of these unfoldings is described in detail by R. I. Bogdanov [7, 8]. He assumes that the matrix of the linear part of the given vector field is equivalent to the Jordan block with 1 above the diagonal and zeros elsewhere and, moreover, some coefficients of the second order terms are nonzero. A similar singularity was also studied by F. Takens [20] (see also [11]) under some symmetry conditions. Versal deformations of two-parameter families of vector fields in the plane, invariant with respect to rotations by an angle $2\pi/n$ about the origin, are discussed by V. I. Arnold [4]. All these results can also be found in Arnold's book [5]. The paper of Takens [19] contains results on the Hopf bifurcation for a class of more-parameter families of vector fields in the plane. The paper of J. Guckenheimer [13] is devoted to two-parameter unfoldings of a vector field, having the matrix of its linearization at a critical point with simple eigenvalues 0, $\pm i\beta$, $\beta \neq 0$ and none of the others pure by imaginary.

We study germs of vector fields under the same assumptions on their linear parts as in [7], however, unlike Bogdanov's assumptions we assume that one coefficient of a second order term is equal to zero ($q_{11} = 0$, $q_{12} \neq 0$, see [7, (5)]). These conditions define a degenerate singularity of codimension 3. This paper is also an attempt to answer Marsden's question "how should one break the symmetry in the Takens bifurcation and produce an associated structurally stable unsymmetric bifurcation" (see [17, p. 1143]). We use in this paper the approach employed by Bogdanov in [7].

2. NOTATION, DEFINITIONS AND MAIN RESULTS

Definition 1. Two mappings $f_1, f_2 \in C^\infty(R^n, R^m)$ are called *0-equivalent at a point* $x \in R^n$, if there exists a neighbourhood U of x such that $f_1/U = f_2/U$. We call the class $\{g \in C^\infty(R^n, R^m) : g$ is 0-equivalent to $f \in C^\infty(R^n, R^m)$ at $x\}$ *the germ of the mapping f at x* and denote it by $\tilde{f}_x$, or $[f]_x$, or simply $\tilde{f}$. The set of all such terms is denoted by $C_x^\infty(R^{\prime}, R^m)$.

Definition 2. Two germs $\tilde{f}, \tilde{g} \in C_x^\infty(R^n, R^m)$ are called *k-equivalent* $(1 \leqq k < \infty)$ if for their representatives f, g we have $f(x) = g(x)$, $D^j f(x) = D^j g(x)$, $j = 1, 2, \ldots, k$. We call the class $j^k \tilde{f}(x) = \{\tilde{g} \in C_x^\infty(R^n, R^m) : \tilde{g}$ is k-equivalent to $\tilde{f}\}$ *the k-jet of the germ $\tilde{f}$ at x or the k-jet of the mapping f at x*, and denote it also by $j^k f(x)$. The set of all such k-jets is denoted by $J_n^k(x)$.

Denote by $\Gamma^\infty = \Gamma_n^\infty$ the set of all C^∞-vector fields on R^n. If $\xi \in \Gamma_n^\infty$, then $\xi(x) = (x, v(x))$, where $v \in C^\infty(R^n, R^n)$. We identify such a vector field with the differential equation $\dot{x} = v(x)$, or with the mapping v. We denote by G_n the set of all germs of vector fields from Γ_n^∞ at 0, for which the origin is their critical point. The set of all k-jets of germs of vector fields from the set G_n is denoted simply by J_n^k.

We can endow the set J_n^k $(k = 1, 2, \ldots)$ with the natural smooth structure induced by the following mappings:

$$\alpha^1 : J_n^1 \to R^{n^2}\,, \quad \alpha^1(j^1\, v(0)) = D\, v(0)\,,$$

$$\alpha^2 : J_n^2 \to R^{n^2} \times R^{(n^2+n)/2}\,, \quad \alpha^2(j^2\, v(0)) = (D\, v(0), D^2\, v(0))\,, \quad \text{etc.}\,,$$

so that the sets J_n^k are smooth manifolds, where $\dim J_n^1 = n^2$, $\dim J_n^2 = n^2 + \frac{1}{2}(n^2 + n)$, etc.

Definition 3. Denote by φ_v: $R^n \times R^1 \to R^n$ the flow of the vector field $v \in \Gamma_n^\infty$. Two germs $\tilde{v}_1, \tilde{v}_2 \in G_n$ are called *topologically*, or *orbitally*, *C^0-equivalent*, if for their representatives v_1, v_2, the following holds: There exist neighbourhoods U and V of $0 \in R^n$ and a homeomorphism h: $U \to V$ such that if $x \in U$ and $\varphi_{v_1}(x, [0, t]) \subset U$ for some $t > 0$, then there exists a $t' > 0$ such that $h(\varphi_{v_1}(x, [0, t]) = \varphi_{v_2}(h(x), [0, t'])$.

Definition 4. Let $v \in \Gamma_n^\infty$. The germ $\tilde{V}$ at 0 of a k-parameter family of vector fields

$V\colon U \to \Gamma_n^\infty$ such that $V(0) = v$, where U is a neighbourhood of $0 \in R^k$, is called a *k-parameter unfolding of the germ* $\tilde{v}$. The neighbourhood U is called the *basis of the unfolding.*

Definition 5. Let $\tilde{V}_1$, $\tilde{V}_2$ be two unfoldings of a given vector field v with the same basis $U \subset R^k$. The unfolding $\tilde{V}_2$ is called *C^0-equivalent to the unfolding* $\tilde{V}_1$, if there exist their representatives V_2 and V_1, respectively, such that for all $\lambda \in U$ the corresponding vector fields $V_2(\lambda)$ and $V_1(\lambda)$ are orbitally C^0-equivalent, where the homemorphism $h(\lambda)$ of this equivalence depends continuously on λ.

Definition 6. Let $v \in \Gamma_u^\infty$ and $\tilde{V}$ be an unfolding of the germ $\tilde{v}$ with the basis $U \subset R^k$. A mapping $\Psi\colon W \to U$, where W is a neighbourhood of $0 \in R^m$, $\Psi(0) = 0$, defines a new unfolding $\Psi^* \tilde{V}$ of the germ $\tilde{v}$, i.e. a germ of the m-parameter family of vector fields defined via $\Psi^* \tilde{V} = \tilde{V} \circ \tilde{\Psi}$. If the mapping Ψ is of the class C^r, we say that the unfolding $\Psi^* \tilde{V}$ is C^r-induced from V.

Definition 7. An unfolding $\tilde{V}$ of the germ $\tilde{v} \in G_n$ is called *topologically versal, or versal,* if any unfolding of the germ $\tilde{v}$ is C^0-equivalent to an unfolding of $\tilde{v}$ which is C^0-induced from $\tilde{V}$.

Definition 8. Let $V\colon U \to \Gamma_n^\infty$, $U \subset R^k$, be a given family of vector fields and let N be a neighbourhood of the origin in the phase space R^n. Assume that the vector field $V(\varepsilon_0)$ (we shall often write V_ε instead of $V(\varepsilon)$) has a critical point $x_0 \in N$. This critical point is called a *nonbifurcation point of the family V* if *there exists a neighbourhood* $N' \subset N$ of the point x_0 and a neighbourhood $U' \subset U$ of the point ε_0 such that for all $\varepsilon \in U'$ the vector field V_ε/N' is orbitally C^0-equivalent to $V_{\varepsilon 0}/N'$ in N'. A critical point which is not nonbifurcation is called *bifurcation.* A point $\varepsilon_0 \in U$ is called *a bifurcation value for the family V and for the neighbourhood N* if there exists an ε in an arbitrary small neighbourhood of ε_0 such that the vector fields V_ε and V_{ε_0} are not orbitally C^0-equivalent in N. The *bifurcation diagram* of critical points of the family V is the set of all bifurcation values for the family V and for the neighbourhood N.

Now let us recall the formulae for the so called first and second Ljapunov's focus number, which will be important for better understanding of our further considerations.

Consider the following plane system of differential equations:

$$\begin{aligned} \dot{x} &= ax + by + P(x, y)\,, \\ \dot{y} &= cx + dy + Q(x, y)\,, \end{aligned} \tag{2.1}$$

where $P(x, y) = \sum_{i=2}^{5} P_i(x, y) + R_1(x, y)$, $Q(x, y) = \sum_{i=2}^{5} Q_i(x, y) + R_2(x, y)$, $P_i(x, y) = a_{i0}x^i + a_{i-1,1}x^{i-1}y + \ldots + a_{0i}y^i$, $Q_i(x, y) = b_{i0}x^i + b_{i-1,1}x^{i-1}y + \ldots + b_{0i}y^i$, $i = 2, 3, \ldots, 5$, $R_j \in C^\infty$, $R_j(x, y) = o(|x|^5 + |y|^5)$, $j = 1, 2$, $\sigma = a + d \leqq 0$, $\Delta = ad - bc > 0$.

If $I_\varepsilon = \{(\varrho, 0) \in R^2 \colon 0 \leqq \varrho < \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small and $I^+ = \{(\varrho, 0) \colon \varrho \geqq 0\}$, then the Poincaré mapping $H \colon I_\varepsilon \to I^+$ is defined and by [2, (25), p. 253], we have

$$(2.2) \qquad G(\varrho) = H(\varrho) - \varrho = \left(e^{(2\pi/b)\sigma} - 1\right) + \alpha_2 \varrho^2 + \alpha_3 \varrho^3 + \ldots .$$

By [2, IX, § 24, Lemma 5], if $d^i G(0)/d\varrho^i = 0$, $i = 1, 2, \ldots, k$, then k must be even. If $\sigma = 0$, then $dG(0)/d\varrho = 0$ and therefore also $\alpha_2 = d^2 G(0)/d\varrho^2 = 0$. In this case the number $L_1 = \alpha_3$ is called the *first Ljapunov's focus number*. If also $\alpha_3 = 0$, then α_4 must be zero and in this case the number $L_2 = \alpha_5$ is called the second Ljapunov's focus number. By [2, IX, (76), p. 263],

$$(2.3) \qquad L_1 = -\frac{\pi}{4b\sqrt{\Delta^3}} \{[ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + ab(b_{11}^2 + b_{11}a_{20} + a_{11}b_{20} + c^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2ac(b_{02}^2 - a_{20}a_{02}) - 2ab(a_{20}^2 - b_{20}b_{02}) - b^2(2a_{20}b_{20} + b_{11}b_{20}) + (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20})] - (a^2 + bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - b_{21}b)]\} .$$

By [6, *p.* 209],

$$(2.4) \qquad L_2 = \tfrac{1}{24}\pi[a_{02}b_{20}(5a_{02}b_{11} + 10a_{02}a_{20} + 4b_{11}^2 + 11a_{20}b_{11} + 6a_{20}^2 - 10b_{20}b_{02} - 4a_{11}^2 - 11a_{11}b_{02} - 6b_{02}^2) + a_{20}b_{02}(6b_{02}^2 - 5a_{11}b_{02} + 10b_{02}b_{20} - 2a_{11}^2 - 5a_{11}b_{20} + 5a_{20}b_{11} - 6a_{20}^2 - 10a_{20}a_{02} + 2b_{11}^2 + 5a_{02}b_{11}) + a_{02}b_{02}(5b_{11}^2 - a_{11}^2 - 6a_{11}a_{02}) - a_{20}b_{20}(5a_{11}^2 - b_{11}^2 - 6a_{20}b_{11}) + a_{11}^3(a_{20} + a_{02}) - b_{11}^3(b_{02} + b_{20}) - 5b_{20}^2(a_{12} + 3b_{03}) + b_{02}^2(3b_{21} - 6a_{12} - 5a_{30}) + a_{11}^2(a_{12} + a_{30}) + b_{20}b_{02}(5b_{21} - 5a_{12} - 9b_{03} + 5a_{30}) - b_{20}a_{11}(4a_{12} + 9b_{03} + 5a_{30}) + b_{02}a_{11}(3b_{21} - a_{12} + 4a_{30}) - 5a_{02}^2(b_{21} + 3a_{30}) + a_{20}^2(3a_{12} - 6b_{21} - 5b_{03}) + b_{11}^2(b_{21} + b_{03}) + a_{20}a_{02}(5a_{12} - 5b_{21} - 9a_{30} + 5b_{03}) - a_{02}b_{11}(4b_{21} + 9a_{30} + 5b_{03}) + a_{20}b_{11}(3a_{12} - b_{21} + 4b_{03}) + 4b_{20}b_{11}(2b_{30} + b_{12}) + b_{02}b_{11}(7b_{30} - a_{21} + 5b_{12} + a_{03}) + 2a_{11}b_{11}(a_{03} + b_{30}) + 2a_{20}b_{20}(8b_{30} - 5a_{21} - b_{12}) + 2a_{20}b_{02}(4b_{30} - 5a_{21} - 5b_{12} + 4a_{03}) + a_{20}a_{11}(b_{30} + 5a_{21} - b_{12} + 7a_{03}) - 2a_{02}b_{20}(a_{21} + b_{12}) + 2a_{02}b_{02}(8a_{03} - 5b_{12} - a_{21}) + 4a_{02}a_{11}(2a_{03} + a_{21}) + b_{11}(5b_{04} - b_{22} + 2a_{13} - 3b_{40}) + a_{02}(2b_{22} + 20b_{04} + 5a_{13} + 3b_{13}) - a_{11}(5a_{40} - a_{22} + 2b_{31} - 3a_{04}) + 3a_{21}(2a_{30} + b_{03} + a_{12}) - 3b_{12}(2b_{03} + a_{30} + b_{21}) + 3a_{03}(a_{12} + 3b_{03}) - 3b_{30}(b_{21} + 3a_{30}) - b_{02}(4a_{22} + 22a_{40} + 7b_{31} - 6a_{04} + 9b_{13}) + 3b_{41} + 3b_{23} + 15b_{05} + 15a_{50} + 3a_{32} + 3a_{14}.$$

We recall this very complicated formula for L_2 because it is not generally known and nonetheless plays an important role in our considerations.

Now we formulate the known Bogdanov's results on two parameter families of plane autonomous ordinary differential equations of the form

$$\dot{y} = g(y, \varepsilon), \tag{2.5}$$

where $y = (y_1, y_2)^*$, $g = (g_1, g_2)^* \in C^\infty$ (i. e. g is smooth in (y, ε); u^* is the transpose of u), $g(y, 0) = Ay + h(y)$, the matrix A is equivalent to the Jordan block

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $h(y) = o(\|y\|^2)$. Consider also the equation

$$\dot{y} = g(y, 0) = Ay + h(y). \tag{2.6}$$

Definition 9. By a *smooth regular transformation* we mean a smooth mapping keeping the origin fixed and having a regular Jacobian matrix at the origin.

Lemma 1 (Bogdanov [7]). (1) *There exists a smooth regular transformation of*

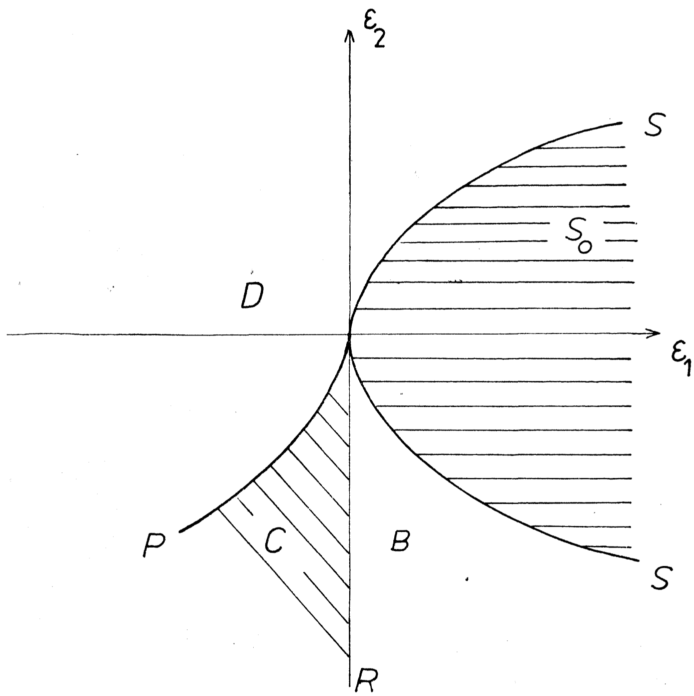

Fig. 1. Bogdanov's bifurcation diagram.

coordinates in the phase space transforming the system (2.6) *into the form*

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = (Tx, x) + R(x)\,, \tag{2.7}$$

$$(Tx, x) = t_{11}x_1^2 + t_{12}x_1x_2 + t_{22}x_2^2\,, \quad R(x) = o(\|x\|^2)\,.$$

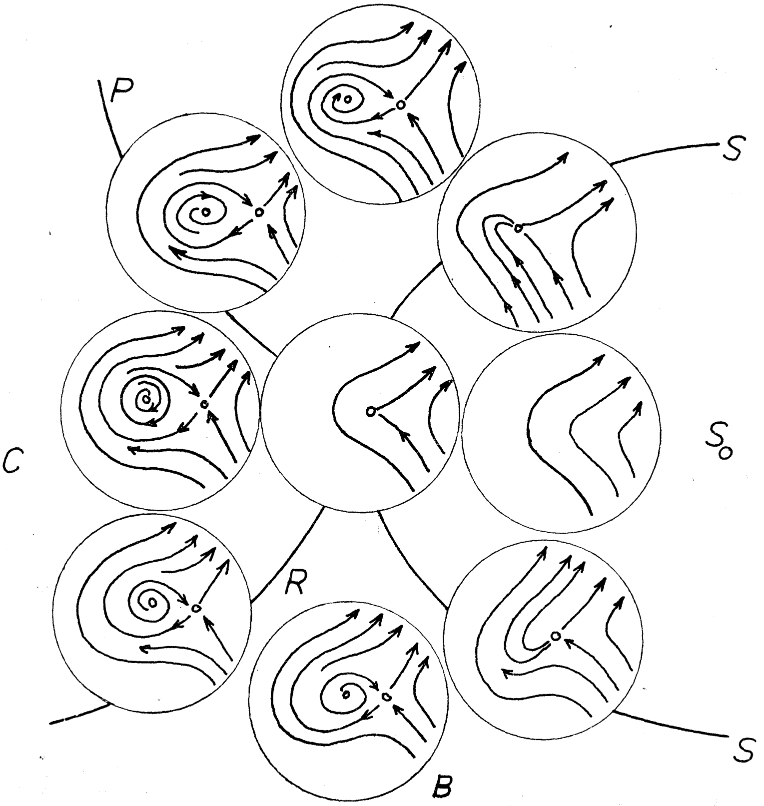

Fig. 2. Bogdanov's bifurcation ($q > 0$).

(2) *If the family* (2.5) *is nondegenerate* (*see Definition* 11; *in this case* $t_{11} \neq 0$, $t_{12} \neq 0$), *then there exists a smooth regular transformation of coordinates* $(x, \mu) = (\Psi_1(y, \varepsilon), \Psi_2(\varepsilon))$ *transforming the family* (2.5) *into the form*

$$\dot{x}_1 = x_2\,, \tag{2.8}$$

$$\dot{x}_2 = \mu_1 + \mu_2 x_1 + x_1^2 + x_1 x_2 Q(x_1, \mu) + x_2^2 \Phi(x, \mu)\,,$$

where $Q, \Phi \in C^\infty$, $Q(0, 0) = q = t_{12}/t_{11}$.

(3) *The family* (2.8) *is versal.*

(4) *The bifurcation diagram of the family* (2.8) *in a sufficiently small neighbourhood U of the origin in the parameter space looks like that in Figure* 1. *If* $q > 0$ *then there are the following bifurcations* (*see Figure* 2): *For* $\mu \in S_0$ *there are no critical points, for* $\mu \in S$ *there is one critical point of the saddle-node type*

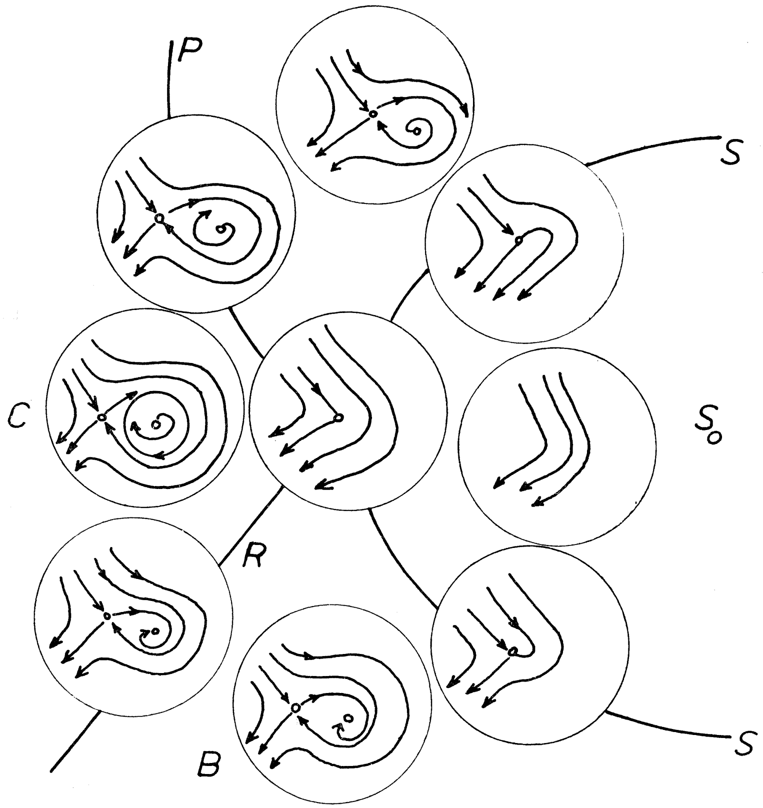

Fig. 3. Bogdanov's bifurcation ($q < 0$).

and for $\mu \in U \setminus \bar{S}_0$ *there are two critical points: one is a saddle while the second is a focus. If* μ *moves in the direction* $B \to C \to D$ *crossing the curves R and P transversally the following bifurcations occur: If* μ *crosses the curve R the stable focus bifurcates into an unstable closed orbit and then this closed orbit bifurcates into a separatrix of the saddle for* $\mu \in P$, *which disappears for* $\mu \in D$. *The first Ljapunov's focus number* L_1 *is positive for* $\mu \in R$.

(5) *The family* (2.8) *with* $q < 0$ *is obtained from a family of the form* (2.8) *with* $q > 0$ *by using the change of variables* $x_2 \to -x_2$, $t \to -t$. *The bifurcations of the family* (2.8) *with* $q < 0$ *looks like that in Figure* 3. *The first Ljapunov's focus number* L_1 *is negative for this family.*

Definition 10. The number sign q is called the *signature of the family* (2.8).

Now let us consider the following two-parameter family of the autonomous system of differential equations

$$(2.9)_\lambda \qquad \dot{x} = a(\lambda)\,x + b(\lambda)\,y + P(x, y, \lambda)\,,$$
$$\dot{y} = c(\lambda)\,x + d(\lambda)\,y + Q(x, y, \lambda)\,,$$

where $\lambda = (\lambda_1, \lambda_2) \in R^2$, a, b, c, d, $P, Q \in C^\infty$ (smooth in (x, y, λ)). Let $\sigma(\lambda) = a(\lambda) + d(\lambda)$, $\Delta(\lambda) = a(\lambda)\,\mathrm{d}(\lambda) - b(\lambda)\,c(\lambda)$. We assume that $\sigma(\lambda^0) = 0$, $\Delta(\lambda^0) > 0$, $\lambda^0 = (\lambda_1^0, \lambda_2^0)$ and U is a neighbourhood of λ^0 in R^2 for which the set $\Sigma = \{\lambda \in U: \sigma(\lambda) = 0\}$ is a curve dividing U into two disjoint regions $\Sigma^+ = \{\lambda \in U: \sigma(\lambda) > 0\}$, $\Sigma^- = \{\lambda \in U: \sigma(\lambda) < 0\}$. The point λ^0 divides the curve Σ into two connected components Σ_1 and Σ_2.

If $\lambda \in \Sigma$, then the first Ljapunov's focus number is defined and we denote it by $L_1(\lambda)$, or simply L_1. If $L_1(\lambda) = 0$, then also the second Ljapunov's focus number is defined and we denote it by $L_2(\lambda)$, or simply L_2.

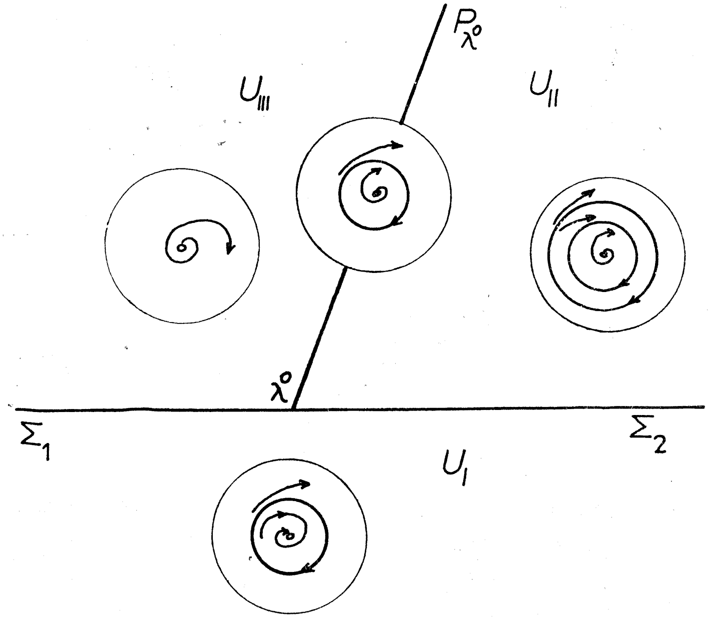

Fig. 4. Bifurcations of the family $(2.9)_\lambda$ in a neighbourhood of λ^0 ($L_1(\lambda^0) = 0$, $L_2(\lambda^0) > 0$).

Lemma 2 ([6, p. 243]). *Assume that* $L_1(\lambda^0) = 0$, $L_1(\lambda) \neq 0$ *for* $\lambda \in \Sigma_1 \cup \Sigma_2$ *and* $L_2(\lambda^0) \neq 0$. *Then for a sufficiently small neighbourhood U of λ^0 in R^2 the following assertions hold*:
(1) *If* $L_2(\lambda^0) > 0$ ($L_2(\lambda^0) < 0$), *then there exists a curve* P_{λ^0} *which has one end-point*

at λ^0 and the other on the boundary ∂U of U. This curve together with the curve divide U into three disjoint regions $U_{\mathrm{I}}, U_{\mathrm{II}}$ and U_{III}, for which $U_{\mathrm{I}} = \Sigma^-$, $\partial U_{\mathrm{II}} = P_{\lambda^0} \cup \Sigma_2 \cup \beta$ $(\partial U_{\mathrm{II}} = P_{\lambda^0} \cup \Sigma_1 \cup \beta)$, $\beta \subset \partial U$, $U_{\mathrm{III}} = \Sigma^+ \setminus U_{\mathrm{II}}$ $(U_{\mathrm{III}} = \Sigma^-)$; see Figures 4,5.

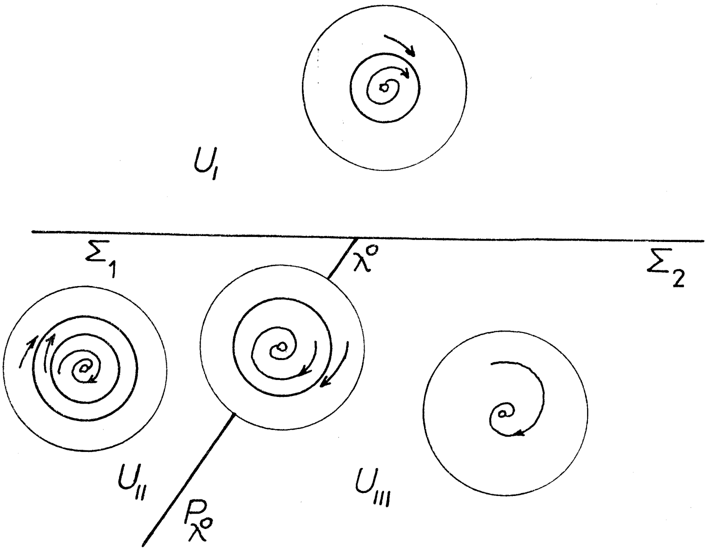

Fig. 5. Bifurcations of the family $(2.9)_\lambda$ in a neighbourhood of λ^0 $(L_1(\lambda^0) = 0,\ L_2(\lambda^0) < 0)$.

(2) *Let $L_1(\lambda) > 0$ for $\lambda \in \Sigma_1$, $L_1(\lambda) < 0$ for $\lambda \in \Sigma_2$ and $L_2 = L_2(\lambda^0) > 0$ Then the system $(2.9)_\lambda$ has one unstable closed orbit and one stable focus for $\lambda \in U_{\mathrm{I}}$. This stable focus bifurcates into a stable closed orbit (Hopf bifurcation) if λ crosses the curve Σ_1, i.e. for $\lambda \in U_{\mathrm{II}}$ there is one stable and one unstable closed orbit and one unstable focus. These two closed orbits bifurcate into one semi-stable closed orbit on the curve P_{λ^0}, which disappears when λ crosses the curve P_{λ^0}, i.e. for $\lambda \in U_{\mathrm{III}}$ there is an unstable focus and no closed orbits.*

(3) *If $L_2(\lambda^0) < 0$, $L_1(\lambda) > 0$ for $\lambda \in \Sigma_1$ and $L_1(\lambda) < 0$ for $\lambda \in \Sigma_2$, then the bifurcation diagram in U looks like that in Figure 5 and the structure of trajectories of $(2.9)_\lambda$ in the corresponding regions $U_{\mathrm{I}}, U_{\mathrm{II}}$ and U_{III} is the same as we have described in (2) for $L_2 > 0$.*

In this paper we consider an unfolding of a germ of vector fields, represented by the following 3-parameter family of vector fields in the plane:

$$(2.10) \qquad \dot{x} = f(x, \varepsilon),$$

where $f = (f_1, f_2)^* \in C^\infty$, $x = (x_1, x_2)^*$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We shall often write $f_\varepsilon(x)$ instead of $f(x, \varepsilon)$. We assume that for $\varepsilon = 0$ the vector field (2.10), denoted by f_0, has the form

$$(2.11)\qquad \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = L\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}(Px, x) + P_1(x) + h_1(x)\\ (Qx, x) + Q_1(x) + h_2(x)\end{bmatrix},$$

where the matrix

$$L = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

is equivalent to the Jordan block

$$S = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

$P = (p_{ij})$, $Q = (q_{ij})$ are symmetric matrices, $(\cdot, \cdot)$ is the scalar product in R^2, $h_i(x) = o(\|x\|^3)$, $i = 1, 2$ and

$$(2.12)\qquad P_1(x) = b_{30}x_1^3 + b_{03}x_2^3 + b_{21}x_1^2x_2 + b_{12}x_1x_2^2,$$

$$(2.13)\qquad Q_1(x) = c_{30}x_1^3 + c_{03}x_2^3 + c_{21}x_1^2x_2 + c_{12}x_1x_1^2.$$

We denote by H^∞ the set of all 3-parameter families of C^∞-vector fields in the plane of the form (2.10) and endow this set with the C^∞-Whitney topology (see [12]). Let us denote by $\tilde{H}^\infty$ the set of all germs at $0 \in R^5$ of all 3-parameter families of vector fields from H^∞.

Now let us formulate the main results of this paper.

Theorem 1. *There exists an open dense subset H_1^∞ of the set H^∞ of all 3-parameter families of vector fields of the form (2.10) such that if $f \in H_1^\infty$, then f is nondegenerate (see Definition 12) and it is possible to transform this family by a smooth regular transformation $(u, \mu) = (\chi(x, \varepsilon), \Psi(\varepsilon))$ in a sufficiently small neighbourhood of the origin into one of the form*

$$v_\mu^\pm:\quad \begin{aligned}\dot{u}_1 &= u_2,\\ \dot{u}_2 &= \gamma_1^\pm(\mu) + \gamma_2^\pm(\mu)\,u_1 + \mu_3u_1^2 \pm u_1^3 + u_1u_2Q(u_1, \mu) + u_2^2\Phi(u, \mu),\end{aligned}$$

where

$$(2.14)\qquad \gamma_1^\pm(\mu) = \pm 2\mu_1 + \mu_2\mu_3 + \tfrac{1}{27}\mu_3^3,$$

$$(2.15)\qquad \gamma_2^\pm(\mu) = \pm(3\mu_2 + \tfrac{1}{3}\mu_3^2),$$

$Q(0,0) = \omega \neq 0.$

Let $D(\mu) = \mu_1^2 + \mu_2^3$, $\mathscr{D} = \{\mu \in R^3: D(\mu) = 0\}$, $\mathscr{D}^+ = \{\mu \in R^3: D(\mu) > 0\}$, $\mathscr{D}^- = \{\mu \in R^3: D(\mu) < 0\}$, $S_1 = \mathscr{D}^+ \cup \{0\}$, $S_2 = \mathscr{D} \setminus \{0\}$, $S_3 = \mathscr{D}^-$, $G_k = \{\mu: \gamma_k^-(\mu) = 0\}$, $G_k^+ = \{\mu: \gamma_k^-(\mu) > 0\}$, $G_k^- = \{\mu: \gamma_k^-(\mu) < 0\}$, $H_k = \{\mu: \gamma_k^+(\mu) = 0\}$, $H_k^+ = \{\mu: \gamma_k^+(\mu) > 0\}$, $H_k^- = \{\mu: \gamma_k^+(\mu) < 0\}$, $k = 1, 2$ and let $\alpha^- = G_1 \cap G_2$, $\alpha^+ = H_1 \cap H_2$ (see Figures 6, 7). By $L(K)$ we denote the matrix of the linear part of a vector field computed at a critical point K.

Theorem 2. *If $f \in H_1^\infty$ and $v_\mu^\pm$ is its normal form (see Theorem 1), then there exists* *a neighbourhood U of the origin in the parameter space and a neighbourhood V of the origin in the phase space such that the following assertions hold:*

(1) *If $\mu \in S_1 \cap U$, then the vector field $v_\mu^+(v_\mu^-)$ has exactly one critical point in V, which is a saddle (a focus or a node; for $\mu = 0$ it may also be a critical point with one elliptic sector, two parabolic and two hyperbolic sectors (see Figure 16)).*

(2) *If $\mu \in S_2 \cap U$, then the vector field $v_\mu^+(v_\mu^-)$ has exactly two critical points: a saddle and a saddle node a saddle node and either a focus or a node).*

(3) *If $\mu \in S_3 \cap U$, then the vector field $v_\mu^+(v_\mu^-)$ has exactly three critical points: two saddles and one focus or three saddles (one saddle and either two foci or two nodes).*

(4) *The sets $H_1, H_2(G_1, G_2)$ are smooth 2-dimensional submanifolds of R^3 and $\alpha^+(\alpha^-)$ is the curve along which the surface $H_1(G_1)$ touches the surface $\mathscr{D}$; see Figures 6, 7.*

(5) *If $\mu \in \mathscr{D}$ and K is the saddle node of the vector field $v_\mu^+(v_\mu^-)$, then the matrix $L(K)$ has zero eigenvalue of multiplicity 2 if and only if $\mu \in \alpha^+$ $(\mu \in \alpha^-)$.*

Theorem 3 *(bifurcations for v_μ^+). If $f \in H_1^\infty$ and U, V are as in Theorem 2, then the following assertions hold:*

(1) *If $\mu \in \mathscr{D}^-$, then the focus K of the vector field v_μ^+ is degenerate (i.e. the matrix $L(K)$ has pure imaginary eigenvalues) if and only if $\mu \in \mathscr{H}^+ = H_1 \cap H_2^- \cap \mathscr{D}^-$.*

(2) *There exists a curve η in the surface $\mathscr{H}^+$, which has one of its end-points at the origin, divides the surface $\mathscr{H}^+$ into two connected components $\mathscr{H}_1^+, \mathscr{H}_2^+$ and the following assertions hold:*

   (a) *The first Ljpunov's focus number $L_1 = L_1(\mu)$ of the focus K is equal to zero if and only if $\mu \in \eta$.*

   (b) *If $\mu \in \mathscr{H}_1^+$ $(\mu \in \mathscr{H}_2^+)$, then $L_1(\mu) > 0$ $(L_1(\mu) < 0)$.*

   (c) *If b_{ij} is the coefficient at $u_1^i u_2^j$ on the right-hand side of the second equation of the vector field v_0^+ and $\mu \in \eta$, then the second Ljapunov's focus number of the focus K is given by the formula*

$$L_2(\mu) = \frac{\pi}{24\sqrt{[-\gamma_2^+(\mu)]}}(N + 0(\|\mu\|)),$$

where $N = -b_{11}^3 b_{02} + b_{11}^2 b_{21} - 7b_{02}b_{11}b_{30} + 3b_{30}b_{21}$, *i.e.* $\operatorname{sign} L_2(\mu) = \operatorname{sign} N$ *for* $\|\mu\|$ *sufficiently small. The number* $\operatorname{sign} N$ *is invariant with respect to regular transformations of coordinates in the phase space.*

(3) *Let P_0 be the plane passing throguh the point $(0, \mu_2^0, 0)$, $\mu_2^0 < 0$, and parallel to the (μ_1, μ_3)-plane. Then the set $P_0 \cap \mathscr{D}$ consists of two lines d_1, d_2 parallel to the μ_3-axis. The closure of the set $P_0 \cap H_1 \cap H_2^- \cap \mathscr{D}^-$ is a curve h, which touches the lines d_1, d_2 at its end-points Q_1 and Q_2, respectively. The set $\eta \cap P_0$ consists of a single point Q (see Figure 6).*

(4) *Let U_1, U_2 and U be sufficiently small neighbourhoods of the points Q_1, Q_2 and Q, respectively, in the plane P_0. Let w_1^+, w_2^+ and w^+ be two-parameter families of vector fields obtained from v_μ^+ by restricting the parameter set to*

the sets U_1, U_2 and U, respectively. Then there exist curves P_i $(i = 1, 2)$ touching the curves $R_i = h \cap U_i$, $s_i = d_i \cap U_i$ at the points Q_i, which form a complete bifurcation diagram for w_i^+ in U_i. (The curves P_i, R_i, s_i correspond to the curves P, R and S_0, respectively, which form Bogdanov's bifurcation diagram.)

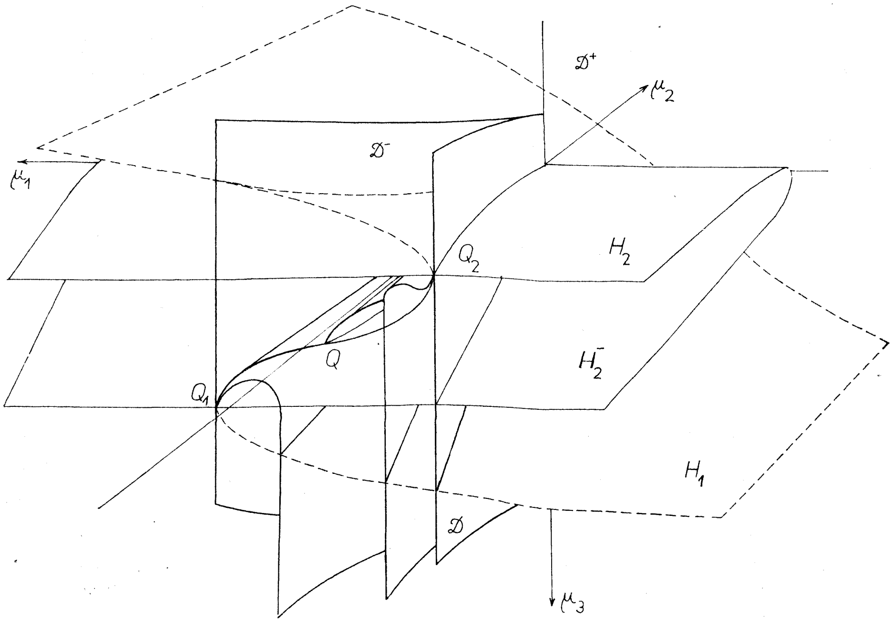

Fig. 6. Bifurcation diagram for v_μ^+.

(5) *If the parameter μ circulates around the point $Q_1(Q_2)$, we obtain bifurcations corresponding to the bifurcations of Bogdanov's normal form with $q > 0$ $(q < 0)$ where, besides the saddle and the focus arising from a saddle-node as in Bogdanov's bifurcation, there is another saddle (see Figures* 10–11).

(6) *The point Q divides the curve $h \cap U$ into two connected components δ^+ and δ^-, where for $\mu \in \delta^+$ $(\mu \in \delta^-)$ we have $L_1(\mu) > 0$ $(L_1(\mu) < 0)$. There is a curve P_Q with one of its end-points at Q, which together with the curve $h \cap U$ divides U into three connected components M_1 M_2, M_3, and the following assertion holds: If $L_2(Q) > 0$, then the bifurcations of the focus are the same as we have described in Lemma* 2, *where the curves P_Q, δ^+, δ^- correspond to the curves P_{λ^0}, Σ_1 and Σ_2, respectively, and the regions M_1, M_2 and M_3 correspond to the regions U_{I}, U_{II} and U_{III}, respectively (similarly for $L_2(Q) < 0$); see Figure* 12.

Theorem 4 (*bifurcations for v_μ^-*). *If $f \in H_1^\infty$ and U, V are as in Theorem* 2, *then the following assertions hold:*

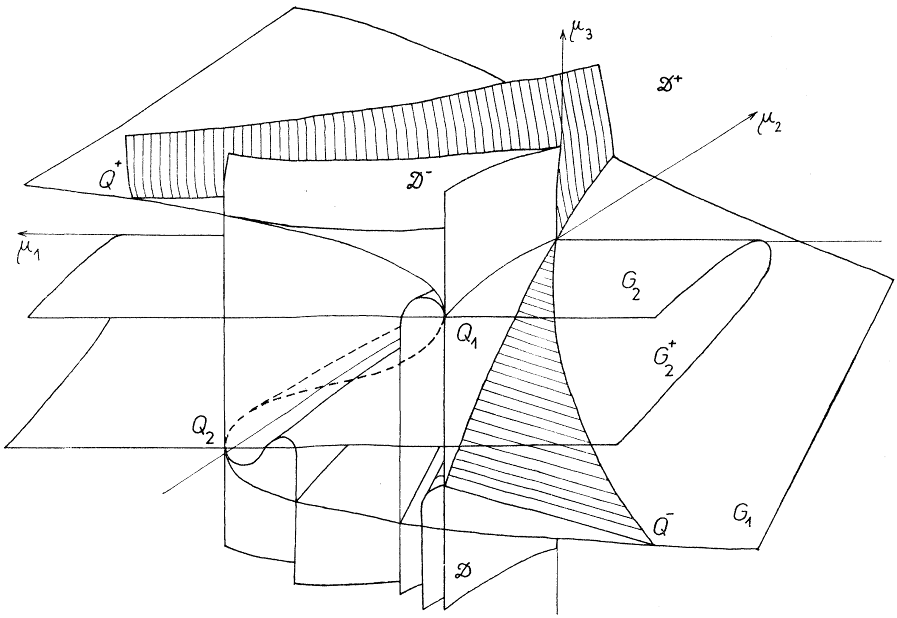

Fig. 7. Bifurcation diagram for v_μ^-.

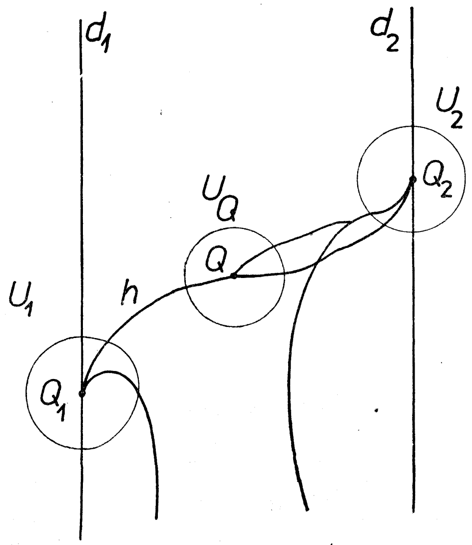

Fig. 8. Bifurcation diagram for v_μ^+ in the plane P_0.

(1) *If $\mu \in \mathscr{D}^-$, then the vector field v_μ^- has two foci K_1 and K_2. The focus $K_1(K_2)$ is degenerate if and only if $\mu \in \mathscr{G}_1 = (G_1 \cap G_2^-) \cap \mathscr{D}^- \cap \{\mu = (\mu_1, \mu_2, \mu_3)$: $\mu_3 > 0\}$ $(\mu \in \mathscr{G}_2 = (G_1 \cap G_2^-) \cap \mathscr{D}^- \cap \{\mu = (\mu_1, \mu_2, \mu_3)$: $\mu_3 < 0\})$. For $\mu \in \mathscr{G}_1$ $(\mu \in \mathscr{G}_2)$ the first Ljapunov's focus number $L_1(\mu)$ is positive (negative).*

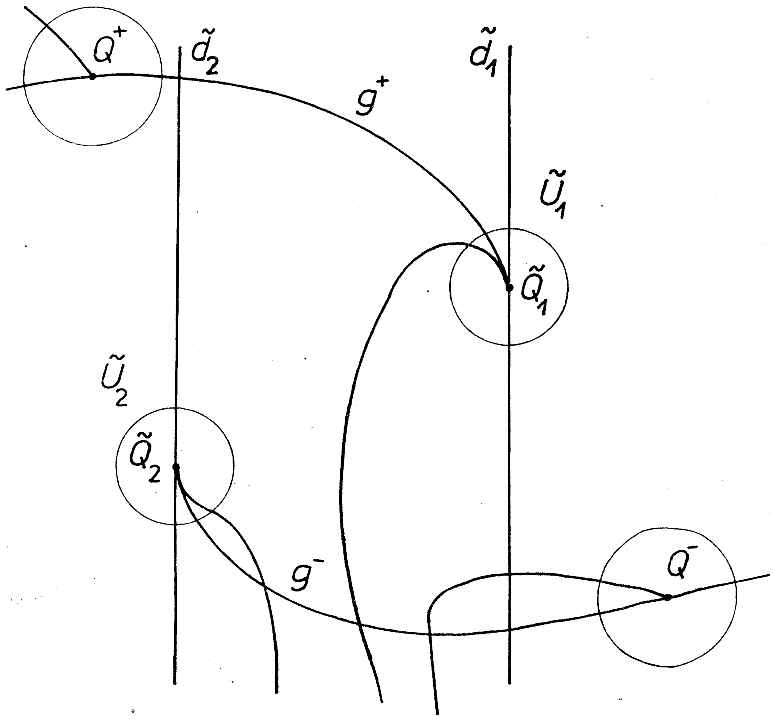

Fig. 9. Bifurcation diagram for v_μ^- in the plane P_0.

(2) *Let P_0, d_1, d_2 be as in Theorem 3(3) and denote $\tilde{d}_1 = d_2$, $\tilde{d}_2 = d_1$. Then the set $g = P_0 \cap G_1 \cap G_2^-$ consists of two connected components g^+ and g^-. The set $g^+(g^-)$ is a curve with the end-point $\tilde{Q}_1 \in \tilde{d}_1(\tilde{Q}_2 \in \tilde{d}_2)$ at which it touches the line $\tilde{d}_1(\tilde{d}_2)$; see Figure 9. Let $\tilde{U}_1, \tilde{U}_2$ be sufficiently small neighbourhoods of the points $\tilde{Q}_1$ and $\tilde{Q}_2$, respectively, in P_0 and let w_1^-, w_2^- be the two-parameter families of vector fields obtained from v_μ^- by restricting the parameter set to the sets $\tilde{U}_1$ and $\tilde{U}_2$, respectively. Then there exist curves $\tilde{P}_i$ $(i = 1, 2)$ touching the curves $\tilde{R}_i = g \cap \tilde{U}_i$, $s_i = \tilde{d}_i \cap \tilde{U}_i$ at the point $\tilde{Q}_i$, which form a complete bifurcation diagram for w_i^- in U_i. (The curves $\tilde{P}_i, \tilde{R}_i, \tilde{s}_i$ correspond to the curves P, R and S_0, respectively, which form Bogdanov's bifurcation diagram). When the parameter μ circulates around the point $\tilde{Q}_1(\tilde{Q}_2)$, we obtain bifurcations corresponding to the bifurcations of Bogdanov's normal form*

with $q > 0$ $(q < 0)$ *where, besides the saddle and the focus arising from a saddle node as in Bogdanov's bifurcation, there is another focus*; *see Figures* 13, 14. *The Hopf bifurcation in Bogdanov's bifurcation near* $\tilde{Q}_1$ *concerns the focus* K_1 (*see the assertion* (1)), *while the same bifurcation near* $\tilde{Q}_2$ *concerns the focus* K_2.

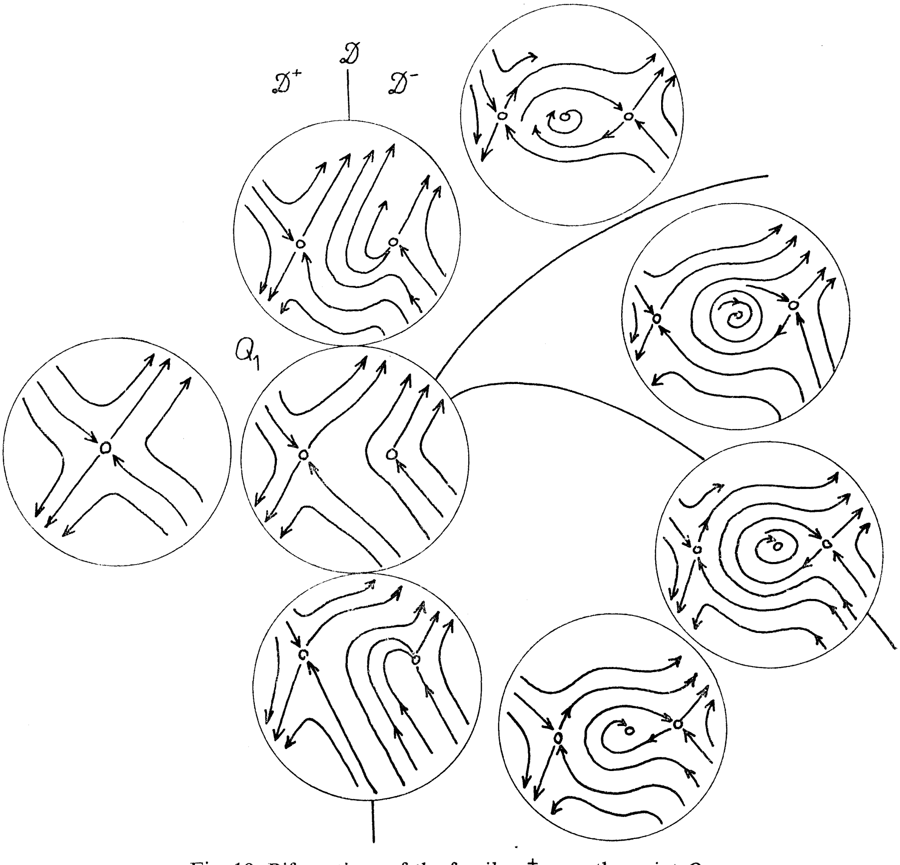

Fig. 10. Bifurcations of the family v_μ^+ near the point Q_1.

(3) *If* $\mu \in \mathscr{D}^+$, *the only focus* K *of the vector field* v_μ^- *is degenerate if and only if* $\mu \in \mathscr{G}^- = G_1 \cap G_2^- \cap \mathscr{D}^+$. *There exist three curves* η_1, η_2, η_3 *in* $\mathscr{G}^-$ *all having one end-point at the origin, which divide the surface* $\mathscr{G}^-$ *into four connected components* $\mathscr{G}_1^-, \mathscr{G}_2^-, \mathscr{G}_3^-, \mathscr{G}_4^-$ *and the following assertions hold*:

(a) *The first Ljapunov's focus number* $L_1(\mu)$ *for the focus* K *is equal to zero if and only if* $\mu \in \eta_1 \cup \eta_2 \cup \eta_3$.

(b) *If* $\mu \in \mathscr{G}_1^- \cup \mathscr{G}_4^-$ $(\mu \in \mathscr{G}_2^- \cup \mathscr{G}_3^-)$, *then* $L_1(\mu) > 0$ $(L_1(\mu) < 0)$; *see Figure* 7.

(c) *If* $\mu \in \eta_1 \cup \eta_2 \cup \eta_3$, *then the same assertion as the assertion* (2)–(c) *from*

Theorem 3 is valid, where in the formula for $L_2(\mu)$ *we have* $\gamma_2^-(\mu)$ *instead of* $\gamma_2^+(\mu)$.

(4) *The curve* $g^+(g^-)$ *intersects the curve* $\eta_1(\eta_2)$ *precisely at one point* $Q^+(Q^-)$. *Let* U^+, U^- *be sufficiently small neighbourhoods of the points* Q^+ *and* Q^-,

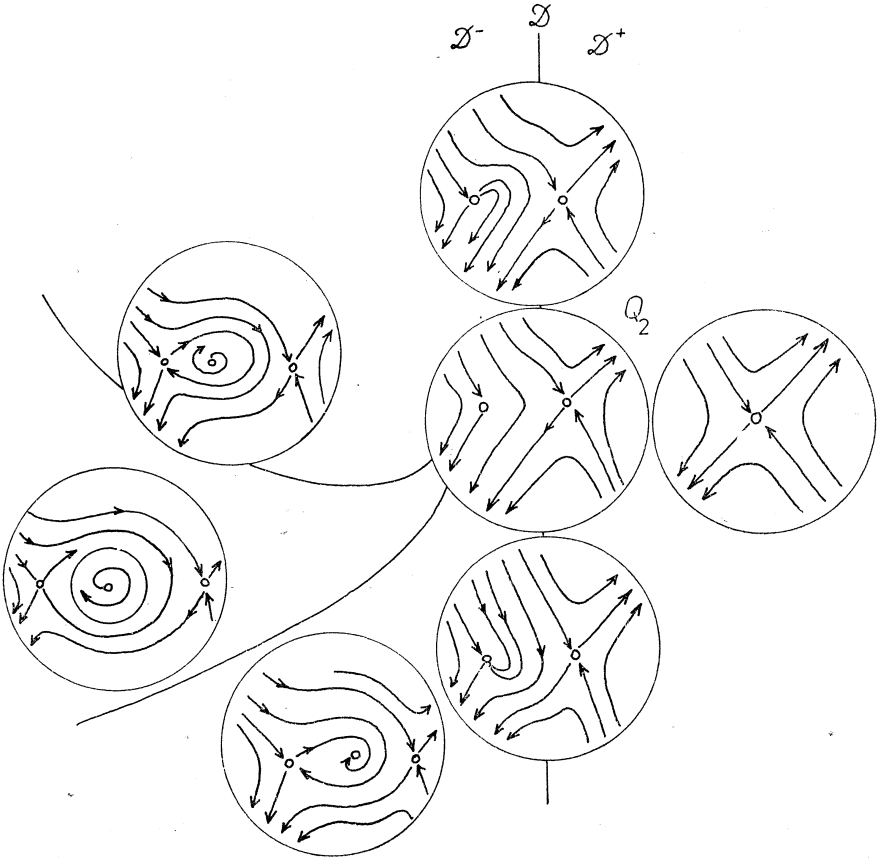

Fig. 11. Bifurcations of the family v_μ^+ near the point Q_2.

respectively. Let w^+, w^- *be the two-parameter families of vector fields obtained from* v_μ^- *by restricting the parameter set to the sets* U^+ *and* U^-, *respectively. Then the following assertion holds: The point* Q^+ *divides the curve* $g^+ \cap U^+$ *into two connected components* δ^+ *and* δ^-, *where we have* $L_1(\mu) > 0$ *for* $\mu \in \delta^+$ *and* $L_1(\mu) < 0$ *for* $\mu \in \delta^-$. *There exists a curve* P_{Q^+} *with one end-point at* Q^+, *which together with the curve* $g^+ \cap U^+$ *divides* U^+ *into three connected components* M_1, M_2, M_3 *and the following holds: If* $L_2(Q^+) > 0$, *then the bifurcations of the focus of the vector field* w^+ *are the same as we have described in*

Lemma 2, *where the curves* P_{Q^+}, δ^+, δ^- *correspond to the curves* P_{λ^0}, Σ_1 *and* Σ_2, *respectively, and the regions* M_1, M_2 *and* M_3 *correspond to the regions* U_{I}, U_{II} *and* U_{III}, *respectively (similarly for* $L_2(Q^+) < 0$*). The same assertion is valid for a neighbourhood* U^- *of the point* Q^- *in* P_0; *see Figures* 4, 15.

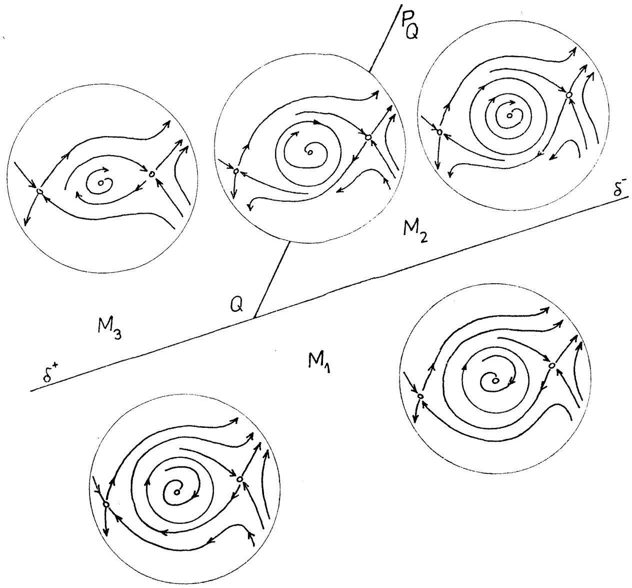

Fig. 12. Bifurcations of the family v_μ^+ near the point Q.

(5) *If* $\hat{P}_1$ *is the plane passing through the point* $(0, \mu_2^1, 0)$, $\mu_2^1 > 0$, *parallel to the plane* P_0, *then* $P_1 \cap G_1$ *is a curve, which intersects the curve* η_3 *precisely at one point* Q^0. *There exists a neighbourhood* U^0 *of the point* Q^0 *in* $\hat{P}_1$ *such that if* w^0 *is the two-parameter family of vector fieds obtained from* v_μ^- *by restricting the parameter set to the set* U^0, *then the same assertion on bifurcations in* U^0 *is valid for* w^0 *as the above assertion* (4) *for the bifurcations of* w^+ *in* U^+.

(6) *The family of vector fields* $v_\mu^\pm$ *with negative Ljapunov's focus number* L_2 *may be obtained from the family of the same form with* $L_2 > 0$ *by using the change of variables* $u_1 \to -u_1$, $\mu_1 \to -\mu_1$, $\mu_3 \to -\mu_3$ *and* $t \to -t$.

Remark 1. Since Theorems 3, 4 are valid for any plane P_0 or $\hat{P}_1$, respectively, sufficiently close to the (μ_1, μ_3)-plane, there must exist surfaces $\mathscr{R}_i$, $\mathscr{P}_i$, $\mathscr{P}(\tilde{\mathscr{R}}_i, \tilde{\mathscr{P}}_i, \tilde{\mathscr{P}})$ such that $\mathscr{R}_i \cap U_i = R_i$, $\mathscr{P}_i \cap U_i = P_i$, $i = 1, 2$, $\mathscr{P} \cap U_Q = P_Q(\tilde{\mathscr{R}}_i \cap \tilde{U}_i = \tilde{R}_i$, $\tilde{\mathscr{P}}_i \cap \tilde{U}_i = \tilde{P}_i$, $i = 1, 2$, $\tilde{\mathscr{P}} \cap U_{Q^+} = P_{Q^+}$, $\tilde{\mathscr{P}} \cap U_{Q^-} = P_{Q^-}$, $\tilde{\mathscr{P}} \cap U_{Q^0} = P_{Q^0})$; see Figure 6 (Figure 7).

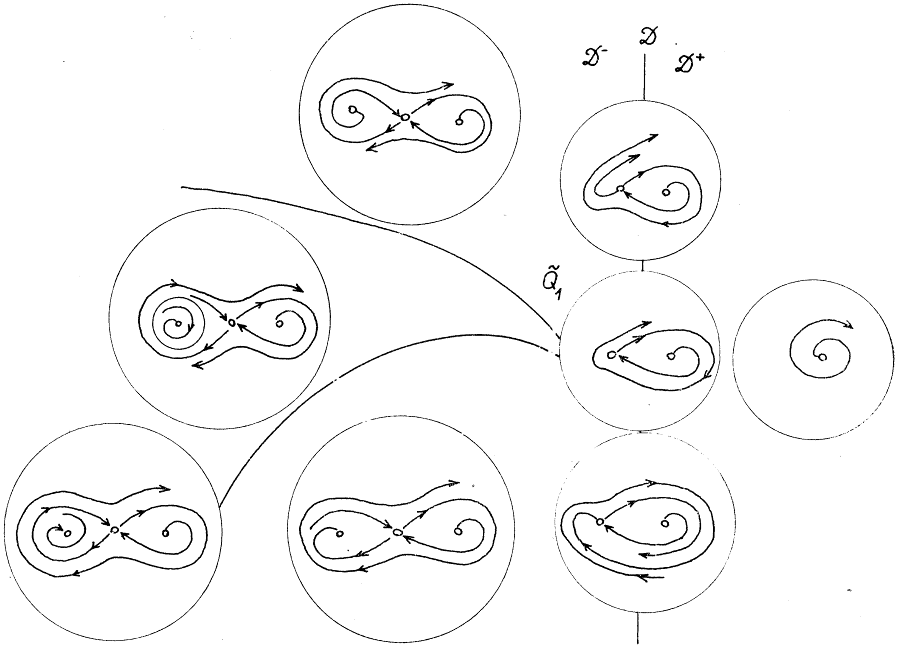

Fig. 13. Bifurcations of the family v_μ^- near the point Q_1.

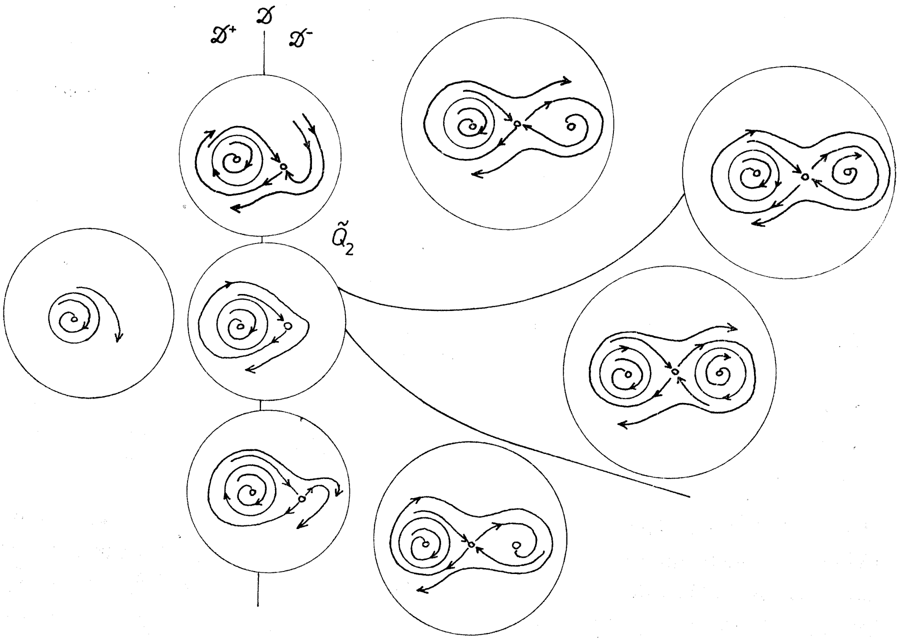

Fig. 14. Bifurcations of the family v_μ^- near the point Q_2.

Remark 2. Since we have imposed no symmetry condition on the families of vector fields, there are simultaneously terms $b_{11}u_1u_2$ and $\pm u_1^3$ in the second equation of the family $v_\mu^\pm$. Therefore there is no scaling like in the cases studied by Bogdanov [7]

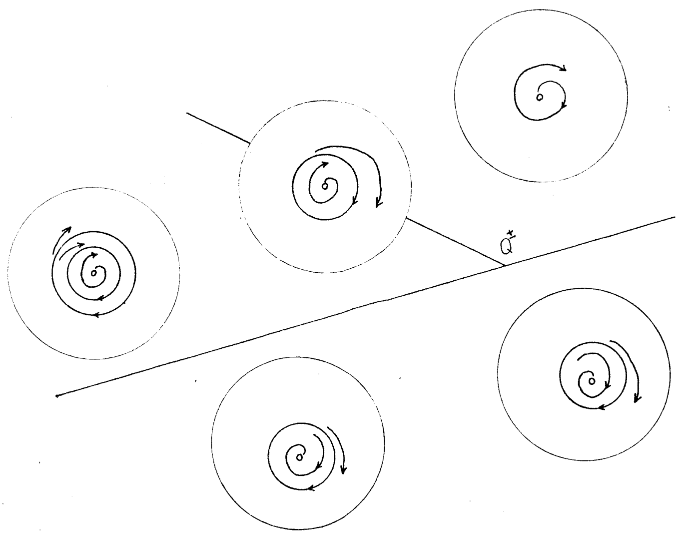

Fig. 15. Bifurcations of the family v_μ^- near the point $Q^\pm$.

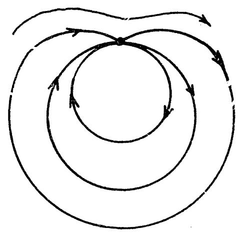

Fig. 16. Critical point of the vector field v_0^- (one of the possibilities; the second on is a focus).

and Takens [20] (see also J. Carr [11]), reducing the families v_μ^+ and v_μ^- to small perturbations of some Hamiltonian systems. This is why the problem concerning the global properties of the surfaces $\mathscr{R}_i, \mathscr{P}_i, \tilde{\mathscr{R}}_i, \tilde{\mathscr{P}}_i, \mathscr{P}, \tilde{\mathscr{P}}$ seems to be not easy. We conjecture that these surfaces probably look like those in Figures 6, 7, because in this case all the local bifurcations described in Theorem 3 and Theorem 4, respectively, form a harmonic whole.

Remark 3. The functions $\gamma_1^\pm(\mu)$, $\gamma_2^\pm(\mu)$ and the terms $b_{20}u_1^2$, $b_{11}u_1u_2$, $b_{02}u_2^2$, $\pm u_1^3$,

$b_{21}u_1^2u_2$ give us all the necessary information for the determination of the local bifurcations of the families $v_\mu^\pm$, as we have described in Theorems 1–4. We conjecture that this information is sufficient for the determination of the global properties of the surfaces $\mathscr{R}_i, \mathscr{P}_i, \tilde{\mathscr{R}}_i, \tilde{\mathscr{P}}_i$, $i = 1, 2$, $\mathscr{P}, \tilde{\mathscr{P}}$ mentioned in Remark 1. Let us regard the functions $f_1^\pm(\mu_1, \mu_2, \mu_3) = \pm 2\mu_1\mu_3 + \frac{1}{2}\mu_2\mu_3^2 + \frac{1}{4.27}\mu_3^4$, $f_2^\pm(\mu_1, \mu_2, \mu_3) = \pm(3\mu_2\mu_3 + \frac{1}{9}\mu_3^3)$ as unfoldings of the functions $\frac{1}{4.27}\mu_3^4$ and $\pm\frac{1}{9}\mu_3^3$, respectively (see e.g. [10, 21]), where μ_3 is a state variable and μ_1, μ_2 are parameters. By Thom's theorem on the seven elementary catastrophes (see [21, Theorem 5.1]) these unfoldings are universal. Since $\partial f_1^\pm/\partial\mu_3 = \gamma_1^\pm(\mu)$ $(\partial f_2^\pm/\partial\mu_3 = \gamma_2^\pm(\mu))$, the sets H_1, G_1 from Theorems 3, 4 (H_2, G_2) are the domains of the catastrophe maps $\chi_1^+ : H_1 \to (\mu_1, \mu_2)$-plane and $\gamma_1^- : G_1 \to (\mu_1, \mu_2)$-plane, respectively, which are defined as the projections of these sets $(X_2^+ : H_2 \to \mu_2$-axis, $\chi_2^- : G_2 \to \mu_2$-axis). The set $\alpha^+(\alpha^-)$ is the set of all non-regular points of the catastrophe map $\chi_1^+(\chi_1^-)$. The projection of the set $\alpha^+(\alpha^-)$ into the (μ_1, μ_2)-plane is the catastrophe map, which is obviously a cusp. The universality of the above mentioned unfoldings is another very weighty argument allowing us to conjecture that the following families are versal:

$$(\pm, \varkappa) \quad \begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= \gamma_1^\pm(\mu) + \gamma_2^\pm(\mu)\, u_1 + \mu_3 u_1^2 \pm u_1^3 + b_{11}u_1u_2 + b_{02}u_2^2 + b_{21}u_1^2u_2, \end{aligned}$$

where $\varkappa = \operatorname{sign} N$, $N = -b_{11}^2 b_{02} + b_{11}^2 b_{21} + 7b_{02}b_{11}b_{30} + 3b_{30}b_{21} \neq 0$. Since the normal forms $(+, -1)$, $(-, -1)$ may be obtained from the families $(+, 1)$ and $(-, 1)$, respectively, by using the change of variables $u_1 \to -u_1$, $\mu_1 \to -\mu_1$, $\mu_3 \to -\mu_3$ and $t \to -t$, it suffices to prove the versality of the families $(+, 1)$ and $(-, 1)$.

3. PRELIMINARY LEMMAS

Lemma 3 ([7, Lemma 1]). *There exists a linear transformation of coordinates $y = Nx$, transforming the system (2.10) into the form*

$$(3.1) \quad \begin{aligned} \dot{y}_1 &= y_2 + (\tilde{P}y, y) + \tilde{P}_1(y) + g_1(y), \\ \dot{y}_2 &= (\tilde{Q}y, y) + \tilde{Q}_1(y) + g_2(y), \end{aligned}$$

where

$$(3.2) \quad \begin{bmatrix} (\tilde{P}y, y) \\ (\tilde{Q}y, y) \end{bmatrix} = N \begin{bmatrix} (N^{-1})^* P N^{-1}y, y) \\ (N^{-1})^* Q N^{-1}y, y) \end{bmatrix},$$

$$(3.3) \quad \begin{bmatrix} \tilde{P}_1(y) \\ \tilde{Q}_1(y) \end{bmatrix} = N \begin{bmatrix} P_1(N^{-1}y) \\ Q_1(N^{-1}y) \end{bmatrix},$$

$$g_i(y) = o(\|y\|^3), \quad i = 1, 2, \quad N = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}, \quad \text{if} \quad b \neq 0 \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 \\ c & -a \end{bmatrix}$$

if $c \neq 0$, $(N^{-1})^*$ is the transpose of N^{-1}.

Let $\tilde{P} = (\tilde{p}_{ij})$, $\tilde{Q} = (\tilde{q}_{ij})$, $\tilde{P}_1(y) = \tilde{b}_{30}y_1^3 + \tilde{b}_{03}y_2^3 + \tilde{b}_{21}y_1^2y_2 + \tilde{b}_{12}y_1y_2^2$, $\tilde{Q}_1(y) = \tilde{c}_{30}y_1^3 + \tilde{c}_{03}y_2^3 + \tilde{c}_{21}y_1^2y_2 + \tilde{c}_{12}y_1y_2^2$.

Lemma 4. *The matrix T and the function $R(x)$ from the righthand side of the system (2.7) have the form*

$$T = \tilde{Q} + \hat{P}\,, \quad \hat{P} = \begin{bmatrix} 0 & \tilde{p}_{11} \\ \tilde{p}_{11} & 2\tilde{p}_{12} \end{bmatrix}, \quad R(x) = t_{30}x_1^3 + t_{21}x_1^2x_2 + t_{12}x_1x_2^2 +$$

$$+ t_{03}x_2^3 + h(x)\,, \quad \textit{where} \quad t_{30} = \tilde{c}_{30} + 2\tilde{p}_{12}\tilde{q}_{11}\,,$$

$$t_{21} = \tilde{q}_{21} + 2(\tilde{p}_{22} - \tilde{p}_{12})\,\tilde{q}_{11} + 3\tilde{b}_{30} \quad \textit{and} \quad h(x) = o(\|x\|^3)\,.$$

Proof. We introduce new coordinates via the following diffeomorphism: $x_1 = y_1$, $x_2 = y_2 + (\tilde{P}y, y) + \tilde{P}_1(y) + g_1(y)$. Obviously, H^{-1}: $y_1 = x_1$, $y_2 = x_2 - (\tilde{P}x, x) + o(\|x\|^2)$ and by direct computation one can easily show that the new system has the form

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = (\tilde{Q}x, x) + \tilde{Q}_1(x) + 2\tilde{p}_{11}x_1x_2 + 2\tilde{p}_{12}x_2^2 + 2\tilde{p}_{12}\tilde{q}_{11}x_1^3 +$$

$$+ [\tilde{q}_{21} + 2(\tilde{p}_{22} - \tilde{p}_{12})\,\tilde{q}_{11} + 3\tilde{c}_{30}]\,x_1^2x_2 + t_{12}x_1x_2^2 + t_{03}x_2^3 + h(x)\,,$$

where $h(x) = o(\|x\|^3)$.

We have obtained a smooth regular transformation of coordinates $\Phi = H \circ N$, which transforms the system (2.11) into the form (2.7).

Lemma 5. *Let $T = (t_{ij})$ and $R(x)$ be as in Lemma 4. Then the following assertions hold:*

(1) *The property $t_{11} = 0$ is invariant with respect to smooth regular transformations of coordinates.*

(2) *If $t_{11} = 0$, then the number $p = t_{30}/t_{12}^2$ is invariant with respect to smooth regular transformations of coordinates, i.e., it does not depend on any choice of coordinates in which the system (2.11) has the form (2.7).*

Proof. The assertion (1) follows immediately from the proof of [7, Lemma 3]. Since we shall use the idea and some relations from this proof also in the proof of the assertion (2), we give the proof of (1).

Let N be the matrix from Lemma 3 and H the mapping from the proof of Lemma 4. Then the mapping $\Phi = H \circ N$ transforms the system (2.11) into the form (2.7). If Φ' is another mapping transforming the system (2.11) into the form (2.7), then $\Phi' \circ \Phi^{-1}$ is the regular transformation, transforming the system (2.7) into the same form. Therefore it suffices to prove the invariance of p with respect to the regular transformations transforming the system (2.7) into the same form. An arbitrary transformation with this property is composed of the mappings $H \circ \varrho$ and R, where ϱ is a linear mapping which does not change the linear part of the system and R is a nonlinear mapping having its linear part equal to the identity and transforming the system (2.7) into the same form. Let the mapping R be defined as follows:

(3.4)
$$R:\ y_1 = x_1 + X_1(x) + Y_1(x) + o(\|x\|^3)\,,$$
$$y_2 = x_2 + X_2(x) + Y_2(x) + o(\|x\|^3)\,,$$

where X , Y , $i = 1, 2$ are homogeneous polynomials of degree 2 and 3, respectively. The mapping ϱ must be of the form

(3.5)
$$\varrho:\ y = \begin{bmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{bmatrix} x\,,$$

where λ, ε are real numbers, $\lambda \neq 0$.

Since the mapping R transforms the system (2.7) into the same form and in new coordinates we obtain that

$$\dot{y}_1 = y_2 - X_2(y) + \frac{\partial X_1(y)}{\partial y_1}\, y_2 + o(\|y\|^2)\,,$$

the function X_2 must satisfy the equality $X_2(y) = (\partial X_1(y)/\partial y_1)\, y_2$. Therefore we have

$$\dot{y}_2 = (Ty, y) + \frac{\partial X_2(y)}{\partial y_1}\, y_2 + o(\|y\|^2) = (Ty, y) + \frac{\partial^2 X_1(y)}{\partial y_1^2} y_2 + o(\|y\|^2)$$

and this proves that the mapping R does not change the numbers t_{11}, t_{12}.

If $y_1 = \lambda x_1 + \varepsilon x_2$, $y_2 = \lambda x_2$, $\lambda, \varepsilon \in R^1$, $\lambda \neq 0$, then $x_2 = \lambda^{-1} y_2$, $x_1 = \lambda^{-1} y_1 - \varepsilon \lambda^{-2} y_2$. In these new coordinates we obtain a system of the form (3.1), where $\tilde{p}_{11} = \varepsilon \lambda^{-2} t_{11}$, $\tilde{p}_{12} = \frac{1}{2}(t_{12}\lambda^{-2}\varepsilon - 2\varepsilon^2\lambda^{-3} t_{11})$, $\tilde{q}_{11} = \lambda^{-1} t_{11}$, $\tilde{q}_{12} = \frac{1}{2}(\lambda^{-1} t_{12} - 2\varepsilon\lambda^{-2} t_{11})$, $\tilde{c}_{30} = \lambda^{-2} t_{30}$. By Lemma 1 and Lemma 4, there is a smooth regular transformation transforming this system into the form $\dot{x}_1 = x_2$, $\dot{x}_2 = (\tilde{T}x, x) + \tilde{t}_{30} x_1^3 + \tilde{T}_3(x) + \tilde{h}(x)$, where

$$\tilde{T} = (\tilde{t}_{ij}) = \tilde{Q} + \begin{bmatrix} 0 & \tilde{p}_{11} \\ \tilde{p}_{11} & 2\tilde{p}_{12} \end{bmatrix}, \quad \tilde{t}_{30} = \tilde{c}_{30} + 2\tilde{p}_{12}\tilde{q}_{11}\,.$$

Therefore we have

(3.6)
$$\tilde{t}_{11} = \lambda^{-1} t_{11}\,, \quad \tilde{t}_{12} = \lambda^{-1} t_{12}\,,$$
$$\tilde{t}_{30} = \lambda^{-2} t_{30} + 2(t_{12}\lambda^{-2}\varepsilon - 2\varepsilon^2\lambda^{-3} t_{11})(\lambda^{-1} t_{11})\,.$$

Thus the property $t_{11} = 0$ is invariant with respect to the mappings R and ϱ. If $t_{11} = 0$, then (3.6) implies that $\tilde{t}_{30} = \lambda^{-2} t_{30}$, $t_{12}^2 = \lambda^{-2} t_{12}^2$ and thus the number p is also invariant with respect to the mapping ϱ. Now, it suffices to prove the invariance of p with respect to the mapping R. In the coordinates defined by R we have

$$\dot{y}_2 = \left\{(Tx, x) + t_{30}x_1^3 + T_3(x) + \frac{\partial X_2}{\partial x_1} x_2 + \frac{\partial X_2}{\partial x_2}(Tx, x) + \frac{\partial Y_2}{\partial x_1} x_2 + \right.$$

$$\left. + \frac{\partial Y_2}{\partial x_2}(Tx, x) + o(\|x\|^3)\right\}_{x=R^{-1}y}.$$

Therefore if $t_{11} = 0$, then $\dot{y}_2 = (Ty, y) + t_{30}y_1^3 + \hat{T}_3(y) + o(\|y\|^3)$, where $\hat{T}_3(y)$ is a homogeneous polynomial of degree 3 in y_1, y_2, which does not contain any term with y_1^3, i.e. the number t_{30} is invariant with respect to the mapping R. We have shown above that the number t_{12} is invariant with respect to R and so the number p is also invariant with respect to this map. This completes the proof.

4. TRANSFORMATION INTO THE NORMAL FORM

Using Lemma 1 we can rewrite the system (2.10) into the form

$$(4.1) \qquad v: \begin{aligned} \dot{x}_1 &= x_2 + v_1(x, \varepsilon), \\ \dot{x}_2 &= t_{12}x_1x_2 + t_{22}x_2^2 + t_{30}x_1^3 + Q_3(x) + v_2(x, \varepsilon), \end{aligned}$$

where $v_1, v_2 \in C^\infty$, $v_1(x, 0) \equiv 0$, $v_2(x, 0) = o(\|x\|^3)$, $Q_3(x)$ is a homogeneous polynomial of degree 3 in x_1, x_2 which does not contain the power x_1^3. We assume $t_{11} = 0$, $t_{12} \neq 0$, $t_{30} \neq 0$.

Let us choose new coordinates: $y = \sqrt{|p|}\, t_{12}x$. Then we obtain

$$\begin{aligned} \dot{y}_1 &= y_2 + \tilde{v}_1(y, \varepsilon), \\ \dot{y}_2 &= \frac{1}{\sqrt{|p|}}\left\{y_1y_2 + \frac{t_{22}}{t_{12}} y_2^2\right\} + (\operatorname{sign} p)\, y_1^3 + \tilde{Q}_3(y) + \tilde{v}_2(y, \varepsilon), \end{aligned}$$

where $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{Q}_3$ have the same properties as v_1, v_2 and Q_3, respectively. Therefore we may assume that (4.1) has the form

$$(4.2) \qquad \begin{aligned} \dot{x}_1 &= x_2 + \tilde{v}_1(x, \varepsilon), \\ \dot{x}_2 &= \omega x_1x_2 + \omega_{02}x_2^2 + \sigma x_1^3 + \hat{Q}_3(x) + \hat{v}_2(x, \varepsilon), \end{aligned}$$

where $\omega = 1/\sqrt{|p|}$ is the invariant of the germ represented by the family (2.10), $\sigma = \operatorname{sign} p$, $\hat{Q}_3$ and $\tilde{v}_1$, $\hat{v}$ have the same properties as Q_3 and v_1, v_2, respectively.

After introducing new coordinates $y_1 = x_1$, $y_2 = x_2 + \tilde{v}_1(x, \varepsilon)$, (4.2) becomes

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \omega y_1y_2 + \omega_{02}y_2^2 + \sigma y_1^3 + Q_3'(y) + v_2'(y, \varepsilon), \end{aligned}$$

where Q_3' and v_2' have the same properties as Q_3 and v_2, respectively. We can rewrite this system into the form

$$(4.3)\qquad \begin{aligned}\dot{y}_1 &= y_2\,,\\ \dot{y}_2 &= \tilde{F}(y,\varepsilon) + y_2\hat{Q}(y_1,\varepsilon) + y_2^2\tilde{\Psi}(y,\varepsilon)\,,\end{aligned}$$

where

$$\tilde{F}, \tilde{Q}, \tilde{\Psi} \in C^\infty\,, \quad \tilde{F}(0,0) = \frac{\partial \tilde{F}(0,0)}{\partial y_1} = \frac{\partial^2 \tilde{F}(0,0)}{\partial y_1^2} = 0\,,$$

$$\frac{\partial^3 \tilde{F}(0,0)}{\partial y_1^3} = 6\sigma\,, \quad \frac{\partial \tilde{Q}(0,0)}{\partial y_1} = \omega\,, \quad \tilde{Q}(0,0) = 0\,.$$

Lemma 6. *If* $\omega \neq 0$, *then there exists a smooth regular transformation* $z = z(y, \varepsilon)$, $z(0, 0) = 0$ *transforming the system* (4.3) *into the form*

$$(4.4)\qquad \begin{aligned}\dot{z}_1 &= z_2\,,\\ \dot{z}_2 &= F(z_1,\varepsilon) + z_1 z_2 G(z_1,\varepsilon) + z_2^2\Psi(z,\varepsilon)\,,\end{aligned}$$

where

$$F, G, \Psi \in C^\infty\,, \quad F(0,0) = \frac{\partial F(0,0)}{\partial z_1} = \frac{\partial^2 F(0,0)}{\partial z_1^2} = 0\,,$$

$$\frac{\partial^3 F(0,0)}{\partial z_1^3} = 6\sigma\,, \quad G(0,0) = \omega\,.$$

Proof. If $z_1 = y_1 - \alpha(\varepsilon)$, $z_2 = y_2$, then

$$\begin{aligned}\dot{z}_1 &= z_2\,,\\ \dot{z}_2 &= \tilde{F}(y_1,\varepsilon) + y_2\tilde{Q}(y_1,\varepsilon) + y_2^2\tilde{\Psi}(y,\varepsilon) = \tilde{F}(z_1 + \alpha(\varepsilon),\varepsilon) +\\ &\quad + y_2\tilde{Q}(z_1 + \alpha(\varepsilon),\varepsilon) + y_2^2\tilde{\Psi}(z + \alpha(\varepsilon),\varepsilon)\,.\end{aligned}$$

We have $\tilde{Q}(z_1 + \alpha(\varepsilon), \varepsilon) = \tilde{Q}(\alpha(\varepsilon), \varepsilon) + z_1\hat{Q}(z_1, \varepsilon)$, where $\tilde{Q}(0, 0) = 0$, $\partial\tilde{Q}(0, 0)/\partial\alpha = \omega$, $\hat{Q}(0, 0) = \omega$. Since $\omega \neq 0$, the Implicit Function Theorem implies that there exists a neighbourhood U of $0 \in R^3$ and a smooth function $\alpha: U \to R^1$ such that $\alpha(0) = 0$, $\tilde{Q}(\alpha(\varepsilon), \varepsilon) = 0$ for all $\varepsilon \in U$ and we obtain a system of the form (4.4).

Lemma 7. *If* $\omega \neq 0$, *then there exists a smooth regular transformation* $u = u(z, \varepsilon)$, $u(0, 0) = 0$ *transforming the system* (4.4) *into the form*

$$(4.5)\qquad \begin{aligned}\dot{u}_1 &= u_2\,,\\ \dot{u}_2 &= \varphi_1(\varepsilon) + \varphi_2(\varepsilon)\,u_1 + \varphi_3(\varepsilon)\,u_1^2 + \sigma u_1^3 +\\ &\quad + u_1 u_2 Q_1(u_1,\varepsilon) + u_2^2\Phi_1(u,\varepsilon)\,,\end{aligned}$$

where $\varphi_i \in C^\infty$, $\varphi_i(0) = 0$, $i = 1, 2, 3$, $Q_1, \Phi_1 \in C^\infty$, $Q_1(0, 0) = \omega$.

Proof. Let the function F be the function from Lemma 6. Then the Malgrange-Weierstrass preparation theorem (see [15, V, p. 82] and also [10,Theorem 6.3]) implies that there exist smooth functions $\varphi_i(\varepsilon)$, $i = 1, 2, 3$, $B(z_1, \varepsilon)$ such that $\varphi_i(0) = 0$, $i = 1, 2, 3$, $B(0, 0) = 1$ and $F(z_1, \varepsilon) = [\sigma z_1^3 + \varphi_3(\varepsilon) z_1^2 + \varphi_2(\varepsilon) z_1 + \varphi_1(\varepsilon)] \cdot B(z_1, \varepsilon)$ for (z_1, ε) from a sufficiently small neighbourhood of the origin. If $u_1 = z_1$, $u_2 = z_2/\sqrt{B(z_1, \varepsilon)}$, then the system (4.4) becomes

$$\dot{u}_1 = u_2 \Theta(u_1, \varepsilon),$$

$$\dot{u}_2 = [\varphi_1(\varepsilon) + \varphi_2(\varepsilon) u_1 + \varphi_3(\varepsilon) u_1^2 + \sigma u_1^3 + u_1 u_2 Q_1(u_1, \varepsilon) + u_2^2 \Phi_1(u, \varepsilon)] \Theta(u_1, \varepsilon),$$

where

$$Q_1(u_1, \varepsilon) = \frac{G(u_1, \varepsilon)}{B(u_1, \varepsilon)}, \quad \Phi_1 \in C^\infty, \quad \Theta(u_1, \varepsilon) = \sqrt{B(u_1, \varepsilon)}.$$

Using the transformation of time $s = \alpha(t) = \int_0^t \Theta(u_1(\tau), \varepsilon) \, d\tau$ we divide the system by $\Theta(u_1, \varepsilon)$ and thus obtain the system (4.5).

5. BASIC ALGEBRAIC MANIFOLDS

Let $M(i, j)$ be the set of all $i \times j$-matrices and $M(k) = M(k, k)$. We can identify any 2-jet $\alpha \in J_2^2(x)$ with a couple of matrices (L, K), where $L \in M(2)$ and $K \in M(2, 3)$. More precisely, if $f: R^2 \to R^2$ is a smooth mapping, then $j^2 f(x) = (L(f)(x), K(f)(x))$, where

$$f = (f_1, f_2), \quad L(f)(x) = Df(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2) \quad \text{and}$$

$$K(f)(x) = \begin{bmatrix} p_{11} & p_{12} & p_{22} \\ q_{11} & q_{12} & q_{22} \end{bmatrix}, \quad p_{11} = \frac{\partial^2 f_1(x)}{\partial x_1^2}, \quad p_{12} = \frac{\partial^2 f_1}{\partial x_1 \, \partial x_2}, \quad p_{22} = \frac{\partial^2 f_1(x)}{\partial x_2^2},$$

$$q_{11} = \frac{\partial^2 f_2(x)}{\partial x_1^2}, \quad q_{12} = \frac{\partial^2 f_2(x)}{\partial x_1 \, \partial x_2}, \quad q_{22} = \frac{\partial^2 f_2(x)}{\partial x_2^2}.$$

Let us define the following subsets of $J_2^2 = J_2^2(0)$:

(5.1) $$T_{jk} = \{(L, K) \in J_2^2 : F_i(L, K) = 0, \quad i = 1, 2, \quad F_3 = F_{jk}(L, K) = t_{jk} = 0, \quad \operatorname{rank} L = 1\}, \quad (j, k) = (1, 1), (1, 2), \quad F_1 = \operatorname{Tr} L = a + b,$$
$$F_2 = \det L = ad - bc,$$

(5.2) $$t_{11} = ap_{11} - 2\frac{a^2}{b} p_{12} + \frac{a^3}{b^2} p_{22} + bq_{11} - 2aq_{12} + \frac{a^2}{b} q_{22},$$

(5.3) $$t_{12} = 2p_{11} - 2\frac{a}{b} p_{12} + 2q_{12} - 2\frac{a}{b} q_{22},$$

under the assumption $b \neq 0$. Since $\operatorname{rank} L = 1$, we have $b^2 + c^2 \neq 0$. If $c \neq 0$, then using Lemma 3 one can show that t_{11} has the form (5.2), where the variables in this expression are changed as follows: $b \to c$, $p_{11} \to q_{22}$, $p_{12} \to q_{12}$, $p_{22} \to q_{11}$, $p_{11} \to p_{22}$, $q_{12} \to p_{12}$, $q_{22} \to p_{11}$. Similarly for t_{12}: $b \to c$, $p_{11} \to q_{22}$, $p_{12} \to q_{12}$, $q_{12} \to p_{12}$, $q_{22} \to p_{11}$. If $c \neq 0$, then similarly to the sets T_{11}, T_{12} we can define the sets $\tilde{T}_{11}$ and $\tilde{T}_{12}$, respectively. Denote $F = (F_1, F_2, F_3)$: $R^{10} \to R^3$.

Lemma 8. *The sets T_{11}, $\tilde{T}_{11}$, T_{12}, $\tilde{T}_{12}$ are smooth submanifolds of J_2^2 of codimension* 3.

Proof. We will prove the assertion of Lemma 8 for the sets T_{11} and T_{12}. The proof of the assertion for the sets $\tilde{T}_{11}$, $\tilde{T}_{12}$ is analogous. It suffices to show that rank $DF = 3$. Let $H_i = (h_{kj})$, $i = 1, 2$, where

$$h_{kl} = \frac{\partial F_k}{\partial a}, \quad h_{k2} = \frac{\partial F_k}{\partial c}, \quad h_{k3} = \frac{\partial F_k}{\partial q_{1i}}, \quad k = 1, 2,$$

$$h_{31} = \frac{\partial F_{1i}}{\partial a}, \quad h_{32} = \frac{\partial F_{1i}}{\partial c}, \quad h_{33} = \frac{\partial F_{1i}}{\partial q_{11}}.$$

Then $\det H_1 = -b^2$ and $\det H_2 = -2b$. Therefore the mapping F corresponding to the set T_{11} and also to the T_{12} satisfies rank $DF = 3$.

We can identify any 3-jet $\beta \in J_2^3(x)$ with a triple of matrices (L, K, M), where $(L, K) \in J_2^2(x)$ and $M \in M(2, 4)$. More precisely, if f: $R^2 \to R^2$ is a smooth mapping, then $j^3 f(x) = (L(f)(x), K(f)(x), M(f)(x))$, where

$$(L(f)(x), K(f)(x)) \in J_2^2(x), \quad M(f)(x) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ s_{11} & s_{12} & s_{13} & s_{14} \end{bmatrix},$$

$$r_{11} = \frac{\partial^3 f_1(x)}{\partial x_1^3}, \quad r_{12} = \frac{\partial^3 f_1(x)}{\partial x_1^2 \partial x_2}, \quad r_{13} = \frac{\partial^3 f_1(x)}{\partial x_1 \partial x_2^2}, \quad r_{14} = \frac{\partial^3 f_1(x)}{\partial x_2^3},$$

$$s_{11} = \frac{\partial^3 f_4(x)}{\partial x_1^3}, \quad s_{12} = \frac{\partial^3 f_2(x)}{\partial x_1^2 \partial x_2}, \quad s_{13} = \frac{\partial^3 f_2(x)}{\partial x_1 \partial x_2^2}, \quad s_{14} = \frac{\partial^3 f_4(x)}{\partial x_2^3}.$$

Let us define the following subsets of $J_2^3 = J_2^3(0)$:

$$T_{30} = \{(L, K, M) \in J_2^3 \colon F_i(L, K) = 0, \ i = 1, 2, \ F_4(L, K, M) = 0, \ \text{rank}\, L = 1\},$$

$$T_{3jk} = \{(L, K, M) \in J_2^3 \colon F_i(L, K) = 0, \ i = 1, 2, \ F_3 = F_{jk}(L, K) = 0,$$
$$F_4 = F_4(L, K, M) = 0, \ \text{rank}\, L = 1\},$$

$(j, k) = (1, 1), (1, 2)$, $F_i(L, K)$, $i = 1, 2$, $F_{jk}(L, K)$ are defined as above and $F_4 = t_{30} = \tilde{c}_{30} + 2\tilde{p}_{12}\tilde{q}_{11}$ (see Lemma 4), where

$$\tilde{p}_{12} = \frac{1}{b}\left(p_{12} - \frac{a}{b} p_{22}\right), \quad \tilde{q}_{11} = -2\frac{a}{b} q_{12} + \left(\frac{a}{b}\right)^2 p_{22},$$

$$\tilde{c}_{30} = -\frac{a}{b} b_{30} - b_{03}\left(\frac{a}{b}\right)^3 + \frac{1}{b} c_{30} - c_{03}\left(\frac{a}{b}\right)^3.$$

If $c \neq 0$, then one can show that

$$\tilde{c}_{30} = b_{30} \frac{a^3}{c^2} + cb_{03} + b_{21} \frac{a^2}{c} + b_{12}a - c_{30} \frac{a^4}{c^3} - ac_{03} - c_{21} \frac{a^3}{c^2} - c_{12} \frac{a^2}{c}.$$

Similarly we can express p_{12} ano q_{11}. In this case we can define sets $\tilde{T}_{30}$ and $\tilde{T}_{3jk}$ in a similar way as we have defined the sets T_{30} and T_{3jk}.

Lemma 9. *The sets T_{311}, T_{312}, $\tilde{T}_{311}$, $\tilde{T}_{312}$ are smooth submanifolds of J_2^3 of codimension 4 and the sets T_{30}, $\tilde{T}_{30}$ are smooth submanifolds of J_2^3 of codimension 3.*

Proof. We will prove the assertion of Lemma 9 for the sets T_{311}, T_{312} and T_{30} only. The proof for the sets $\tilde{T}_{311}$, $\tilde{T}_{312}$, $\tilde{T}_{30}$ is analogous. Let $F_i = (F_1, F_2, F_{1i}, F_4)$: $R^{18} \to R^4$ be the mapping with the components defined in the definition of the set T_{31i} $(i = 1, 2)$ and let $H_i \doteq (h_{kj})$, where

$$h_{kl} = \frac{\partial F_k}{\partial a}, \quad h_{k2} = \frac{\partial F_k}{\partial c}, \quad h_{k3} = \frac{\partial F_k}{\partial q_{1i}}, \quad h_{k4} = \frac{\partial F_k}{\partial c_{30}},$$

$$k = 1, 2, 4, \quad h_{31} = \frac{\partial F_{1i}}{\partial a}, \quad h_{32} = \frac{\partial F_{1i}}{\partial c}, \quad h_{33} = \frac{\partial F_{1i}}{\partial q_{1i}}, \quad h_{34} = \frac{\partial F_{1i}}{\partial c_{30}}.$$

Then $\det H_1 = -b$ and $\det H_2 = -2$. Therefore rank $DF_1 = 4$ and rank $DF_2 = 4$ and thus the sets T_{311} and T_{312} are smooth submanifolds of J_2^3 of codimension 4. The proof for the set T_{30} is similar to the cases of the sets T_{11} and T_{12} (see the proof of Lemma 8).

Let us define the following sets: $T_i = \{(0, 0)\} \times T_{1i}$, $\tilde{T}_i = \{(0, 0)\} \times \tilde{T}_{1i} \subset R^2 \times J_2^2$, $i = 1, 2$, $T_3 = \{(0, 0)\} \times T_{30}$, $\tilde{T}_3 = \{(0, 0)\} \times \tilde{T}_{30} \subset R^2 \times J_2^3$, $T_{3j} = \{(0, 0)\} \times T_{31}j$, $\tilde{T}_{3j} = \{(0, 0)\} \times \tilde{T}_{31j} \subset R^2 \times J_2^3$, $j = 1, 2$. As a consequence of Lemmas 8 ano 9 we have

Lemma 10. *The sets T_1, T_2, $\tilde{T}_1$, $\tilde{T}_2$ are smooth submanifolds of $R^2 \times J_2^2$ of codimension 5 and the sets T_3, T_{31}, T_{32}, $\tilde{T}_3$, $\tilde{T}_{31}$, $\tilde{T}_{32}$ are smooth submanifolds of $R^2 \times J_2^3$, where* codim $T_3 = 5$, codim $\tilde{T}_3 = 5$, codim $T_{31} =$ codim $\tilde{T}_{31} =$ codim $T_{32} =$ codim $\tilde{T}_{32} = 6$.

Denote by H_2^∞ the set of all 2-parameter families of smooth vector fields of the form (2.5) and by $H_3^\infty = H^\infty$ the set of all 3-parameter families of smooth vector fields of the form (2.10).

Given any $g \in H_2^\infty$ we define the mapping $\varrho(g)\colon (x, \varepsilon) \to (g(x, \varepsilon), \pi_2 \tilde{G}_{(x,\varepsilon)})$, where $G_{(x,\varepsilon)}\colon (y, \mu) \to g(x + y, \varepsilon + \mu) - g(x, \varepsilon)$, $\tilde{G}_{(x,\varepsilon)}$ is the germ of $G_{(x,\varepsilon)}$ at $(0, 0) \in R^2 \times R^2$ and $\pi_2\colon G_2 \to J_2^2$ is the natural projection (here we have dim $\varepsilon = 2$!).

Lemma 11 ([7, Lemma 4]). *The set*

$$\Sigma^2 = L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2)\colon a + d = 0, \ ad - bc = 0, \quad a^2 + b^2 + c^2 + d^2 \neq 0\}$$

is a smooth submanifold in $M(2)$ of codimension 2.

This lemma implies that the se: $\Sigma = \{(0, 0)\} \times \Sigma^2 \subset R^2 \times R^2 \times M(2)$ is a smooth submanifold in $R^2 \times R^2 \times M(2)$ of codimension 4.

Definition 11. A two-parameter family $g \in H_2^\infty$ is called *nondegenerate*, if we have $t_{11} \cdot t_{12} \neq 0$ for the vector field g_0 (see (2.7)) and

(5.4) $$\varrho(g) \mathrel{\bar{\pitchfork}}_{(0,0)} \Sigma ,$$

i.e. the mapping $\varrho(g)$ transversally intersects the set Σ at $(0, 0) \in R^2 \times R^2$.

Given any 3-parameter family $f \in H^\infty$ and any natural number i, we define the mapping

(5.5) $$\varrho_i(f): (x, \varepsilon) \to (f(x, \varepsilon), \pi_i \tilde{F}_{(x,\varepsilon)}) ,$$

where $F_{(x,\varepsilon)}: (y, \mu) \to f(x + y, \ \varepsilon + \mu) - f(x, \varepsilon), \tilde{F}_{(x,\varepsilon)}$ is the germ of $F_{(x,\varepsilon)}$ at $(0, 0) \in R^2 \times R^3$ and $\pi_i: G_2 \to J_2^i$ is the natural projection (here we have $\dim \varepsilon = 3$!).

Definition 12. A 3-parameter family $f \in H^\infty$ is called *nondegenerate*, if we have $t_{12} \cdot t_{30} \neq 0$ for the vector field f_0 (see (2.7) and Lemma 4) and

(5.6) $$\varrho_2(f) \mathrel{\bar{\pitchfork}}_{(0,0)} T_1 , \quad \varrho_2(f) \mathrel{\bar{\pitchfork}}_{(0,0)} \tilde{T}_1 .$$

As a consequence of Lemma 10 and Thom's transversality theorem (see e.g. [21, Theorem 3.1]) we obtain

Lemma 12. (1) *There exists a residual subset H_0 of H^∞ such that if $f \in H_0$, then the sets $(\varrho_2(f))^{-1} (T_1), (\varrho_2(f))^{-1} (\tilde{T}_1), (\varrho_2(f))^{-1} (T_2), (\varrho_2(f))^{-1} (\tilde{T}_2), (\varrho_3(f))^{-1}(T_3)$, $(\varrho_3(f))^{-1} (\tilde{T}_3)$ consist of isolated points and are mutually disjoint. The sets $(\varrho_3(f))^{-1} (T_{3j}), (\varrho_3(f))^{-1} (\tilde{T}_{3j}), j = 1, 2$ are empty.*

(2) *If $X \subset R^2 \times R^3$ is a compact set, then there is an open dense subset $H_0(X)$ of H^∞ such that if $f \in H_0(X)$, then the sets $((\varrho_2(f))^{-1} (T_i)) \cap X, ((\varrho_2(f))^{-1}(\tilde{T}_i)) \cap$ $\cap X$, $i = 1, 2$, $((\varrho_3(f))^{-1} (T_3)) \cap X$, $((\varrho_3(f))^{-1} (\tilde{T}_3)) \cap X$ consist of a finite number of points and are mutually disjoint. The sets $((\varrho_3(f))^{-1} (T_{3j})) \cap X$, $((\varrho_3(f))^{-1} (\tilde{T}_{3j})) \cap X, j = 1, 2$ are empty.*

As a direct consequence of this lemma we have

Lemma 13. *The set of all nondegenerate 3-parameter families of vector fields $H_1 \subset H^\infty$ is open dense in H^∞. If $f \in H_1$, then the set $\{(x, \varepsilon) \in R^2 \times R^3: \varrho_2(f) (x, \varepsilon) \in$ $\in T_1 \cup \tilde{T}_1\}$ consists of isolated points.*

For each $g \in H_1$ we can find its normal form of the form (4.5). Let us denote by $t_{11}(g_\varepsilon)$ the coefficient at u_1^2 in the second equation of this normal form. For f from Lemma 7 we have $t_{11}(f_\varepsilon) = \varphi_3(\varepsilon)$.

For any $g \in H_1$ define the mapping $\sigma_g: R^5 \to R^5$, $\sigma_g(x, \varepsilon) = (g(x, \varepsilon), \operatorname{Tr} D_x g_\varepsilon(x)$, $\det D_x g_\varepsilon(x), t_{11}(g_\varepsilon))$. The condition (5.6) implies that $\det D\sigma_g(0, 0) \neq 0$.

Let us compute the Jacobian matrix of the mapping σ_f for the family f which is in the normal form (4.5). Since

$$\operatorname{Tr} D_u f = \frac{\partial f_2}{\partial u_2}, \quad \det D_u f = \frac{\partial f_2}{\partial u_1}, \quad t_{11}(f_\varepsilon) = \varphi_3(\varepsilon),$$

we have

$$D\,\sigma_f(0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial\varphi_1(0)}{\partial\varepsilon_1} & \dfrac{\partial\varphi_1(0)}{\partial\varepsilon_2} & \dfrac{\partial\varphi_1(0)}{\partial\varepsilon_3} \\ \dfrac{\partial^2 f_2(0)}{\partial u_1 \partial u_2} & \dfrac{\partial^2 f_2(0)}{\partial u_2^2} & * & * & * \\ 0 & \dfrac{\partial^2 f_2(0)}{\partial u_1 \partial u_2} & \dfrac{\partial\varphi_2(0)}{\partial\varepsilon_1} & \dfrac{\partial\varphi_2(0)}{\partial\varepsilon_2} & \dfrac{\partial\varphi_2(0)}{\partial\varepsilon_3} \\ 0 & 0 & \dfrac{\partial\varphi_3(0)}{\partial\varepsilon_1} & \dfrac{\partial\varphi_3(0)}{\partial\varepsilon_2} & \dfrac{\partial\varphi_3(0)}{\partial\varepsilon_3} \end{bmatrix}$$

and therefore

$$\det D\,\sigma_f(0) = -\frac{\partial^2 f_2(0)}{\partial u_1 \partial u_2} \det D\,\varphi(0) \neq 0,$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Since $f \in H_1$ we have $\partial^2 f_2(0)/\partial u_1\, \partial u_2 = t_{12} \neq 0$ and therefore $\det D\varphi(0) \neq 0$. This enables us to introduce new coordinates in the parameter space

(5.7) $$v_i = \varphi_i(\varepsilon), \quad i = 1, 2, 3.$$

The family (4.5) can be written in the form

(5.8) $$\begin{aligned} \dot u_1 &= u_2, \\ \dot u_2 &= v_1 + v_2 u_1 + v_3 u_1^2 + \sigma u_1^3 + u_1 u_2 \tilde{Q}(u_1, v) + u_2^2 \tilde{\Phi}(u, v), \end{aligned}$$

where $\tilde{Q}, \tilde{\Phi} \in C^\infty$, $\tilde{Q}(0, 0) = \omega$.

The critical points of the family (5.8) have the form $(z, 0)$, where z is a real root of the algebraic equation

(5.9) $$\sigma x^3 + v_3 x^2 + v_2 x + v_1 = 0.$$

If $y = x + (1/3\sigma)\, v_3$, then

(5.10) $$y^3 + 3py + 2q = 0,$$

where $p = p(v) = \frac{1}{3}(\sigma v_2 - \frac{1}{3} v_3^2)$, $q = q(v) = \frac{1}{2}(\sigma v_1 - \frac{1}{3} v_2 v_3 + \sigma \frac{2}{27} v_3^3)$, $v = (v_1, v_2, v_3)$. Let us introduce new coordinates in the parameter space via the diffeomorphism

(5.11) $$U\colon \mu_1 = q(v), \quad \mu_2 = p(v), \quad \mu_3 = v_3.$$

Direct computation shows that

(5.12) $$\begin{aligned} U^{-1}\colon v_1 &= \gamma_1^\sigma(\mu) = 2\sigma\mu_1 + \mu_2\mu_3 + \tfrac{1}{27}\mu_3^3, \\ v_2 &= \gamma_2^\sigma(\mu) = \sigma(3\mu_2 + \tfrac{1}{3}\mu_3^2), \quad v_3 = \mu_3. \end{aligned}$$

In these new coordinates the discriminant of the equation (5.10) has the form $D = D(\mu) = \mu_1^2 + \mu_2^3$. In the μ-coordinates the family (5.8) has the form

$$(5.13) \qquad \begin{aligned} \dot{u}_1 &= u_2\,, \\ \dot{u}_2 &= \gamma_1^\sigma(\mu) + \gamma_2^\sigma(\mu)\,u_1 + \mu_3 u_1^2 + \sigma u_1^3 + u_1 u_2 Q(u_1, \mu) + u_2^2 \Phi(u, \mu)\,, \end{aligned}$$

where $Q, \Phi \in C^\infty$, $Q(0, 0) = \omega$. This is the normal form from Theorem 1 and thus Lemma 13 completes the proof of Theorem 1.

6. BIFURCATIONS NEAR CRITICAL POINTS

Now we are interested in bifurcations of the vector field $v_\mu^\pm$ (see Theorem 1), which we also denote by v_μ^σ. The first coordinates of critical points of this vector field are real roots of the equation

$$(6.1^\sigma) \qquad \sigma x^3 + \mu_3 x^2 + \gamma_2^\sigma(\mu)\,x + \gamma_1^\sigma(\mu) = 0\,,$$

where $\gamma_1^\sigma, \gamma_2^\sigma$ (denoted also by $\gamma_1^\pm, \gamma_2^\pm$) are defined by (2.14), (2.15).

Let $\mathscr{D}, \mathscr{D}^+, \mathscr{D}^-, S_1\; S_2, S_3, G_k, G_k^+, G_k^-, H_k, H_k^+, H_k^-$, $k = 1, 2$ and $\alpha^\pm$ (denoted also by α^σ) be defined as in Section 2 before Theorem 1. We remark that $\mathscr{D} = H^+ \cup H^-$, where $H^\pm = \{\mu\colon \mu_1 = \pm h(\mu_2)\}$, $h(\mu_2) = (-\mu_2)^{3/2}$, $\mu_2 \leqq 0$ (see Figure 6).

Denote by $R(u, \mu)$ the right hand side of the second equation of the family (5.13) and let $L(K)$ be the matrix of the linear part of the vector field v_μ^σ at a critical point K. Then

$$(6.2) \qquad L(K) = \begin{bmatrix} 0 & 1 \\ \dfrac{\partial R(K, \mu)}{\partial u_1} & \dfrac{\partial R(K, \mu)}{\partial u_2} \end{bmatrix}.$$

Since $\det L(K) = -\partial R(K, \mu)/\partial u_1$, the matrix $L(K)$ has at least one zero eigenvalue if and only if

$$(6.3) \qquad \frac{\partial R(K, \mu)}{\partial u_1} = 0\,.$$

If z, z_1, z_2 are the roots of the equation (6.1^σ), then $R(u, \mu) = \sigma(u_1 - z)(u_1 - z_1)\,.\,(u_1 - z_2) + u_1 u_2 Q(u, \mu) + u_2^2 \Phi(u, \mu)$. Therefore, for $K = (z, 0)$ we have $\partial R(K, \mu)/\partial u_1 = \sigma(z - z_1)(z - z_2)$, $\partial R(K, \mu)/\partial u_2 = z Q(z, \mu)$. Since $Q(0, 0) = \omega \neq 0$, there is a sufficiently small neighbourhood U of the point $0 \in R^3$ such that the matrix $L(K)$ has zero eigenvalue of multiplicity 1 (2) if and only if $z \neq 0$ ($z = 0$) is the roof of the equation (6.1^σ) of multiplicity 2. For $\mu \in U \setminus \mathscr{D}$ the matrix $L(K)$ has no zero eigenvalue. Obviously, the matrix $L(K)$ has zero eigenvalue of multiplicity 2(1) if and only if $\mu \in \alpha^\sigma(\mu \in \mathscr{D} \setminus \alpha^\sigma)$.

The matrix $L(K)$ has pure by imaginary eigenvalues if and only if

$$(6.4) \qquad \operatorname{Tr} L(K) = \frac{\partial R(K, \mu)}{\partial u_2} = z Q(z, \mu) = 0\,,$$

(6.5) $$\frac{\partial R(K, \mu)}{\partial u_1} < 0 .$$

Since $\omega \neq 0$, the equality (6.4) is satisfied in a sufficiently small neighbourhood U of the point $0 \in R^3$ if and only if $z = z(\mu) = 0$. However, z is a real root of the equation (6.1$^\sigma$) and therefore $z = 0$ if and only if $\gamma_1^\sigma(\mu) = 0$. If $z = 0$, then $\partial R(K, \mu)/\partial u_1 = \sigma z_1 z_2$ and z_1, z_2 are the roots of the equation $\sigma x^2 + \mu_3 x + \gamma_2^\sigma(\mu) = 0$. Therefore $z_1 z_2 = \sigma \gamma_2^\sigma(\mu)$ and thus $\partial R(K, \mu)/\partial u_1 = \gamma_2^\sigma(\mu)$. We have obtained that the conditions (6.4), (6.5) are simultaneously satisfied for μ sufficiently small if and only if

(6.6) $$\gamma_1^\sigma(\mu) = 0 , \quad \gamma_2^\sigma(\mu) < 0 .$$

Proof of Theorem 2. Since $\mathscr{D}$ is the set of zeros of the discriminant of the cubic equation (6.1$^\sigma$), the well known results concerning the roots of a cubic equation imply the assertion of Theorem 2 concerning the number of critical points of the vector field v_μ^σ. Let U and V be as in Theorem 2. By [1, Theorem 6.2.1 (1)] the only critical point (0, 0) of the vector field v_0^+ is a saddle and since for $\mu \in S_1 \cap U \setminus \{0\}$ the vector field v_μ^+ has exactly one critical point, this must also be a saddle. If $\omega^2 - 8 < 0$, then by [1, Theorem 6.2.1 (3), (6)] the only critical point (0, 0) of the vector field v_0^- is a focus and if $\omega^2 - 8 \geqq 0$, then this point is a critical point of v_0^- with one elliptic sector, two parabolic and two hyperbolic sectors (see Figure 15). For $\mu \in S_1 \cap \cap U \setminus \{0\}$ the vector field v_μ^- has exactly one critical point K and it suffices to examine some μ with $\gamma_1^-(\mu) = 0$. In this case we have $K = (0, 0)$, $\gamma_2^-(\mu) < 0$,

$$L(K) = \begin{bmatrix} 0 & 1 \\ \gamma_2^-(\mu) & 0 \end{bmatrix}$$

and therefore K must be a focus. This means that for $\mu \in S_1 \cap U \setminus \{0\}$ near the set G_1 the only critical point of v_μ^- is a focus. This focus may be changed into a node for some $\mu \in S^1 \cap U \setminus \{0\}$ far from the set G_1.

If $\mu \in S_4 \cap U$, then the vector field v_μ^σ has two critical points $K = (z, 0)$, $K_1 = (z_1, 0)$ (the roots z_1, z_2 of (6.1$^\sigma$) coincide) and from the considerations before this proof we obtain that

$$L(K) = \begin{bmatrix} 0 & 1 \\ 0 & zQ(z, \mu) \end{bmatrix}, \qquad L(K_1) = \begin{bmatrix} 0 & 1 \\ \sigma(z - z_1)^2 & z_1 Q(z_1, \mu) \end{bmatrix}.$$

Therefore K is a saddle node of the vector field v_μ^σ. The eigenvalues of the matrix $L(K_1)$ are $\lambda_{1,2} = \frac{1}{2}(z_1 Q(z_1, \mu) \pm \sqrt{d(\mu)})$, where $d(\mu) = z_1^2(Q(z_1, \mu))^2 + 4\sigma(z - z_1)^2$. Therefore K_1 is a saddle of the vector field v_μ^+ for each $\mu \in S_2 \cap U$. If $\sigma = -1$, then K_1 is a focus fos $d(\mu) < 0$ (this is valid e.g. if $\omega^2 - 4 < 0$ 1nd $\mu \in \alpha^-$) and K_1 is a node of the vector field v_μ^- for $d(\mu) \geqq 0$.

If $\mu \in S_3 \cap U$, then the vector field v_μ^σ has three critical points $K = (z, 0)$, $K_1 = (z_1, 0)$, $K_2 = (z_2, 0)$. It suffices to examine some $\mu \in S_3 \cap U$, for which $\gamma_1^\sigma(\mu) = 0$. In this case we have $\gamma_2^+(\mu) < 0$ and $\gamma_2^-(\mu) > 0$. Direct computation of eigenvalues of the matrices $L(K)$, $L(K_1)$, $L(K_2)$ shows that in this case K is a focus (a saddle) and K_1, K_2 are saddles (foci) of the vector field $v_\mu^+(v_\mu^-)$. The foci may be changed into nodes fos $\mu \in S_3 \cap U$ far from the sets H_1 and G_1, respectively. This completes the proof of the theorem, except the assertion (4). The sets H_1, H_2, G_1, G_2 are obviously graphs of smooth functions, the forms of which are well known from Thom's catastrophe theory and whose pictures may be found e.g. in the book of T. Bröcker and L. Lander [10]. The assertion concerning the sets α^+ and α^- is then obvious. Thus the proof of Theorem 2 is complete.

Bifurcations for v_μ^+. As we have shown in the proof of Theorem 2, the only critical point of v_μ^+ for $\mu \in S_1$ is a saddle. Let P_0 be the plane passing through the point $(0, \mu_2^0, 0) \in H_2^-$ and parallel to the (μ_1, μ_3)-plane. Let $w_\mu^+ = v_\mu^+$ for $\mu \in P_0$, i.e., w_μ^+ is a two-parameter family of vector fields with the parameter set P_0. The set $P_0 \cap \mathcal{D}$ consists of two lines $d_1 \subset H^+$, $d_2 \subset H^-$ parallel to the μ_3-axis. The curve $h = P_0 \cap H_1 \cap (\mathcal{D} \cup \mathcal{D}^-)$ is the piece of the graph of the function $\mu_1 = h_0(\mu_3) = -\frac{1}{2}(\mu_2^0\mu_3 + \frac{1}{27}\mu_3^3)$ included in the set $P_0 \cap (\mathcal{D} \cup \mathcal{D}^-)$. For $\mu \in \operatorname{Int} h$, the matrix $L(K)$ corresponding to the focus K has pure by imaginary eigenvalues. Obviously (see Figure 6), there are $\mu_3' > 0$, $\mu_3'' < 0$ such that the points $Q_1 = (h_0(\mu_3), \mu_2^0, \mu_3) \in d_1$, $Q_2(h_0(\mu_3), \mu_2^0, \mu_3) \in d_2$ are the and-points of the curve h (we have $d_1 \subset \{\mu: \mu_1 < 0\}$, $d_2 \subset \{\mu: \mu_1 > 0\}$). Obviously, the curve h touches the lines d_1, d_2 at the points Q_1 and Q_2, respectively. Each of the vector fields $w_{Q_1}^+$ and $w_{Q_2}^+$ has two critical points: a saddle K_1 and a saddle node K_2 for which the matrix $L(K_2)$ has zero eigenvalue of multiplicity 2. Since the signature (see Definition 10) corresponding to the vector field $w_{Q_1}^+(w_{Q_2}^+)$ is equal to $\omega \,.\, \mu_3'(\omega \,.\, \mu_3'')$ and $\omega = 1/\sqrt{|p|} > 0$ (see Section 4), we obtain that the signature corresponding to the vector field $w_{Q_1}^+(w_{Q_2}^+)$ is positive (negative). Therefore by Lemma 1 there exist neighbourhoods U_1, U_2, V of Q_1, Q_2 and K_2, respectively, such that the bifurcation diagram for the vector field $w_{Q_1}^+(w_{Q_2}^+)$ in $U_1(U_2)$ and the corresponding bifurcations in V correspond to the bifurcation diagram and the bifurcations of Bogdanov's normal form (2.8) with positive (negative) signature, i.e. with $q > 0$ $(q < 0)$. Denote $\beta_i = h \cap U_i$, $i = 1, 2$. For $\mu \in \beta_1$ $(\mu \in \beta_2)$ two critical points are saddles and the third is a focus, which we denote by K. The matrix $L(K)$ has a couple of pure by imaginary eigenvalues. Now we shall compute the sign of the first Ljapunov's focus number $L_1 = L_1(\mu)$ corresponding to the focus K. Since for $\mu \in \beta_i$ $(i = 1, 2)$ we have $\gamma_1^+(\mu) = 0$ and $\gamma_2^+(\mu) < 0$, the focus must be the point $(0, 0)$. Using the formula (2.3) one can obtain that $L_1(\mu) = -(\pi/4\sqrt{\Delta^3})(-\omega\mu_3 + \gamma_2^+(\mu)(\omega b_{02} + b_{21} - 3b_{03}\gamma_2^+(\mu))$, where b_{02}, b_{21}, b_{03} are the coefficients at $u_2^2, u_1^2u_2, u_2^3$, respectively, an the right-hand side of the second equation of the system (5.13) and $\Delta = -\gamma_2^+(\mu)$. Since $\lim_{\mu \to Q_i} \gamma_2^+(\mu) = 0$ $(i = 1, 2)$, we obtain that $\operatorname{sign} L_1(\mu) = \operatorname{sign} \omega\mu_3$ for μ sufficiently close to Q_i. Therefore, if the neighbourhoods U_1, U_2 are sufficiently small, then $L_1(\mu) > 0$ for $\mu \in \beta_1$ and $L_1(\mu) < 0$ for $\mu \in \beta_2$. This implies that the function $L_1(\mu)$ must change its sign somewhere in the interior of the curve h.

Lemma 14. *There is exactly one point $Q \in h$ where the function $L_1 = L_1(\mu)$ changes its sign.*

Proof. If $\mu \in \mathscr{D}^-$, then the equation (6.1^+) has three roots ξ_1, ξ_2, ξ_3. In this case the known Cardano's formulae are not suitable for the computation of the roots. We shall use the known goniometric formulae for the roots of a cubic equation. Using these formulae one can obtain that $\xi_1 = -2\cos\frac{1}{3}\varphi - \frac{1}{3}\mu_3$, $\xi_2 = 2r\cos(60° - \frac{1}{3}\varphi) - \frac{1}{3}\mu_3$, $\xi_3 = 2r\cos(60° + \frac{1}{3}\varphi) - \frac{1}{3}\mu_3$, where $\cos\varphi = 1/\pm\sqrt{-\mu_2^3}$, $r = \pm\sqrt{|\mu_2|}$, $\operatorname{sign} r = \operatorname{sign}\mu_1$. Let $K = (\xi_1, 0)$ be the focus (if ξ_2 or ξ_3 is the first coordinate of the focus, the proof is similar).

If $y_1 = u_1 - \xi_1$, $y_2 = u_2$, then the family v_μ^+ becomes

$$(6.7)\qquad \begin{aligned} \dot{y}_1 &= y_2\,, \\ \dot{y}_2 &= y_1(y_1 - \varrho_1)(y_1 - \varrho_2) + \xi_1 y_2 \tilde{Q}(y_1, \mu) + y_1 y_2 \tilde{Q}(y_1, \mu) + y_2^2 \tilde{\Phi}(y, \mu)\,, \end{aligned}$$

where $\varrho_1 = \xi_2 - \xi_1$, $\varrho_2 = \xi_3 - \xi_1$, $\tilde{Q}(y_1, \mu) = Q(y_1 + \xi_1, \mu)$, $\tilde{\Phi}(y, \mu) = \Phi(y_1 + \xi_1, y_2, \mu)$, $\varrho_1 = \xi_2 - \xi_1 = r(3\cos\frac{1}{3}\varphi + \sqrt{3}\sin\frac{1}{3}\varphi)$, $\varrho_2 = \xi_3 - \xi_1 = (r\cos\frac{1}{3}\varphi - \sqrt{3}\sin\frac{1}{3}\varphi)$. From the formulae for ϱ_1, ϱ_2 one can simply obtain the following relations, which will be useful later:

$$(6.8)\qquad \begin{aligned} \varrho_1 + \varrho_2 &= 6r\cos\tfrac{1}{3}\varphi\,, \\ \varrho_1\varrho_2 &= 3r^2(4\cos^2\tfrac{1}{3}\varphi - 1)\,. \end{aligned}$$

Since we have expressed $\cos\varphi$ as a function of the parameters μ_1 and μ_3, it will be suitable to use the following trigonometrical identity:

$$(6.9)\qquad \cos\varphi = 4\cos^3\tfrac{1}{3}\varphi - 3\cos\tfrac{1}{3}\varphi\,.$$

Let us rewrite the family (6.7) in the form

$$(6.10)\qquad \begin{aligned} \dot{y}_1 &= y_2\,, \\ \dot{y}_2 &= \varrho_1\varrho_2 y_1 + b_{11}\tilde{\xi}_1 y_2 - (\varrho_1 + \varrho_2)\, y_1^2 + y_1^3 + \\ &\quad + (b_{11} + 2\tilde{\xi}_1 b_{21})\, y_1 y_2 + (b_{02} + b_{11}\tilde{\xi}_1)\, y_2^2 + \\ &\quad + b_{12} y_1 y_2^2 + b_{21} y_1^2 y_2 + b_{03} y_2^3 + S(y, \mu)\,, \end{aligned}$$

where b_{ij} are the coefficients at $y_1^i y_2^j$ of Taylor's expansion of the right hand side of the second equation (6.7), $b_{11} = \omega$, $\tilde{\xi}_1 = \xi_1 \tilde{Q}(\xi_1, \mu)$ and $S(y, \mu)$ contains only terms of orders higher than 3. Using the formula (2.3) one can obtain that

$$(6.11)\qquad \begin{aligned} L_1 = -\frac{\pi}{4\sqrt{\Delta^3}} &[(b_{11} + 2\tilde{\xi}_1 b_{21})(\varrho_1 + \varrho_2 + b_{02}\varrho_1\varrho_2) - \\ &- 3b_{03}(\varrho_1\varrho_2)^2 + b_{21}\varrho_1\varrho_2)]\,, \end{aligned}$$

where $\Delta = -\varrho_1\varrho_2$. We have assumed that $K = (\xi_1, 0)$ is a focus of the system (5.13) and therefore the origin must be a focus of the system (6.10). Thus $\Delta > 0$. Using the above formulae for $\xi_1, \varrho_1, \varrho_2$ we obtain

$$(6.12) \qquad L_1 = -\frac{\pi}{4\sqrt{\Delta^3}} F(r, \varphi, \mu),$$

where $F(r, \varphi, \mu) = rG(\cos \frac{1}{3}\varphi, r, \mu)$, $G(z, r, \mu) = (b_{11} + b_{21}h(z, r, \mu))(6z + 3r(4z^2 - 1) b_{02} - 27b_{03}r^3(4z^2 - 1)^2 + 3b_{21}r(4z^2 - 1))$, $h(z, r, \mu) = 2(-2rz - \frac{1}{3}\mu_3) Q(-2rz - \frac{1}{3}\mu_3, \mu))$, $r = \pm\sqrt{(-\mu_2)}$, $\mu_2 < 0$, where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$) and $\cos \varphi = \mu_1/r^3$. The function G is obviously smooth, $G(0, 0, 0) = 0$ and $\partial G(0, 0, 0)/\partial z = 6b_{11} = 6\omega \neq 0$. The Implicit Function Theorem implies that there exists a smooth function $z = \Psi_0(r, \mu)$ such that $\Psi_0(0, 0) = 0$ and $G(\Psi_0(r, \mu), r, \mu) = 0$ in a sufficiently small neighbourhood of the origin.

We are interested in a solution of the equation $L_1(\mu) = 0$ for $\mu_2 < 0$. This equation is obviously equivalent to the equation $G(\cos \frac{1}{3}\varphi, \pm\sqrt{(-\mu_2)}, \mu) = 0$, $\mu_2 < 0$, where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$). From the uniqueness of the implicit function Ψ_0 it follows that this equation is equivalent to the equation

$$(6.13) \qquad \cos \tfrac{1}{3}\varphi = \Psi_0(\pm \sqrt{(-\mu_2)}, \mu), \quad \mu_2 < 0.$$

Therefore, (6.9) and the definition of φ yield that the equation (6.13) is equivalent to the equation

$$(6.14) \qquad \frac{\mu_1}{\pm\sqrt{(-\mu_2)^3}} = 4\Psi_0^3(\pm\sqrt{(-\mu_2)}, \mu) - 3\Psi_0(\pm\sqrt{(-\mu_2)}, \mu),$$

where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$). Let us define a function $\Psi(\mu)$ as follows:

$$\Psi(\mu) = \mu_1 + (-\mu_2)^{3/2} (4\Psi_0^3(-\sqrt{(-\mu_2)}, \mu) - 3\Psi_0(-\sqrt{(-\mu_2)}, \mu))$$
$$\text{for} \quad \mu_1 < 0, \quad \mu_2 < 0,$$
$$\Psi(\mu) = \mu_1 - (-\mu_2)^{3/2} (4\Psi_0^3(\sqrt{(-\mu_2)}, \mu) - 3\Psi_0(\sqrt{(-\mu_2)}, \mu))$$
$$\text{for} \quad \mu_1 > 0, \quad \mu_2 < 0 \quad \text{and} \quad \Psi(\mu) \equiv \mu_1 \quad \text{for} \quad \mu_2 \geqq 0.$$

Obviously, the function Ψ is of the class C^1, $\Psi(0, 0, 0) = 0$, $\partial\Psi(0, 0, 0)/\partial\mu_1 = 1$. The Implicit Function Theorem implies that there exists a C^1-function $\mu_1 = H(\mu_2, \mu_3)$ such that $H(0, 0) = 0$ and $\Psi(H(\mu_2, \mu_3), \mu_2, \mu_3) = 0$ in a sufficiently small neighbourhood of the origin. Thus we have obtained that $L_1(\mu) = 0$ if and only if μ is situated on that part of the graph of the function H for which $\mu_2 < 0$. Since $H \in C^1$ and obvisouly $H(\mu_2, \mu_3) \equiv 0$ for $\mu_2 \geqq 0$, we obtain that if U is a sufficiently small neighbourhood of the origin, then the graph of H transversally intersects the survace $U \cap H_1 \cap (\mathscr{D}^- \cup \{0\})$ at a curve η passing through the origin. Obviously, there is exactly one point Q at which the curve η intersects the curve h. Thus we have proved that there is exactly one point Q where the function $L_1(\mu)$ changes its sign.

Let Q and h be as in Lemma 14. If we wish to describe the bifurcations for μ near the point Q, we need to compute the sign of the second Ljapunov's focus number $L_2 = L_2(Q)$ at Q (see Lemma 2). If $Q = \mu \in \eta \cap h$, then $\gamma_1^+(\mu) = 0$ and the vector field v_μ^+ has the form

$$
(6.15)\qquad \begin{aligned}
\dot{u}_1 &= u_2\,, \\
\dot{u}_2 &= \gamma_2^+(\mu)\, u_1 + \mu_3 u_1^2 + b_{11}u_1u_2 + b_{02}u_2^2 + b_{30}u_1^3 + b_{03}u_2^3 + \\
&\quad + b_{21}u_1^2u_2 + b_{12}u_1u_2^2 + g_4(u_1, u_2) + g_5(u_1, u_2) + g(u, \mu)\,,
\end{aligned}
$$

where $g(u, 0) = o(\|u\|^5)$, g_4, g_5 are homogeneous polynomials in u_1, u_2 of degree 4 and 5, respectively, with coefficients b_{ij} at $u_1^i u_2^j$, $b_{30} = 1$, $b_{11} = \omega$, $b_{40} = b_{50} = 0$. Obviously, the point $K = (0, 0)$ is the only focus of the system (5.13), for which we have computed that $L_1(Q) = 0$. Let us introduce new variables: $\vartheta_1 = u_1$, $\vartheta_2 = -u_2/\varkappa$, $\tau = \varkappa t$, $\varkappa = \sqrt{(-\gamma_2^+(\mu))}$. Then the vector field (6.15) becomes

$$
(6.16)\qquad \begin{aligned}
\dot{\vartheta}_1 &= -\vartheta\,, \\
\dot{\vartheta}_2 &= \vartheta_1 + \tilde{b}_{20}\vartheta_1^2 + \tilde{b}_{11}\vartheta_1\vartheta_2 + \tilde{b}_{02}\vartheta_2^2 + \tilde{b}_{30}\vartheta_1^3 + \tilde{b}_{03}\vartheta_2^3 + \\
&\quad + \tilde{b}_{21}\vartheta_1^2\vartheta_2 + \tilde{b}_{12}\vartheta_1\vartheta_2^2 + \tilde{g}_4(\vartheta_1, \vartheta_2) + \tilde{g}_5(\vartheta_1, \vartheta_2) + \tilde{g}(\vartheta_1, \vartheta_2, \mu)\,,
\end{aligned}
$$

where $g(\vartheta_1, \vartheta_2, 0) = o((\sqrt{(\vartheta_1^2 + \vartheta_2^2)})^5)$, $\tilde{g}_4, \tilde{g}_5$ are homogeneous polynomials of degree 4 and 5, respectively, with coefficients $\tilde{b}_{ij}$ at $\vartheta_1^i\vartheta_2^j$,

$$
\tilde{b}_{20} = \frac{\mu_3}{\varkappa^2}, \quad \tilde{b}_{11} = -\frac{b_{11}}{\varkappa}, \quad \tilde{b}_{02} = b_{02}\,, \quad \tilde{b}_{30} = -\frac{b_{30}}{\varkappa^2}, \quad \tilde{b}_{21} = \frac{b_{21}}{\varkappa}\,,
$$

$$
\tilde{b}_{12} = -b_{12}\,, \quad \tilde{b}_{03} = \varkappa b_{03}\,, \quad \tilde{b}_{40} = 0\,, \quad \tilde{b}_{31} = -\frac{b_{31}}{\varkappa}\,,
$$

$$
\tilde{b}_{22} = -b_{22}\,, \quad \tilde{b}_{13} = \varkappa b_{13}\,, \quad \tilde{b}_{04} = -\varkappa^2 b_{04}\,, \quad \tilde{b}_{50} = 0\,, \quad \tilde{b}_{41} = -\frac{b_{41}}{\varkappa}\,,
$$

$$
\tilde{b}_{23} = \varkappa b_{23}\,, \quad \tilde{b}_{32} = b_{32}\,, \quad \tilde{b}_{14} = \varkappa^2 b_{14}\,, \quad \tilde{b}_{05} = \varkappa^3 b_{05}\,.
$$

Putting the coefficients $\tilde{b}_{ij}$ into the formula (2.4) one can obtain that

$$
(6.17)\qquad L_2(Q) = L_2(\mu) = \frac{\pi}{24\varkappa^3}\,[N + O(\|\mu\|)]\,.
$$

where $N = -b_{11}^3 b_{02} + b_{11}^2 b_{21} - 7b_{02}b_{11}b_{30} + 3b_{30}b_{21}$ nad therefore

$$
(6.18)\qquad \operatorname{sign} L_2(Q) = \operatorname{sign} N
$$

for Q sufficiently close to the origin.

Lemma 15. *The number* sign N *is invariant with respect to regular transformations of coordinates in the phase space.*

Proof. It suffices to consider the system (6.15) for $\mu = 0$, i.e. the system

$$
(6.18)\qquad \begin{aligned}
\dot{u}_1 &= u_2\,, \\
\dot{u}_2 &= b_{02}u_2^2 + b_{11}u_1u_2 + b_{30}u_1^3 + b_{21}u_1^2u_2 + b_{12}u_1u_2^2 + b_{03}u_2^3 + \\
&\quad + g_4(u_1, u_2) + g_5(u_1, u_2) + g(u, 0)\,,
\end{aligned}
$$

where g_4, g_5 and g are as above, and to prove the invariance of the number sign N with respect to the mappings R and ϱ (see (3.4) and (3.5)). First let us prove the invariance of sign N with respect to the mapping ϱ. This mapping transforms the system (6.18) into the form (3.1), where $\tilde{p}_{11} = 0$, $\tilde{p}_{12} = \frac{1}{2}b_{11}\varepsilon\lambda^{-2}$, $\tilde{q}_{11} = 0$, $\tilde{p}_{30} = b_{30}\varepsilon\lambda^{-3}$, $\tilde{q}_{30} = b_{30}\lambda^{-2}$, $\tilde{q}_{21} = b_{21}\lambda^{-2} - 3\lambda^{-3}\varepsilon b_{30}$, $\tilde{q}_{22} = b_{02}\lambda^{-1} - b_{11}\varepsilon\lambda^{-2}$. By Lemmas 1 and 4 there exists a smooth regular transformation of coordinates transforming this system into the form (2.7), where $t_{11} = 0$, $t_{12} = 2\tilde{q}_{12} + \tilde{p}_{11} = b_{11}\lambda^{-1}$, $t_2 = \tilde{q}_{22} + 2\tilde{p}_{12} = b_{02}\lambda^{-1}$, $t_{30} = \tilde{q}_{30} + 2\tilde{p}_{12}\tilde{q}_{11} = \tilde{q}_{30} = b_{30}\lambda^{-2}$, $t_{21} = \tilde{q}_{21} + 2(\tilde{p}_{22} - \tilde{p}_{12})\tilde{q}_{11} + 3\tilde{p}_{30} = \tilde{q}_{21} + 3\tilde{p}_{30} = b_{21}\lambda^{-2}$. Thus we have obtained a system of the form (6.18), where instead of b_{ij} we have the coefficients $\hat{b}_{ii}, \hat{b}_{02} = t_{22} = b_{02}\lambda^{-1}$, $\hat{b}_{11} = t_{14} = b_{11}\lambda^{-1}$, $\hat{b}_{30} = t_{30} = b_{30}\lambda^{-2}$, $\hat{b}_{21} = t_{21} = b_{21}\lambda^{-2}$. This means that the number N is chlnged by the mapping ϱ into the number $\hat{N} = -\hat{b}_{11}^3\hat{b}_{02} + \hat{b}_{11}^2\hat{b}_{21} - 7\hat{b}_{02}\hat{b}_{11}\hat{b}_{30} + 3\hat{b}_{30}\hat{b}_{21} = \lambda^{-4}N$ and thus sign $\hat{N}$ = sign N.

Let the mapping R have the form

$$R:\quad \begin{aligned} z_1 &= x_1 + \alpha_{20}x_1^2 + \alpha_{11}x_1x_2 + \alpha_{02}x_2^2 + \alpha_{30}x_1^3 + \alpha_{21}x_1^2x_2 + \dots,\\ z_2 &= x_2 + \alpha_{20}x_1^2 + \alpha_{11}x_1x_2 + \alpha_{02}x_2^2 + \alpha_{30}x_1^3 + \alpha_{21}x_1^2x_2 + \dots. \end{aligned}$$

One can easily show that this mapping transforms the system (6.18) into the form (3.1), where $\tilde{p}_{11} = -\beta_{20}$, $\tilde{p}_{12} = 2\alpha_{20} - \beta_{11}$, $\tilde{p}_{22} = \alpha_{11} - \beta_{02}$, $\tilde{b}_{30} = 2\beta_{20}^2 + \beta_{11}\alpha_{20} - 2\alpha_{20}\beta_{40}$, $\tilde{q}_{12} = b_{11} + 2\beta_{20}$, $\tilde{q}_{22} = b_{02} + \beta_{11}$, $\tilde{c}_{30} = b_{30} - 2\beta_{20}^2 - \beta_{20}b_{11}$, $\tilde{c}_{21} = b_{21} + \alpha_{20}b_{11} - 2\beta_{20}b_{02}$. Since the mapping R does not change the form of the system (6.18), the coefficients $\alpha_{20}, \beta_{20}, \beta_{02}, \alpha_{11}, \beta_{11}$ must satisfy the identities: $2\alpha_{20} - \beta_{11} = 0, \alpha_{11} - \beta_{02} = 0, \beta_{20} = 0, 2\beta_{20}^2 + \beta_{11}\alpha_{20} - 2\alpha_{20}\beta_{20} = 0$. These identities are obviously satisfied if $\alpha_{20} = \beta_{20} = \beta_{02} = \alpha_{11} = \beta_{11} = 0$. This implies that the mapping R does not change the number N at all and thus the proof is complete.

Proof of Theorem 3. The assertions of Theorem 3 are consequences of Lemmas 1, 2, 14, 15 and the considerations presented in Section 6.

Bifurcations for v_μ^-. By Theorem 2 the only critical point $K = (0, 0)$ of the vector field v_0^- is either a focus or a critical point with one elliptic sector, two parabolic and two hyperbolic sectors. For $\mu \in \mathscr{D}^+$ the only critical point $K_\mu = (z(\mu), 0)$ of v_μ^- is a focus. From the equation (6.1$^-$) we obtain that $\partial z(\mu)/\partial\mu_3 = \frac{1}{3}$ and this implies that $z(\mu) > 0$ for $\mu \in G_1^+$ and $z(\mu) < 0$ for $\mu \in G_1^-$. Since $-x^3 + \mu_3x^2 + \gamma_2^-(\mu)/x + \gamma_1^-(\mu) = -(x - z)P(x)$, where $P(x) > 0$, we have $L(K_\mu) = (c_{ij})$, where $c_{11} = 0$, $c_{12} = 1$, $c_{21} = -P(z)$, $c_{22} = zQ(z, \mu)$, $z = z(\mu)$, $Q(0, 0) = \omega > 0$. This yields that for μ sufficiently close to the set $G_1 \cap \mathscr{D}^+$, the matrix $L(K_\mu)$ has complex eigenvalues with the real parts equal to $\frac{1}{2}z(\mu)\,Q(z(\mu), \mu)$ and therefore the focus K_μ is unstable for $\mu \in G_1^+$ and stable for $\mu \in G_1^-$.

Let $L_1 = L_1(\mu)$ be the first Ljapunov's focus number of the focus $K = (0, 0)$ for

$\mu \in G_1 \cap \mathscr{D}^+$, and let $L_2 = L_2(\mu)$ be the second Ljapunov's focus number, which is defined for μ satisfying the identity $L_1(\mu) = 0$.

The vector field v_μ^-, $\mu \in G_1$ has the form

$$
(6.19) \quad \begin{aligned}
\dot{u}_1 &= u_2 , \\
\dot{u}_2 &= \gamma_2^-(\mu)\, u_1 + \mu_3 u_1^2 + b_{11} u_1 u_2 + b_{02} u_2^2 + b_{30} u_1^3 + b_{03} u_2^3 + \\
&\quad + b_{21} u_1^2 u_2 + b_{12} u_1 u_2^2 + g_4(u_1, u_2) + g_5(u_1, u_2) + g(u, \mu) ,
\end{aligned}
$$

where $b_{30} = -1$, $b_{11} = \omega > 0$ and g_4, g_5, g are functions as in the system (6.15).

Lemma 16. *There exist two C^1-curves η_1, η_2 in $G_1 \cap \mathscr{D}^+ \cap \{\mu: \mu_2 \leqq 0\}$ such that the following assertions hold:*

(1) *The origin is an end-point of the curves η_1, η_2.*

(2) *The curve $\eta_1(\eta_2)$ divides the set $G_1 \cap \mathscr{D}^+ \cap \{\mu: \mu_2 \leqq 0,\ \mu_1 > 0\}$ $(G_1 \cap \mathscr{D}^+ \cap \cap \{\mu: \mu_2 \leqq 0,\ \mu_1 < 0\})$ into two connected components $F_1, F_2(F_3, F_4)$, where $\partial F_1 = \eta_1 \cup \alpha_1 \cup \{0\}$ $(\partial F_3 = \eta_2 \cup \alpha_2 \cup \{0\})$, $\alpha_1 = \{\mu \in \alpha^-: \mu_3 > 0\}$, $\alpha_2 = \{\mu \in \alpha^-: \mu_3 < 0\}$, $\alpha^- = G_1 \cap G_2$.*

(3) *If $L_1(\mu)$ is the Ljapunov's focus number of the only focus K of the vector field v_μ^-, $\mu \in \mathscr{D}^+$, then $L_1(\mu) = 0$ for $\mu \in \mathscr{D}^+$, $\mu_2 < 0$ if and only if $\mu \in \eta_1 \cup \eta_2$.*

(4) *$L_1(\mu) > 0$ for $\mu \in F_1 \cup F_4$ and $L_1(\mu) < 0$ for $\mu \in F_2 \cup F_3$.*

(5) *If $\mu \in \eta_1 \cup \eta_2$, then the second Ljapunov's focus number of the focus $K = (0, 0)$ is given by the formula*

$$
L_2(\mu) = \frac{\pi}{24\sqrt{(-\gamma_2^-(\mu)^3)}}\,(N + O(\|\mu\|)) ,
$$

where $N = -b_{11}^3 b_{02} + b_{11}^2 b_{21} - 7 b_{02} b_{11} b_{30} + 3 b_{30} b_{21}$, the number $\operatorname{sign} N$ is invariant with respect to regular transformations of coordinates in the phase space.

Proof. If $\mu \in \mathscr{D}^+$, then the equation (6.1^-) has one real root ξ_1 and two complex conjugate roots $\xi_2, \xi_3 = \bar{\xi}_2$. We shall use their goniometric form. For $\mu_2 < 0$ they are given by the formulae

$$
\xi_1 = -2r \operatorname{ch} \tfrac{1}{3}\varphi + \tfrac{1}{3}\mu_3 , \quad \xi_{2,3} = r \operatorname{ch} \tfrac{1}{3}\varphi + \tfrac{1}{3}\mu_3 \pm \mathrm{i}\sqrt{(3)}\, r \operatorname{sh} \tfrac{1}{3}\varphi ,
$$

$$
\operatorname{ch} \varphi = \frac{\mu_1}{\pm\sqrt{(-\mu_2)^3}} , \quad r = \pm\sqrt{(-\mu_2)} , \quad \operatorname{sign} r = \operatorname{sign} \mu_1 .
$$

If $y_1 = u_1 - \xi_1$, $y_2 = u_2$ and $\varrho = \xi_2 - \xi_1$, then the vector field v_μ^- becomes

$$
(6.20) \quad \begin{aligned}
\dot{y}_1 &= = y_2 , \\
\dot{y}_2 &= -|\varrho|^2 y_1 + b_{11}\xi_1 y_2 + 2(\operatorname{Re} \varrho)\, y_1^2 - y_1^3 + \\
&\quad + (b_{11} + 2\xi_1 b_{21})\, y_1 y_2 + (b_{02} + b_{11}\xi_1)\, y_2^2 + b_{12} y_1 y_2^2 + \\
&\quad + b_{21} y_1^2 y_2 + b_{03} y_2^3 + \tilde{g}_4(y_1, y_2) + \tilde{g}_5(y_1, y_2) + \tilde{g}(y, \mu) ,
\end{aligned}
$$

where $\tilde{\xi}_1 = \xi_1 \tilde{Q}(y_1, \mu)$, $\tilde{Q}(y_1, \mu) = Q(y_1 + \xi_1, \mu)$, $b_{11} = \omega$, $\tilde{g}_4, \tilde{g}_5$ are homogeneous polynomials of degrees 4 and 5, respectively, and $\tilde{g}(y, 0) = o(\|y\|^5)$.

Using the formula (2.3) we obtain that

$$(6.21)\quad L_1(\mu) = -\frac{\pi}{4|\varrho|^3}\left[(b_{11} + 2\tilde{\xi}_1 b_{21})(-2\operatorname{Re}\varrho - b_{02}|\varrho|^3) - b_{03}|\varrho|^4 - b_{21}|\varrho|^2\right],$$

$|\varrho|^2 = 3r^2(4\operatorname{ch}^2\frac{1}{3}\varphi - 1)$, $2\operatorname{Re}\varrho = 6r\operatorname{ch}\frac{1}{3}\varphi$ and therefore

$$L_1(\mu) = -\frac{\pi}{4|\varrho|^3}\tilde{F}(r, \varphi, \mu), \quad \tilde{F}(r, \varphi, \mu) = rG(\operatorname{ch}\tfrac{1}{3}\varphi, r, \mu),$$

where $G(z, r, \mu)$ is the function defined in the proof of Lemma 14, $r = \pm\sqrt{(-\mu_2)}$, $\mu_2 < 0$, where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$) and $\operatorname{ch}\varphi = \mu_1/r^3$. Let $z = \Psi_0(r, \mu)$ be the function from the proof of Lemma 14, defined as a solution of the implicit equation $G(z, r, \mu) = 0$.

We are interested in a solution of the equation $L_1(\mu) = 0$ for $\mu \in \mathscr{D}^+$, $\mu_2 < 0$. From the uniqueness of the implicit function ψ_0 it follows that this equation is equivalent to the equation

$$(6.22)\qquad \operatorname{ch}\tfrac{1}{3}\varphi = \Psi_0(\pm\sqrt{(-\mu_2)}, \mu), \quad \mu \in \mathscr{D}^+, \quad \mu_2 < 0,$$

where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$).

Now using the known identity $\operatorname{ch}\varphi = 4\operatorname{ch}^3\frac{1}{3}\varphi - 3\operatorname{ch}\frac{1}{3}\varphi$ (compare with (6.9)) and the definition of φ, we obtain that the equation (6.22) is equivalent to the equation (6.14). Let $\Psi = \Psi(\mu)$ and $\mu_1 = H(\mu_2, \mu_3)$ be the functions from the proof of Lemma 14. We remark that the function $L_1(\mu)$ given by the formula (6.21) and the functions defining the equation (6.22) are defined not only for $\mu \in \mathscr{D}^+ \cap \{\mu\colon \mu_2 < 0\}$ but on the whole set $\{\mu\colon \mu_2 \leqq 0\}$, and we consider the equation $L_1(\mu) = 0$ on this set. The results obtained in the proof of Lemma 14 immediately yield that $L_1(\mu) = 0$ if and only if μ is situated on that part of the graph of the function H where $\mu_2 < 0$. Since $H(\mu_2, \mu_3) \equiv 0$ for $\mu_2 \geqq 0$, there exists a neighbourhood U of the origin such that the graph of the function H does not intersect the surface $G_1 \cap G_2^- \cap (\mathscr{D} \cup \mathscr{D}^-) \cap U$ and it must intersect the surface $G_1 \cap \mathscr{D}^+ \cap U$ exactly at two curves η_1, η_2 in such a way that the assertions (1)–(3) of the lemma hold.

If $\gamma_1^-(\mu) = 0$, i.e. if $\mu \in G_1$, then the equation (6.1$^-$) has one zero root and the other roots can be computed from the equation $-x^2 + \mu_3 x + \gamma_2^-(\mu) = 0$. Using these formulae for the roots, one can easily show that for $\mu \in G_1 \cap \mathscr{D}^+$ we have

$$L_1(\mu) = -\frac{\pi}{4\sqrt{(-\gamma_2^-(\mu))^3}}\left[-\omega\mu_3 + \gamma_2^-(\mu)(\omega b_{02} + b_{21}) - 3(\gamma_2^-(\mu))^2 b_{03}\right] =$$

$$= -\frac{\pi}{4\sqrt{(-\gamma_2^-(\mu))}}\left[\frac{\omega\mu_3}{\gamma_2^-(\mu)} + (\omega b_{02} + b_{21}) - 3\gamma_2^-(\mu)\, b_{03}\right].$$

Since $\gamma_2^-(\mu) = 0$ for $\mu \in \alpha^-$, $\omega > 0$ and $\gamma_2^-(\mu) < 0$ for $\mu \in G_1 \cap \mathscr{D}^+$ we obtain that sign $L_1(\mu)$ = sign μ_3 for $\mu \in G_1 \cap \mathscr{D}^+$ sufficiently close to the curve α^-. This proves the assertion (4) of the lemma. The proof of the assertion (5) of the lemma is the same as the proof of the assertion (2)–(c) of Theorem 3, where the invariance of the number sign N follows from Lemma 15, and thus the proof is complete.

Lemma 17. *If $\mu \in \mathscr{D}^-$, then the vector field v_μ^- has three critical points: a saddle K_1 and critical points K_2, K_3 which are either nodes or focuses, and the following assertions hold:*

(1) *If $\mu \in G_2^+ \cap \mathscr{D}^-$, then the critical points K_2, K_3 are either nodes or nondegenerate foci, where K_2 is stable and K_3 is unstable.*
(2) *The focus $K_2(K_3)$ is degenerate if and only if $\mu \in F^+ = G_1 \cap G_2^- \cap \mathscr{D}^- \cap \cap \{\mu: \mu_3 > 0\}$ $(\mu \in F^- = G_1 \cap G_2^- \cap \mathscr{D}^- \cap \{\mu: \mu_3 < 0\}$.*
(3) *If $L_1(\mu)$ is the first Ljapunov's focus number of the focus $K_2(K_3)$ for $\mu \in F^+$ $(\mu \in F^-)$, then $L_1(\mu) > 0$ for all $\mu \in F^+$ $(L_1(\mu) < 0$ for all $\mu \in F^-)$.*

Proof. From the results proved at the beginning of this section it follows that if the matrix $L(K_i)$ $(i \in \{1, 2, 3\}$, $K_i = (z_i, 0))$ has purely imaginary eigenvalues, then $z_i = 0$. This implies that it suffices to find out the type of the critical points for $\mu \in G_1 \cap \mathscr{D}^-$.

If $\mu \in G_1 \cap G_2^+ \cap \mathscr{D}^-$, then $K_1 = (0, 0)$ is a saddle and $K_2 = (z_2, 0)$, $K_3 = (z_3, 0)$, where $z_{2,3} = \frac{1}{2}(\mu_3 \pm \sqrt{\delta})$, $\delta = \mu_3^2 + 4\gamma_2^-(\mu) > 0$. Since $z_{2,3} \neq 0$, the real parts of the eigenvalues of the matrices $L(K_2)$, $L(K_3)$ are nonzero for all $\mu \in G_1 \cap G_2^+ \cap \mathscr{D}^-$ and therefore it suffices to find out the type of the critical points K_2, K_3 for some $\mu \in G_1 \cap G_2^+ \cap \mathscr{D}^-$ with $\mu_3 = 0$. Under the assumption $\mu_3 = 0$ we have $z_{2,3} = \pm\sqrt{\gamma_2^-(\mu)}$ and $L(K_i) = (c_{jk}^i)$ $(i = 2, 3)$, where $c_{11}^i = 0$, $c_{12}^i = 1$, $c_{21}^i = -z_i^2 - \gamma_2^-(\mu)$, $c_{22}^i = z_i Q(z_i, \mu)$. The matrix $L(K_2)\,(L(K_3))$ has the eigenvalues $\lambda_{1,2} = \frac{1}{2}(\varkappa_1 \pm \sqrt{\delta_1})$ $(\frac{1}{2}(\varkappa_2 \pm \sqrt{\delta_2}))$, where $\varkappa_{1,2} = \pm\sqrt{\gamma_2^-(\mu)}\, Q(\pm\sqrt{\gamma_2^-(\mu)}, \mu)$, $\delta_{1,2} = ((Q(\pm\sqrt{\gamma_2^-(\mu)}, \mu))^2 - 8)\,\gamma_2^-(\mu)$. This implies that K_2, K_3 are either nodes or nondegenerate focuses, according to the signs of δ_1 and δ_2, respectively. Since $\omega > 0$, we also obtain that K_2 is stable and K_3 is unstable. This proves the assertion (1).

Let F^+ and F^- be as in the lemma. First assume $\mu \in F^+$. Then the vector field v_μ^- has three critical points: $K_1 = (z_1, 0)$, $K_2 = (0, 0)$, $K_3 = (z_3, 0)$, where $z_1 = \frac{1}{2}(\mu_3 - \sqrt{\delta})$, $z_3 = \frac{1}{2}(\mu_3 + \sqrt{\delta})$, $\delta = \mu_3^2 + 4\,\gamma_2^-(\mu) > 0$. Since $\gamma_2^-(\mu) < 0$, the critical point $K_2 = (0, 0)$ must be a degenerate focus. The matrix $L(K_1)$ has eigenvalues $\beta_{1,2} = \frac{1}{2}(z_1 Q(z_1, \mu) \pm \sqrt{\Delta})$, where $\Delta = z_1^2(Q(z_1, \mu))^2 + 4z_1\sqrt{\delta}$. Obviously $z_1 > 0$ and therefore β_1, β_2 are real, $\beta_1 > 0$, $\beta_2 < 0$. This means that the critical point K_1 is a saddle. Without any computation we already know that the third critical point K_3 must be either a node or a nondegenerate focus. The proof for $\mu \in F^-$ is analogous.

It remains to prove the assertion (3). We may use the same method which we have used in the proof of Lemma 14. Using the formula (2.3) one can obtain a formula for the function $L_1(\mu)$ corresponding to the focus K_2, which does not essentially differ from the formula (6.11). By the same argument as that used in the proof of Lemma 14, it is possible to show that the set of zeros of the function $L_1(\mu)$ is a C^1-surface which does not intersect the surface F^+. The same is valid for the function $L_1(\mu)$ corresponding to the focus K_3, $\mu \in F^-$. Similarly to the proof of the assertion (4) of Lemma 16, one can show that $L_1(\mu) > 0$ ($L_1(\mu) < 0$) for $\mu \in F^+$ ($\mu \in F^-$) sufficiently close to the curve α^-. Since the function $L_1(\mu)$ does not change its sign on the surface F^+ or F^-, respectively, the proof of the assertion (3) is complete.

Lemma 18. *There exists a C^1-curve η_3 in the set $E = G_1 \cap \mathscr{D}^+ \cap \{\mu: \mu_2 \geqq 0\}$ such that the following assertions hold:*

(1) *The origin is an end-point of the curve η_3 and this curve divides the set E into two connected components E^+, E^-.*

(2) *If $L_1(\mu)$ is the first Ljapunov's focus number of the only focus K of the vector field v_μ^-, $\mu \in E$, then $L_1(\mu) = 0$ for $\mu \in E$ if and only if $\mu \in \eta_3$.*

(3) *$L_1(\mu) > 0$ for $\mu \in E^+$ and $L_1(\mu) < 0$ for $\mu \in E^-$.*

(4) *If $\mu \in \eta_3$, then the second Ljapunov s focus number of the focus $K = (0, 0)$ is given by the same formula as in the assertion (5) of Lemma 16.*

Proof. If $\mu \in \mathscr{D}^+ \cap \{\mu: \mu_2 > 0\}$, then the equation (6.1$^-$) has one real root ξ_1 and two complex conjugate roots $\xi_2, \xi_3 = \bar{\xi}_2$, which may be expressed by the following formulae (compare with the case $\mu \in \mathscr{D}^+ \cap \{\mu: \mu_2 \leqq 0\}$):

$$\xi_1 = -2r \operatorname{sh} \tfrac{1}{3}\varphi + \tfrac{1}{3}\mu_3\,, \quad \xi_{2,3} = r \operatorname{sh} \tfrac{1}{3}\varphi + \tfrac{1}{3}\mu_3 \pm \mathrm{i}\sqrt{(3)}\, r \operatorname{ch} \tfrac{1}{3}\varphi\,,$$

$$\operatorname{sh} \varphi = \frac{\mu_1}{\pm\sqrt{\mu_2^3}}\,, \quad r = \pm\sqrt{\mu_2}\,, \quad \operatorname{sign} r = \operatorname{sign} \mu_1\,.$$

Analogously to the case $\mu_2 \leqq 0$, one can show that

$$L_1(\mu) = -\frac{\pi}{4|\varrho|^3}\hat{F}(r, \varphi, \mu)\,,$$

where $|\varrho|^2 = 3r^2(4 \operatorname{sh}^2 \tfrac{1}{3}\varphi - 1)$. $\hat{F}(r, \varphi, \mu) = rG(\operatorname{sh} \tfrac{1}{3}\varphi, r, \mu)$, $G(z, r, \mu)$ is the function defined in the proof of Lemma 14. Let $z = \Psi_0(r, \mu)$ be the solution of the implicit equation $G(z, r, \mu) = 0$ (see the proof of Lemma 14). Then $L_1(\mu) = 0$ for $\mu \in \mathscr{D}^+ \cap \{\mu: \mu_2 > 0\}$ if and only if $\operatorname{sh} \tfrac{1}{3}\varphi = \Psi_0(r, \mu)$. Since $\operatorname{sh} \varphi = 4 \operatorname{sh}^3 \tfrac{1}{3}\varphi - 3 \operatorname{sh} \tfrac{1}{3}\varphi$ we obtain that the equation $L_1(\mu) = 0$ is equivalent to the equation

$$\frac{\mu_1}{\pm\sqrt{\mu_2^3}} = 4\Psi_0^3(\pm\sqrt{\mu_2}, \mu) - 3\Psi_0(\pm\mu_2, \mu)\,, \tag{6.23}$$

where we have $+(-)$ if $\mu_1 > 0$ ($\mu_1 < 0$). Let us define a function $\tilde{\Psi}(\mu)$ (compare with the function $\Psi(\mu)$ from the proof of Lemma 14) as follows:

$$\tilde{\Psi}(\mu) = \mu_1 + \mu_2^{3/2}(4\Psi_0^3(-\sqrt{(\mu_2)}, \mu) - 3\Psi_0(-\sqrt{(\mu_2)}, \mu)) \quad \text{for} \quad \mu_1 < 0\,, \quad \mu_2 > 0\,,$$

$$\tilde{\Psi}(\mu) = \mu_1 - \mu_2^{3/2}(4\Psi_0^3(\sqrt{(\mu_2)}, \mu) - 3\Psi_0(\sqrt{(\mu_2)}, \mu) \quad \text{for} \quad \mu_1 > 0\,, \quad \mu_2 > 0 \quad \text{and}$$

$$\tilde{\Psi}(\mu) \equiv \mu_1 \quad \text{for} \quad \mu_2 \leqq 0\,.$$

Obviously, the function $\tilde{\Psi}$ is of the class C^1, $\Psi(0, 0, 0) = 0$, $\partial\tilde{\Psi}(0, 0, 0)/\partial\mu_1 = 1$. The Implicit Function Theorem implies that there exists a C^1-function $\mu_1 = \tilde{H}(\mu_2, \mu_3)$ such that $\tilde{H}(0, 0) = 0$ and $\tilde{\Psi}(\tilde{H}(\mu_2, \mu_3), \mu_2, \mu_3) = 0$ in a sufficiently small neighbourhood of the origin. Thus we have obtained that $L_1(\mu) = 0$ for $\mu \in \mathscr{D}^+ \cap \cap \{\mu: \mu_2 > 0\}$ if and only if μ is situated on that part of the graph of the function $\tilde{H}$ for which $\mu_2 > 0$. Since $\tilde{H} \in C^1$ and obviously $\tilde{H}(\mu_2, \mu_3) \equiv 0$ for $\mu_2 \leqq 0$, we obtain that if U is a sufficiently small neighbourhood of the origin, then the graph of the function $\tilde{H}$ transversally intersects the surface $U \cap G_1 \cap (\mathscr{D}^+ \cup \{0\})$ exactly at one curve, which we denote by η_3. The origin is an end-point of this curve and this proves the assertions (1) and (2) of the lemma.

Let E^+ be the component which is situated on the left of the curve η_3 (see Figure 7). Let $\eta^+ \subset E^+$ ($\eta^- \subset E^-$) be a curve with an end-point at the origin and sufficiently close to the curve $\beta = \{\mu \in G_1: \mu_2 = 0\}$. If $\mu = (\mu_1, \mu_2, \mu_3) \in \eta^+$ ($\mu \in \eta^-$), then obviously $\mu_3 > 0$ ($\mu_3 < 0$). For $\mu \in G_1 \cap \mathscr{D}^+$ we have the formula for the first Ljapunov's focus number $L_1(\mu)$ given in the proof of Lemma 16. This formula implies that $L_1(\mu) > 0$ ($L_1(\mu) < 0$) for each $\mu \in \eta^+$ ($\mu \in \eta^-$) sufficiently close to the origin. Since the function $L_1(\mu)$ changes its sign on the curve η_3 only, we obtain that $L_1(\mu) > 0$ for all $\mu \in E^+$ and $L_1(\mu) < 0$ for all $\mu \in E^-$. This proves the assertion (3) of the lemma.

The proof of the assertion (5) is the same as the proof of the assertion (2)—(c) of Theorem 3 and thus the proof of the lemma is complete.

Proof of Theorem 4. The assertions of Theorem 4 are consequences of Lemmas 1, 2, 16, 17, 18.

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