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ON OSCILLATION OF SOLUTIONS OF NONLINEAR RETARDED DIFFERENTIAL EQUATION OF EMDEN-FOWLER TYPE

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We want to study the oscillatory behaviour of solutions of the nonlinear retarded differential equation of Emden-Fowler type

$$(1) \qquad y^{(n)}(t) + p(t)\,|y(g(t))|^{\gamma}\operatorname{sgn} y(g(t)) = 0\,, \quad n \geqq 2\,, \quad \gamma \geqq 1\,,$$

where $p(t)$ and $g(t)$ are continuous on $[0, \infty)$, $p(t) > 0$, $g(t) \leqq t$, $\lim_{t\to\infty} g(t) = \infty$ and $g(t)$ is nondecreasing on $[0, \infty)$.

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some interval $[T_y, \infty)$ and satisfy

$$(2) \qquad \sup\{|y(s)| : 0 \leqq t \leqq s < \infty\} > 0 \quad \text{for any} \quad t \in [T_y, \infty)\,.$$

A solution $y(t)$ of the equation (1) is called *oscillatory* if it has arbitrarily large zeros, and it is called *nonoscillatory* otherwise.

Lemma 1 (Kiguradze). *Let $y(t)$ be a solution of the equation (1) satisfying the condition*

$$y(t) > 0 \quad \textit{for} \quad t \in [0, \infty)\,,$$

and

$$y^{(n)}(t) \leqq 0 \quad \textit{for} \quad t \in [0, \infty)\,.$$

Then there exist $t_1 \in [0, \infty)$ and an integer $l \in \{0, 1, \ldots, n-1\}$ such that $n + l$ is odd and

$$(3) \qquad y^{(i)}(t) > 0 \quad \textit{for} \quad t \in [t_1, \infty) \quad (i = 0, \ldots, l-1)\,,$$

$$(-1)^{i+l}\, y^{(i)}(t) > 0 \quad \textit{for} \quad t \in [t_1, \infty) \quad (i = l, \ldots, n-1)\,,$$

$$(4) \qquad (t - t_1)\,|y^{(l-i)}(t)| \leqq (1 + i)\,|y^{(l-i-1)}(t)| \quad \textit{for} \quad t \in [t_1, \infty)$$

$$(i = 0, \ldots, l-1)\,, \quad 1 \leqq l \leqq n-1\,.$$

An analogous statement can be made if $y(t) < 0$ and $y^{(n)}(t) \geqq 0$ for $t \in [0, \infty)$.

Theorem 1. *Suppose that $\gamma > 1$, $g(t)$ is nondecreasing and for every $l \in$*

$\in \{1, \ldots, n-1\}$ *such that* $n + l$ *is odd the following inequality holds*:

$$\limsup_{t \to \infty} g(t) \int_t^\infty s^{n-l-1} [g(s)]^{(l-1)\gamma} p(s) \, ds > 0 . \tag{5}$$

Then every solution of the equation (1) *is oscillatory provided n is even.*

Moreover, let the following condition be fulfilled:

$$\int^\infty s^{n-1} p(s) \, ds = \infty . \tag{6}$$

If n is odd, then every solution of the equation (1) *is either oscillatory or* $\lim_{t \to \infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$.

Proof. Let $y(t)$ be a nonoscillatory solution of the equation (1) such that $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq 0$. Then with regard to Lemma 1 there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \ldots, n-1\}$ such that $n + l$ is odd and (3), (4) hold.

Let $l \in \{1, \ldots, n-1\}$. For sufficiently large $t_2 \in [t_1, \infty)$, in view of (4) we obtain

$$y(g(t)) \geqq \frac{[g(t) - t_1]^{l-1}}{l!} y^{(l-1)}(g(t)) , \quad t \geqq t_2 . \tag{7}$$

From the identity

$$z^{(j)}(t) = \sum_{i=j}^{k-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} z^{(i)}(s) + \tag{8}$$

$$+ \frac{(-1)^{k-j}}{(k-j-1)!} \int_t^s (u-t)^{k-j-1} z^{(k)}(u) \, du , \quad s \geqq t \geqq t_2 ,$$

where $1 \leqq k \leqq n$, $z \in C_n([0, \infty), R)$, we obtain for $k = n - l + 1$, $j = 1$

$$z'(t) = \sum_{i=1}^{n-l} (-1)^{i-1} \frac{(s-t)^{i-1}}{(i-1)!} z^{(i)}(s) + \tag{9}$$

$$+ \frac{(-1)^{n-l}}{(n-l-1)!} \int_t^s (u-t)^{n-l-1} z^{(n-l+1)}(u) \, du .$$

Choose $z(t) = y^{(l-1)}(t)$. Then with regard to (3) we have $(-1)^{i-1} z^{(i)}(s) > 0$, $i = 1, \ldots, n-l$ and from (9) we conclude

$$z'(t) \geqq \frac{(-1)^{n-l}}{(n-l-1)!} \int_t^\infty (u-t)^{n-l-1} z^{(n-l+1)}(u) \, du .$$

From the last inequality using (7) and the equation (1) we obtain

$$z'(t) \geqq \frac{1}{(n-l-1)! \, (l!)^\gamma} \int_t^\infty (u-t)^{n-l-1} [g(u) - t_1]^{(l-1)\gamma} p(u) \, z^\gamma(g(u)) \, du . \tag{10}$$

We integrate the inequality (10) from T to t, $t > T \geqq t_2$,

$$z(t) \geqq \frac{1}{(n-l-1)!\,(l!)^\gamma} \int_t^\infty [g(u) - t_1]^{(l-1)\gamma} p(u)\, z^\gamma(g(u)) \int_T^t (u-s)^{n-l-1} \,\mathrm{d}s\, \mathrm{d}u\,,$$

$$z(t) \geqq \frac{1}{(n-l)!\,(l!)^\gamma} (t-T) \int_t^\infty (u-T)^{n-l-1} [g(u) - t_1]^{(l-1)\gamma} p(u)\, z^\gamma(g(u))\, \mathrm{d}u\,.$$

Then, for sufficiently large t,

$$(n-l)!\,(l!)^\gamma z(g(t)) \geqq [g(t) - T] \int_t^\infty (u-T)^{n-l-1} [g(u) - T]^{(l-1)\gamma} p(u)\, z^\gamma(g(u))\, \mathrm{d}u\,.$$

Since $z(t)$ is nondecreasing we have

(11) $$(n-l)!\,(l!)^\gamma \frac{z(g(t))}{z^\gamma(g(t))} \geqq [g(t) - T] \int_t^\infty (u-T)^{n-l-1} [g(u) - T]^{(l-1)\gamma} p(u)\, \mathrm{d}u\,.$$

If $z(t)$ increases to infinity as $t \to \infty$, then (11) yields a contradiction with (5).

We recall that the condition (5) implies

(12) $$\int^\infty t^{n-l} [g(t)]^{(l-1)\gamma} p(t)\, \mathrm{d}t = \infty\,.$$

Otherwise, if the integral in (12) converges, then

$$0 < \limsup_{t \to \infty} g(t) \int_t^\infty s^{n-l-1} [g(s)]^{(l-1)\gamma} p(s)\, \mathrm{d}s \leqq \limsup_{t \to \infty} \int_t^\infty s^{n-l} [g(s)]^{(l-1)\gamma} p(s)\, \mathrm{d}s = 0,$$

which is a contradiction.

The condition (12) implies that $z(t)$ cannot be bounded above by a constant (see Foster and Grimmer [3], Theorem 2).

Let $l = 0$. From (8) for $j = 0$, $k = n$, we obtain

$$y(t) \geqq -\frac{1}{(n-1)!} \int_t^\infty (u-t)^{n-1} y^{(n)}(u)\, \mathrm{d}u\,, \quad t \geqq t_2\,,$$

and

$$y(t) \geqq \frac{1}{(n-1)!} \int_t^\infty (u-t)^{n-1} p(u)\, y^\gamma(g(u))\, \mathrm{d}u\,.$$

If $y(t)$ is bounded below by a positive constant c, then

$$y(T) \geqq \frac{c^\gamma}{(n-1)!} \int_T^\infty (u-T)^{n-1} p(u)\, \mathrm{d}u\,, \quad T \geqq t_2\,,$$

which is a contradiction with (6). This completes the proof.

Corollary 1. *Let $g(t)$ be nondecreasing and let*

(13) $$t \leqq [g(t)]^\gamma \quad \text{for} \quad t \geqq T\,, \quad T \in [0, \infty)\,, \quad \gamma > 1\,,$$

and

$$\limsup_{t\to\infty} g(t) \int_t^\infty s^{n-2}\, p(s)\, ds > 0 \tag{14}$$

hold. Then every solution of the equation (1) *is oscillatory if n is even, and every solution of the equation* (1) *is either oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$ *if n is odd.*

Proof. In view of (13) the condition (14) implies (5), (6) and we can apply Theorem 1.

Corollary 2. *Let* $g(t)$ *be nondecreasing and let*

$$t \geqq [g(t)]^{\gamma} \quad \text{for} \quad t \geqq T, \quad T \in [0, \infty), \quad \gamma > 1, \tag{15}$$

$$\limsup_{t\to\infty} g(t) \int_t^\infty [g(s)]^{(n-2)\gamma}\, p(s)\, ds > 0 \tag{16}$$

hold. Then every solution of the equation (1) *is oscillatory if n is even, and every solution of the equation* (1) *is either oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$ *if n is odd.*

Proof. In view of (15) the condition (16) implies (5), (6) and we can apply Theorem 1.

Theorem 2. *Suppose that* $\gamma = 1$, $g(t)$ *is nondecreasing and*

$$\limsup_{t\to\infty} g(t) \int_t^\infty [g(s)]^{n-2}\, p(s)\, ds > (n-1)!\,. \tag{17}$$

Then every solution of the equation (1) *is oscillatory if n is even, and every solution of the equation* (1) *is either oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$ *if n is odd.*

Proof is similar to that of Theorem 1. Let $l \in \{1, \ldots, n-1\}$. From (11) we get

$$(n-1)! \geqq (n-l)!\, l! \geqq [g(t) - T] \int_t^\infty (u - T)^{n-l-1} [g(u) - T]^{l-1}\, p(u)\, du\,,$$

and since $g(t) \leqq t$ we have a contradiction with (17). Let $l = 0$. The condition (17) implies (6) and we find that $\lim_{t\to\infty} y(t) = 0$. The proof is complete.

If $g(t) = t$, then (17) implies the result of Čanturia [2, Theorem 2.3].

Example. Consider the retarded differential equation

$$y^{(4)}(t) + t^{-10/3}\, y^3(t^{1/3}) = 0\,, \quad t > 0\,. \tag{18}$$

The condition

$$\int^\infty [g(t)]^{n-1}\, p(t)\, dt = \infty$$

which guarantees that every solution of the equation (1) is oscillatory is not satisfied for the equation (18). Nevertheless, the conditions (13), (14) or (15), (16) are satisfied. So every solution of the equation (18) is oscillatory.

Now we return to integral sufficient conditions.

Let S denote the set of functions $\varphi \in C([0, \infty) \times R, R)$ such that for every $t \in [0, \infty)$ the function $\varphi(t, \cdot)$ is nondecreasing,

$$\varphi(t, x)\, x \geqq 0 \quad \text{for} \quad t \in [0, \infty)\,, \quad x \in R\,,$$

and the equation

$$x'(t) = -\varphi(t, x(t))$$

has no solution which satisfies (2).

Lemma 2 [7]. *Suppose that $\varphi \in S$, c, $t_0 \in [0, \infty)$. Then the inequality*

$$|x(t)| \geqq c + \int_t^\infty |\varphi(s, x(s))|\, \mathrm{d}s$$

has in the set $C([t_0, \infty), R)$ no solution which satisfies the condition

$$x(t) \neq 0 \quad \text{for} \quad t \in [t_0, \infty)\,.$$

For a proof see [7, Lemma 1.6].

Theorem 3. *Suppose that $\gamma \geqq 1$ and that for every $l \in \{1, \ldots, n-1\}$ such that $n + l$ is odd and for arbitrarily small $\varepsilon > 0$ the following identity holds:*

$$(19) \qquad \int^\infty t^{n-l-1}[g(t)]^{(l-1)\gamma+1-\varepsilon}\, p(t)\, \mathrm{d}t = \infty\,.$$

Then every solution of the equation (1) is oscillatory if n is even.

If in addition (6) holds, then every solution of the equation (1) is either oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$ if n is odd.

Proof. Let $y(t)$ be a nonoscillatory solution of the equation (1) such that $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq 0$. Then with regard to Lemma 1 there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \ldots, n-1\}$ such that $n + l$ is odd and (3), (4) hold. Let $l \in \{1, \ldots, n-1\}$. Now in the same way as in the proof of Theorem 1 we get

$$z(g(t)) \geqq K[g(t) - T] \int_t^\infty (u - T)^{n-l-1} [g(u) - T]^{(l-1)\gamma}\, p(u)\, z^\gamma(g(u))\, \mathrm{d}u\,,$$

$T > t_1$, where $K = 1/(n-l)!\,(l!)^\gamma$.

The condition (19) implies (12) and the condition (12) implies that $z(t)$ increases to infinity as $t \to \infty$. Then for sufficiently large t we have

$$z^\gamma(g(t)) \geqq K[g(t) - T] \int_t^\infty (u - T)^{n-l-1} [g(u) - T]^{(l-1)\gamma}\, p(u)\, [z(g(u))]^{(1-\varepsilon)\gamma} \mathrm{d}u\,,$$

where $0 < \varepsilon < 1$, and

$$\frac{z^\gamma(g(t))}{g(t) - T} \geqq K \int_t^\infty (u - T)^{n-l-1} [g(u) - T]^{(l-1)\gamma+1-\varepsilon} p(u) \left[\frac{z^\gamma(g(u))}{g(u) - T}\right]^{1-\varepsilon} du .$$

Choose $x(t) = z^\gamma(g(u))/(g(u) - T)$. So $x(t)$ is a solution of the above inequality and with regard to Lemma 2, $\varphi \notin S$ where

$$\varphi(t, x(t)) = K(t - T)^{n-l-1} [g(t) - T]^{(l-1)\gamma+1-\varepsilon} p(t) |x(t)|^{1-\varepsilon} \operatorname{sgn} x(t) ,$$

and the equation

$$x'(t) = -\varphi(t, x(t)) \tag{20}$$

has a solution $x(t)$ which satisfies (2). From the equation (20) we have

$$K \int_{t_2}^{t_3} (t - T)^{n-l-1} [g(t) - T]^{(l-1)\gamma+1-\varepsilon} p(t) \, dt = \int_{x(t_3)}^{x(t_2)} \frac{1}{v^{1-\varepsilon}} \, dv , \quad t_3 > t_2 \geqq T ,$$

and as $t_3 \to \infty$ we get a contradiction with (19).

Let $l = 0$. In the same way as in the proof of Theorem 1 we find that $\lim_{t\to\infty} y(t) = 0$. This proves the theorem.

Corollary 3. *Suppose $\gamma \geqq 1$. Let*

$$\liminf_{t\to\infty} \frac{t}{g^\gamma(t)} > 0 , \tag{21}$$

$$\int^\infty [g(t)]^{(n-2)\gamma+1-\varepsilon} p(t) \, dt = \infty , \quad \varepsilon > 0 \tag{22}$$

hold. Then every solution of the equation (1) is oscillatory if n is even, and every solution of the equation (1) is either oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots$ $\ldots, n - 1$ if n is odd.

Proof. With regard to (21) the condition (22) implies (19), (6) and we can apply Theorem 3.

Corollary 4. *Suppose $\gamma \geqq 1$. Let*

$$\liminf_{t\to\infty} \frac{g^\gamma(t)}{t} > 0 , \tag{23}$$

$$\int^\infty t^{n-2}[g(t)]^{1-\varepsilon} p(t) \, dt = \infty , \quad \varepsilon > 0 \tag{24}$$

hold. Then every solution of the equation (1) is oscillatory if n is even, and every solution of the equation (1) is either oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n - 1$ if n is odd.

Proof. With regard to (23) the condition (24) implies (19), (6) and we can apply Theorem 3.

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