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EDGE-DISJOINT 1-FACTORS IN POWERS OF CONNECTED GRAPHS

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In the present paper it will be proved that if n is a positive integer and G is a connected graph of an even order $\geqq n$, then G^n contains at least $n - 1$ edge-disjoint 1-factors. (As follows from the example given in [6], this lower bound cannot be improved).

By a graph we mean a graph in the sense of the books [1] and [4]. Let G be a graph; we denote by $V(G)$ its vertex set and by $E(G)$ its edge set; the integer $|V(G)|$ is referred to as the order of G; if $A \neq \emptyset$ is a subset of $V(G)$, then $\langle A \rangle_G$ denotes the subgraph of G induced by A; we say that F is an *m-factor of* G if F is a regular graph of degree m and it is a spanning subgraph of G; if $u, v \in V(G)$, then the distance between u and v in G will be denoted by $\|u, v\|_G$. If G is a graph and $n \geqq 1$ is an integer, then the n-th power G^n of G is defined as follows: $V(G^n) = V(G)$, and for every $u, v \in V(G)$, $uv \in E(G^n)$ if and only if $1 \leqq \|u, v\|_G \leqq n$.

The following theorem was proved in [6]:

Theorem 0. *Let n be a positive integer, and let G be a connected graph of order $p \geqq n$. Assume that if n is even, then p is also even. Then G^n has an $(n - 1)$-factor.*

(Moreover, it was shown in [6] that for any integers $n \geqq 1$ and $p > n(n + 1)$, there exists a tree T of order p such that T^n has no n-factor).

In the present paper Theorem 0 will be improved for the case when the order of G is even. If H is a graph, then we denote by $\varphi(H)$ the maximum integer m such that there exists a set of m mutually edge-disjoint 1-factors of H; clearly, if $\varphi(H) \neq 0$, then the order of H is even.

The main result of the present paper is the following:

Theorem 1. *Let n be a positive integer, and let G be a connected graph of an even order $p \geqq n$. Then $\varphi(G^n) \geqq n - 1$.*

To prove Theorem 1 we make use of three lemmas (one of them was proved in [6]).

Let T be a tree, and let $u \in V(T)$. Similarly as in [6], we shall say that a subset V_0 of $V(T)$ is a u-set in T if either $V_0 = \{u\}$ or there exist distinct components $T_1, \ldots, T_k$

($k \geqq 1$) of $T - u$ and either $V_0 = V(T_1) \cup \ldots \cup V(T_k)$ or $V_0 = V(T_1) \cup \ldots \ldots \cup V(T_k) \cup \{u\}$.

Let A be a finite nonempty set. Consider an arbitrary object x; if $x \in A$, then we write $\varepsilon(x, A) = 1$, and if $x \notin A$, then we write $\varepsilon(x, A) = 0$. We denote by $K(A)$ the complete graph with the vertex set A. Recall that the edge-chromatic number of the complete graph K_e equals $e - 1$ or e if e is even or odd, respectively; see the construction in [2], pp. 249–250. It follows from that construction that $\varphi(K_e \times K_2) = e$, where $F \times H$ denotes the cartesian product of graphs F and H.

Lemma 1. *Let $n \geqq 2$ be an integer, T a tree, $u \in V(T)$, and let A and B be disjoint u-sets in T. Assume that $|A| \leqq |B| \leqq n \leqq |A \cup B|$ and that $|A \cup B|$ is even. Then $\varphi(\langle A \cup B \rangle_{T^n}) \geqq n - 1$.*

Proof. Denote $a = |A|$, $b = |B|$, and $e = (a + b)/2$. It is easy to see (cf. [6]) that there exist vertices $r_1, \ldots, r_a$ and $s_1, \ldots, s_b$ such that $A = \{r_1, \ldots, r_a\}$ and $B = \{s_1, \ldots, s_b\}$ and that

$$\|r_i, u\|_T \leqq i - \varepsilon(u, A) \quad \text{for every} \quad i \in \{1, \ldots, a\}, \quad \text{and}$$
$$\|s_j, u\|_T \leqq j - \varepsilon(u, B) \quad \text{for every} \quad j \in \{1, \ldots, b\}.$$

If $e = a$, then we put $u_1 = r_a, u_2 = r_{a-1}, \ldots, u_e = r_1, v_1 = s_1, v_2 = s_2, \ldots, v_e = s_b$. If $e < a$, then we put $u_1 = r_a, u_2 = r_{a-1}, \ldots, u_a = r_1, u_{a+1} = s_1, \ldots, u_e = s_{(b-a)/2}$, $v_1 = s_{(b-a)/2+1}, \ldots, v_e = s_b$. Moreover, for every integer $m \notin \{1, \ldots, e\}$, we define $u_m = u_k$ and $v_m = v_k$, where $k \in \{1, \ldots, e\}$ and $m \equiv k \pmod e$.

We distinguish two cases:

1. Assume that $e = n$. Let H_1 be the graph induced by

$$E(K(\{u_1, \ldots, u_n\}) \cup E(K(\{v_1, \ldots, v_n\}) \cup \{u_2v_1, u_3v_2, \ldots, u_nv_{n-1}\}.$$

It is clear that H_1 is a subgraph of T^n. Since $H_1 + u_1v_n$ is isomorphic to $K_n \times K_2$, $\varphi(H_1 + u_1v_n) = n$. This means that $\varphi(H_1) = n - 1$. Hence, $\varphi(\langle A \cup B \rangle_{T^n}) \geqq n - 1$.

2. Assume that $e < n$. Let H_2 be the graph induced by $E(K(\{u_1, \ldots, u_e\}) \cup E(K(\{v_1, \ldots, v_e\})) \cup E_*$, where E_* is the set of edges

$$u_1v_1, u_2v_2, \ldots, u_ev_e,$$
$$u_1v_2, u_2v_3, \ldots, u_ev_{e+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_1v_{n-e}, u_2v_{n+1-e}, \ldots, u_ev_{n-1}.$$

It is easy to see that H_2 is a subgraph of T^n. Let H_{21} be the graph induced by the set of edges $E(K(\{u_1, \ldots, u_e\})) \cup E(K(\{v_1, \ldots, v_e\})) \cup \{u_1v_{n-e}, u_2v_{n+1-e}, \ldots, u_ev_{n-1}\}$. Obviously, H_{21} is isomorphic to $K_e \times K_2$, and therefore, $\varphi(H_{21}) = e$. This implies that $\varphi(H_2) = e + (n - e - 1) = n - 1$. Hence, $\varphi(\langle A \cup B \rangle_{T^n}) \geqq n - 1$.

Lemma 2 ([6]). *Let $n \geqq 2$ be an integer, and let T be a tree of an order $>n$. Then there exists $u \in V(T)$ and disjoint u-sets A and B in T such that*

(a) $A \cup B \neq V(T)$,
(b) $T - (A \cup B)$ *is connected,*
(c) $|A| \leqq |B| < n \leqq |A \cup B|$, *and*
(d) *if* $|A \cup B| \neq n$, *then* $|A \cup B|$ *is even.*

The following lemma plays the main rôle in the proof of Theorem 1:

Lemma 3. *Let $n \geqq 2$ be an integer, and let T be a tree of an even order $p > n$. Assume that there exists $u \in V(G)$ and disjoint u-sets A, B and C in T such that $A \cup B \cup C = V(T)$, $|A| \leqq |B| \leqq |C| \leqq n \leqq |A \cup B|$, $|A \cup B|$ is even, and if $|A| = 1$, then $\varepsilon(u, B) = 0$. Then $\varphi(T^n) \geqq n - 1$.*

Proof. Similarly as in the proof of Lemma 1, we denote $a = |A|$, $b = |B|$, and $e = (a + b)/2$. Moreover, we denote $c = |C|$ and $d = n - c$. Thus $a + b + c = p$, $a \leqq b \leqq c \leqq n$, $e \geqq n/2$, and c is even.

If $c = n$, then $\langle C \rangle_{T^n} = K_n$ and thus $\varphi(\langle C \rangle_{T^n}) = n - 1$. It follows from Lemma 1 that $\varphi(\langle A \cup B \rangle_{T^n}) \geqq n - 1$, and thus $\varphi(T^n) \geqq n - 1$. We shall assume that $c < n$. Hence, $e < n$.

If $b + c \leqq n$, then T^n is complete, and thus $\varphi(T^n) = p - 1 \geqq n - 1$. We shall assume that $b + c > n$. Hence, $c > n/2$.

If $e \leqq d$, then $n \leqq 2e \leqq 2d = 2n - 2c < n$, which is a contradiction. Thus we have proved that $d < e$.

Similarly as in the proof of Lemma 1, we introduce vertices $u_1, \ldots, u_e$ and $v_1, \ldots, v_e$. Since $e < n$, we define the graphs H_2 and H_{21} similarly as in the proof of Lemma 1 (Case 2). Recall that $V(H_2) = V(H_{21}) = A \cup B$, $E(H_2) = E(K(\{u_1, \ldots, u_e\})) \cup \cup E(K(\{v_1, \ldots, v_e\})) \cup E_*$, $E(H_{21}) = E(K(\{u_1, \ldots, u_e\})) \cup E(K(\{v_1, \ldots, v_e\})) \cup \cup \{u_1 v_{n-e}, u_2 v_{n+1-e}, \ldots, u_e v_{n-1}\}$, and that H_2 is a subgraph of T^n.

We now introduce further auxiliary notions. There exist vertices $t_1, \ldots, t_c$ of T such that $C = \{t_1, \ldots, t_c\}$ and that

$$\|t_k, u\|_T \leqq k - \varepsilon(u, C), \quad \text{for every} \quad k \in \{1, \ldots, c\}.$$

Define $g = \max(0, (a + b + c)/2 - n + \varepsilon(u, B))$ and $h = \max(1, a + c - n - - g + \varepsilon(u, B))$. If $h > 1$, then $h \leqq a + c - n + \varepsilon(u, B) - ((a + b + c)/2 - n + + \varepsilon(u, B)) = (a - b + c)/2$. Hence, $1 \leqq h \leqq c/2$. We now define the vertices $x_1, \ldots, x_{c/2}, y_1, \ldots, y_{c/2}$ as follows: $x_1 = t_{c-h+1}, x_2 = t_{c-h+2}, \ldots, x_h = t_c$; if $h < c/2$, then we also define $x_{h+1} = t_{c-h}$, $x_{h+2} = t_{c-h-1}, \ldots, x_{c/2} = t_{(c/2)+1}$; moreover, we define $y_1 = t_{c/2}$, $y_2 = t_{(c/2)-1}, \ldots, y_{c/2} = t_1$. Finally, we define mapping f of the set $\{1, 2, \ldots, d\}$ into the set of integers as follows:

$$f(1) = 2[\![d/2]\!] - 1,\ f(2) = 2[\![d/2]\!] - 2, \ldots, f([\![d/2]\!]) = [\![d/2]\!],$$
$$f([\![d/2]\!] + 1) = d + [d/2],\ f([\![d/2]\!] + 2) = d + [d/2] - 1, \ldots, f(d) = d + 1;$$

note that for a real number ϱ, $[\![\varrho]\!]$ denotes the smallest integer m_1 with the property that $\varrho \leqq m_1$, and $[\varrho]$ denotes the greatest integer m_2 with the property that $m_2 \leqq \varrho$. We can see that the minimum (or maximum) value of f equals $[\![d/2]\!]$ (or $d + [d/2]$, respectively). Obviously, the number of integers m such that $[\![d/2]\!] \leqq m \leqq d + [d/2]$ equals d (or $d + 1$) if d is odd (or even, respectively). The following property of f will be important for us:

$$\{f(1), f(2) - 1, \ldots, f(d) - d + 1\} = \{1, 2, \ldots, d\}\,.$$

We shall construct an $(n-1)$-factor H of T^n such that $\varphi(H) = n - 1$. We shall color the edges of H to obtain $n - 1$ edge-disjoint 1-factors of H, each colored by one of $n - 1$ distinct colors, say colors $\omega_1, \ldots, \omega_{n-1}$.

Since H_{21} is isomorphic to $K_e \times K_2$, it can be divided into e edge-disjoint 1-factors, say $F_{n-e}, F_{n-e+1}, \ldots, F_{n-1}$. Clearly, $F_{n-e}, F_{n-e+1}, \ldots, F_{n-1}$ are 1-factors of H_2, and we color $E(F_{n-e}), E(F_{n-e+1}), \ldots, E(F_{n-1})$ by colors $\omega_{n-e}, \omega_{n-e+1}, \ldots, \omega_{n-1}$, respectively. If $e = n - 1$, then we have obtained $n - 1$ edge-disjoint 1-factors of H_2. If $e < n - 1$, then we denote by $F_1, F_2, \ldots, F_{n-e-1}$ the 1-factors of H_2 induced by

$$\begin{aligned}
&\{u_1v_1, \ldots, u_ev_e\}\,,\\
&\{u_1v_2, \ldots, u_ev_{e+1}\}\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\{u_1v_{n-e-1}, \ldots, u_ev_{n-2}\}\,,
\end{aligned}$$

respectively; we color $E(F_1), E(F_2), \ldots$, and $E(F_{n-e-1})$ by $\omega_1, \omega_2, \ldots$, and ω_{n-e-1}, respectively. Clearly, $\varphi(H_2) = n - 1$. Since c is even, $K(C)$ can be divided into $c - 1$ 1-factors, say $F'_{n-c+1}, F'_{n-c+2}, \ldots, F'_{n-1}$. We color $E(F'_{n-c+1}), E(F'_{n-c+2}), \ldots, E(F'_{n-1})$ by $\omega_{n-c+1}, \omega_{n-c+2}, \ldots$, and ω_{n-1}, respectively.

We first note that for every $m \in \{1, \ldots, d\}$, the edges

$$u_{g+m}v_{g+f(m)}, u_{g+1+m}v_{g+1+f(m)}, \ldots, u_{g+(c/2)-1+m}v_{g+(c/2)-1+f(m)}$$

belong to H_2 and are colored by $\omega_{f(m)-m+1}$. We next note that for every $k \in \{1, \ldots, c/2\}$, the vertices

$$v_{g+k-1+f(1)}, v_{g+k-1+f(2)}, \ldots, v_{g+k-1+f(d)}$$

are mutually distinct; this follows from the fact that $d < e$. The edges

$$u_{g+k}v_{g+k-1+f(1)}, u_{g+k+1}v_{g+k-1+f(2)}, \ldots, u_{g+k+d-1}v_{g+k-1+f(d)}$$

belong to H_2 and are colored by distinct colors $\omega_{f(1)}, \omega_{f(2)-1}, \ldots, \omega_{f(d)-d+1}$, respectively.

The graph H (with colored edges) will be constructed from the graph induced by $E(H_2) \cup E(K(C))$ (with colored edges) in such a way that for every $m \in \{1, \ldots, d\}$, the edges

$$u_{g+m}v_{g+f(m)}, u_{g+1+m}v_{g+1+f(m)}, \ldots, u_{g+(c/2)-1+m}v_{g+(c/2)-1+f(m)}$$

(colored by $\omega_{f(m)-m+1}$) will be replaced by the edges

$$u_{g+m}x_1,\ y_1v_{g+f(m)},\ u_{g+1+m}x_2,\ y_2v_{g+1+f(m)},\ \ldots,\ u_{g+(c/2)-1+m}x_{c/2},\ y_{c/2}v_{g+(c/2)-1+f(m)}$$

(colored by the same color).

We can see that H is a regular graph of order $n-1$ and that it can be divided into $n-1$ edge-disjoint 1-factors (colored by $\omega_1, \ldots, \omega_{n-1}$). We wish to show that H is a subgraph of T^n. It is sufficient to prove that every edge of $H - (E(H_2) \cup E(K(C)))$ belongs to $E(T^n)$.

For every $k \in \{1, \ldots, c/2\}$,

$$u_{g+k}x_k,\ u_{g+k+1}x_k,\ \ldots,\ u_{g+k+d-1}x_k$$

are the edges incident with x_k in $H - (E(H_2) \cup E(K(C)))$. (The edges $u_{g+k}x_k$, $u_{g+k+1}x_k, \ldots$, and $u_{g+k+d-1}x_k$ are colored by mutually distinct colors $\omega_{f(1)}$, $\omega_{f(2)-1}, \ldots$, and $\omega_{f(d)-d+1}$, respectively.) We shall show that these edges belong to T^n.

We first show that $g + h \leqq a$ and $g + h + d - 1 \leqq e$. Recall that $d = n - c$. If $h = a + c - n - g + \varepsilon(u, B)$, then $g + h \leqq a + c - (n - 1) \leqq a$ and $g + h + d - 1 \leqq a \leqq e$. Let $h = 1$. If $g = 0$, then $g + h \leqq 1 \leqq a$ and $g + h + d - 1 = n - c < n/2 \leqq e$. Let $g = (a + b + c)/2 - n + \varepsilon(u, B)$. Then $g + h = (a + b + c)/2 - (n - 1) + \varepsilon(u, B) \leqq a/2 + \varepsilon(u, B)$. It follows from the assumption of the lemma that $a/2 + \varepsilon(u, B) \leqq a$, and therefore, $g + h \leqq a$. Since $c \geqq 2$, we have $g + h + d - 1 = (a + b + c)/2 - c + \varepsilon(u, B) \leqq e + 1 - c/2 \leqq e$.

We first consider an arbitrary $k \in \{1, \ldots, h\}$. Clearly, $g + k \leqq a$ and $g + k + d - 1 \leqq e$. To prove that $u_{g+k+m}x_k$ belongs to T^n for any m, $0 \leqq m \leqq d - 1$, it is sufficient to prove that $\|x_{g+k}, x_k\|_T \leqq n$, and that if $g + k + d - 1 > a$, then $\|u_{g+k+d-1}, x_k\|_T \leqq n$. Clearly, $\|u_{g+k}, x_k\|_T = (a - g - k) + \varepsilon(u, B) + (c - h + k) = (a + c - g + \varepsilon(u, B)) - h \leqq (a + c - g + \varepsilon(u, B)) - (a + c - n - g + \varepsilon(u, B)) = n$. It remains to show that $\|u_{g+k+d-1}, x_k\|_T \leqq n$ under the condition that $g + k + d - 1 > a$, since $g + h \leqq a$, we have $\|u_{g+k+d-1}, x_k\|_T = (g + k + d - 2 - a) + \varepsilon(u, A) + (c - h + k) \leqq (g + h + d - 2 - a) + \varepsilon(u, A) + c < g + h - a + n \leqq n$.

Assume that $g + c/2 > e$. If $g = 0$, then $c > 2e \geqq n$, which is a contradiction. If $g = (a + b + c)/2 - n + \varepsilon(u, B)$, then $c > n - \varepsilon(u, B) \geqq n - 1$; a contradiction. This means that $g + c/2 \leqq e$.

We now consider an arbitrary $k \in \{h + 1, \ldots, c/2\}$. Thus $g + k \leqq e$. If $g + k \leqq a$, then $\|u_{g+k}, x_k\|_T = \|u_{g+1}, x_1\|_T \leqq n$. We wish to show that $\|u_{g+k+m}, x_k\|_T \leqq n$ for every $0 \leqq m \leqq d - 1$. If $g + k + d - 1 \leqq a$, then the result is obvious. Let now $g + k + d - 1 > a$. Denote $\bar{e} = \min(e, g + k + d - 1)$. We have that $\|u_{\bar{e}}, x_k\|_T = (\bar{e} - a) + \varepsilon(u, A) + (c - k) \leqq (g + k + n - c - 1 - a) + 1 + c - k = g + n - a < g + h + n - a \leqq n$. If $g + k + d - 1 \leqq e$, we have that $\|u_{g+k+m}, x_k\|_T \leqq n$ for every $m \in \{0, \ldots, d - 1\}$. Let $g + k + d - 1 > e$. Then $\|u_e, x_k\|_T \leqq n$. We shall prove that $\|u_1, x_k\|_T \leqq n$, and thus $\|u_{m'}, x_k\|_T \leqq n$ for every

$m' \in \{1, \ldots, e\}$. Assume that $g = (a + b + c)/2 - n + \varepsilon(u, B)$. Since $g + k + d - 1 > e$, we have that $k > c/2 + 1 - \varepsilon(u, B) \geqq c/2$, which is a contradiction. This means that $g = 0$. Hence, $\|u_1, x_k\|_T < \|u_1, x_1\|_T \leqq n$.

For every $k \in \{1, \ldots, c/2\}$,

$$y_k v_{g+k-1+g(1)}, y_k v_{g+k-1+f(2)}, \ldots, y_k v_{g+k-1+f(d)}$$

are the edges incident with y_k in $H - (E(H_2) \cup E(K(C)))$. (The edges $y_k v_{g+k-1+f(1)}, \ldots$ $\ldots$, and $y_k v_{g+k-1+f(d)}$ are colored by mutually distinct colors $\omega_{f(1)}, \ldots$, and $\omega_{f(d)-d+1}$, respectively.) We shall show that these edges belong to T^n.

Consider an arbitrary $k \in \{1, \ldots, c/2\}$ and define $e' = \min(e, g + k - 1 + d + [d/2])$. To prove that each of the edges $y_k v_{g+k-1+f(1)}, \ldots, y_k v_{g+k-1+f(d)}$ belongs to T^n, it is sufficient to prove that $\|y_k, v_{e'}\|_T \leqq n$. Clearly, $\|y_k, v_{e'}\|_T = (c/2 - k) + \varepsilon(u, A) + (b - a)/2 + e' \leqq c/2 - k + \varepsilon(u, A) + (b - a)/2 + (g + k - 1 + d + [d/2]) \leqq g + n - c + (n + b - a)/2 + \varepsilon(u, A) - 1$. If $g = 0$, then $\|y_k, v_{e'}\|_T \leqq n - c + (n + b - a)/2 \leqq n - b + (n + b - a)/2 = n - (n - (a + b))/2 \leqq n$. If $g = (a + b + c)/2 - n + \varepsilon(u, B)$, then $\|y_k, v_{e'}\|_T \leqq (a + b + c)/2 - c + (n + b - a)/2 + \varepsilon(u, A) + \varepsilon(u, B) - 1 \leqq b + (n - c)/2 \leqq b + (n - b)/2 = (n + b)/2 < n$.

Thus the proof of the lemma is complete.

Now, we are ready to present the proof proper of Theorem 1.

Proof of Theorem 1. The case when $n = 1$ is obvious. We shall assume that $n \geqq 2$. Let m denote the odd integer with the property that $n - 1 \leqq m \leqq n$. Since p is even, we have $p \geqq m + 1$. If $p = m + 1$, then $G^n = K_p$, and therefore $\varphi(G^n) = p - 1 \geqq n - 1$.

Let now $p > m + 1$. Assume that for every connected graph G' of an even order p' with the property that $m + 1 \leqq p' < p$, it is proved that $\varphi((G')^n) \geqq n - 1$.

Let T be a spanning tree of G. Since p is even and $p > m + 1$, we have that $p \geqq m + 3$. It follows from Lemma 2 that there exists $u \in V(G)$ and disjoint u-sets U and W in T such that $U \cup W \neq V(T)$, $G - (U \cup W)$ is connected, $|U| \leqq |W| \leqq m < |U \cup W|$, and $|U \cup W|$ is even. Since $m \leqq n$, according to Lemma 1 we have $\varphi(\langle U \cup W \rangle_{G^n}) \geqq n - 1$.

We distinguish the following cases and subcases:

1. Assume that $|V(G) - (U \cup W)| \geqq m + 1$. Since p and $|U \cup W|$ are even, it follows from the induction hypothesis that $\varphi(\langle V(G) - (U \cup W) \rangle_{G^n}) \geqq n - 1$. Hence, $\varphi(G^n) \geqq n - 1$.

2. Assume that $|V(G) - (U \cup W)| \leqq m$. Since $|U \cup W|$ is even, $|V(G) - (U \cup W)| \leqq m - 1$.

2.1. Assume that there exist disjoint u-sets V_1 and V_2 in T such that $|V_1| \leqq |V_2| \leqq m$ and $V_1 \cup V_2 = V(T) - \{u\}$. Since $p - 1$ is odd, $|V_1| < |V_2|$. Denote $V_0 = V_1 \cup$

$\cup \{u\}$. Then $|V_0| \leqq |V_2| \leqq n$. Since $V_0 \cup V_2 = V(G)$ and $p \geqq n + 2$, it follows from Lemma 1 that $\varphi(G^n) \geqq n - 1$.

2.2. Assume that for arbitrary disjoint u-sets V_1 and V_2 in T such that $|V_1| \leqq |V_2| \leqq m$, we have $V_1 \cup V_2 \neq V(T) - \{u\}$. Since $|U| \leqq |W| \leqq m < |U \cup W|$, it is not difficult to see that there exist disjoint u-sets A', B', and C' in T such that $|A'| \leqq |B'| \leqq |C'| \leqq m < |A' \cup B'|$ and $A' \cup B' \cup C' = V(G) - \{u\}$.

2.2.1. Assume that n is odd and $|C'| = n$. Then $K(C' \cup \{u\})$ is the subgraph of G^n induced by $C' \cup \{u\}$ and $\varphi(\langle C \cup \{u\}\rangle_{G^n}) = n$. It follows from Lemma 1 that $\varphi(\langle A' \cup B'\rangle_{G^n}) \geqq n - 1$, and therefore, $\varphi(G^n) \geqq n - 1$.

2.2.2. Assume that either n is even or $|C'| < n$. If $|C'|$ is odd, then we put $A = A'$, $B = B'$ and $C' \cup \{u\}$. If C' is even, then $|A'| < |B'|$, and we put $A = A' \cup \{u\}$, $B = B'$ and $C = C'$.

We have that A, B, and C are disjoint u-sets in T which fulfil the assumptions of Lemma 3. This implies that $\varphi(T^n) \geqq n - 1$. Hence, $\varphi(G^n) \geqq n - 1$, which completes the proof.

Let $n \geqq 2$ be an integer. Theorem 1 asserts that for every connected graph G of an even order $\geqq n$, there exists a set of $n - 1$ edge-disjoint 1-factors of G^n. For the cases $n = 2, 3$, and 4 Theorem 1 was known before: the case when $n = 2$ was proved in [3] and [8], the case when $n = 4$ was proved in [5]; the case when $n = 3$ follows from the fact that the third power of every connected graph is hamiltonian-connected ([7]).

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