Ján Jakubík

On lattices determined up to isomorphisms by their graphs

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 2, 305–314

Persistent URL: http://dml.cz/dmlcz/101952

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

ON LATTICES DETERMINED UP TO ISOMORPHISMS BY THEIR GRAPHS

JÁN JAKUBÍK, Košice

(Received March 28, 1983)

G. Birkhoff ([1], Problem 8) proposed the question of characterizing lattices which are determined up to isomorphisms by their (undirected) graphs.

All lattices dealt with in this paper are assumed to be of locally finite length and all graphs considered here are undirected.

Let C_0 be the class of all lattices which are determined up to isomorphisms by their graphs (i.e., a lattice $\mathscr{L}$ belongs to C_0 iff, whenever the graph of $\mathscr{L}$ is isomorphic to the graph of a lattice $\mathscr{L}_1$, then $\mathscr{L}$ is isomorphic to $\mathscr{L}_1$). Further let C_1 be the class of all lattices $\mathscr{L}$ having the property that whenever h is an isomorphism of the graph of $\mathscr{L}$ onto the graph of a lattice $\mathscr{L}_1$, then h turns out to be either an isomorphism or a dual isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$.

The classes C_0 and C_1 are incomparable (i.e., we have $C_0 \setminus C_1 \neq \emptyset$ and $C_1 \setminus C_0 \neq \emptyset$). It will be shown that both these classes are rather large. The following results will be established:

(i) Each lattice can be embedded into a lattice belonging to C_1.

(ii) Each bounded lattice can be embedded into a lattice belonging to $C_0 \cap C_1$.

(iii) Each bounded modular (distributive) lattice can be embedded as a convex sublattice into a bounded modular (distributive) lattice belonging to $C_0 \cap C_1$.

Isomorphisms of graphs of distributive lattices were investigated by M. Kolibiar and the author [8]; for the case of modular lattices cf. G. Birkhoff [2] and the author [3], [5]. Cf. also [4], [7] (the case of semimodular lattices) and [6].

1. PRELIMINARIES

We start by recalling some notions concerning graphs of lattices. Let $\mathscr{L} = (L; \leqq)$ be a lattice. $\mathscr{L}$ is said to be of locally finite length if each bounded chain in $\mathscr{L}$ is finite. In what follows, all lattices are supposed to be of locally finite length. If $a, b \in L$ and a is covered by b (i.e., the interval $[a, b]$ is prime), then we write $a \prec b$ or $b \succ a$.

By the graph $\mathscr{G}(\mathscr{L})$ we mean the undirected graph whose set of vertices is L and whose edges are those pairs $\{a, b\}$ which satisfy either $a \prec b$ or $b \succ a$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are graphs with sets of vertices G_1 and G_2 and if $h: G_1 \to G_2$ is a bijection such that, for every x and y from G_1, the pair $\{x, y\}$ is an edge in $\mathcal{G}_1$ if and only if $\{h(x), h(y)\}$ is an edge in $\mathcal{G}_2$, then h is said to be an *isomorphism of* $\mathcal{G}_1$ *onto* $\mathcal{G}_2$.

If $\mathcal{L}_1 = (L_1; \leqq_1)$ is a lattice and h is an isomorphism of $\mathcal{G}(\mathcal{L})$ onto $\mathcal{G}(\mathcal{L}_1)$, then h is called a *graph isomorphism of the lattice* $\mathcal{L}$ *onto* $\mathcal{L}_1$.

Let $h: L \to L_1$ be any bijection and let $T \subseteq L$. The set T is said to be *preserved* (*reversed*) *under* h if, whenever $x_1, x_2 \in T$ and $x_1 < x_2$, then $h(x_1) <_1 h(x_2)$ (or $h(x_1) >_1 h(x_2)$, respectively).

1.1. Lemma. (Cf. [3], Lemma 5.) *Let h be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$ and let $u, v, a_1, a_2 \in L$, $u \prec a_i \prec v$ $(i = 1, 2)$. Then we have either* (i) *the sets $\{u, a_1\}$ and $\{a_2, v\}$ are preserved under h, or* (ii) *the sets $\{u, a_1\}$ and $\{a_2, v\}$ are reversed under h.*

(In [3] it was assumed that $\mathcal{L}$ and $\mathcal{L}_1$ ar modular, but the proof remains valid in the general case as well.)

1.2. Lemma. *Let h be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Let u, v, a_1, a_2, a_3 be distinct elements of L, $u \prec a_i \prec v$ $(i = 1, 2, 3)$. Then the set $\{u, v, a_1, a_2, a_3\}$ is either preserved or reversed under h.*

Proof. Assume that the set $\{u, a_1\}$ is preserved under h. By a repeated use of 1.1 we obtain that the following sets are preserved under h:

$$\{a_2, v\};\ \{u, a_3\};\ \{a_1, v\};\ \{u, a_2\};\ \{a_3, v\}\,.$$

Hence $\{u, v, a_1, a_2, a_3\}$ is preserved under h. The case when $\{u, a_1\}$ is reversed under h can be dealt with analogously.

Let $[u_1, v_1]$ and $[u_2, v_2]$ be prime intervals of $\mathcal{L}$. These prime intervals will be called equivalent in $\mathcal{L}$ if, whenever h is a graph isomorphism of $\mathcal{L}$ onto a lattice $\mathcal{L}_1$, then $[u_1, v_1]$ is preserved under h iff $[u_2, v_2]$ is preserved under h.

Our considerations being trivial in the case card $L = 1$, we suppose (in the whole paper) that card $L > 1$. Let L' be a nonempty subset of L. We denote by $\mathcal{G}'(L')$ the graph whose set of vertices is L' and for $a, b \in L'$, the pair $\{a, b\}$ is an edge in $\mathcal{G}'(L')$ iff it is an edge $\mathcal{G}(\mathcal{L})$.

Let $D(\mathcal{L})$ be the set of all elements $x \in L$ such that the graph $\mathcal{G}'(L \setminus \{x\})$ fails to be connected.

1.3. Lemma. *An element x of L belongs to $D(\mathcal{L})$ if and only if the following conditions are fulfilled:*

(i) *there exist $x_1, x_2 \in L$ with $x_1 < x < x_2$;*

(ii) *x is comparable with each element of L.*

Proof. If the conditions (i) and (ii) are valid, then clearly the graph $\mathcal{G}'(L \setminus \{x\})$ fails to be connected. Conversely, suppose that the graph $\mathcal{G}'(L \setminus \{x\})$ is not connected.

Then x cannot be the least element of $\mathscr{L}$; similarly, x cannot be the greatest element of $\mathscr{L}$. Hence (i) holds.

For each $y \in L \setminus \{x\}$ we denote by y^- the connected component of the graph $\mathscr{G}'(L \setminus \{x\})$ containing the element y. Since $x \in D(\mathscr{L})$, there are $a, b \in L \setminus \{x\}$ such that $a^- \neq b^-$. Then clearly a must be comparable with b and we have either $a < x < b$ or $b < x < a$. Let $a < x < b$. By way of contradiction, assume that (ii) fails to be valid. Hence there is $c \in L$ such that c is incomparable with x. Put $u = a \wedge c$, $v = b \vee c$.

If $x_1, x_2, \ldots, x_n$ are elements of L such that $x_1 \prec x_2 \prec \ldots \prec x_n$, then $R = \{x_1, x_2, \ldots, x_n\}$ is said to be a *maximal chain* of the interval $[x_1, x_n]$. Let R_1, R_2, R_3 and R_4 be maximal chains of $[u, a]$, $[u, c]$, $[c, v]$ and $[b, v]$, respectively. Since $a < x < b$, the element x cannot belong to $R_1 \cup R_4$; because x is incomparable with c, we infer that x cannot belong to $R_2 \cup R_3$. Hence $x \notin R = R_1 \cup R_2 \cup R_3 \cup R_4$ and the set R connects the elements a and b in the graph $\mathscr{G}'(L \setminus \{x\})$, thus $a^- = b^-$, which is a contradiction.

1.4. Corollary. *Let $x \in D(\mathscr{L})$. Then the graph $\mathscr{G}'(L \setminus \{x\})$ has two connected components (namely $\{z \in L : z < x\}$ and $\{z \in L : z > x\}$).*

1.5. Corollary. *Let x_1 and x_2 be distinct elements of $D(\mathscr{L})$ such that $\{x_1, x_2\}$ fails to be an edge in $G(L)$. Then the graph $\mathscr{G}'(L \setminus \{x_1, x_2\})$ has three connected components.*

2. THE FIRST EMBEDDING

The aim of this section is to prove that each lattice can be embedded into a lattice belonging to C_1. To this end let us consider the following construction:

Let $\mathscr{L} = (L; \leqq)$ be a lattice and let $\{(u_s, v_s)\}_{s \in S}$ be the set of all pairs of elements u_s, v_s of L such that there is $x \in L$ with $u_s \prec x \prec v_s$. Let $A = \{a_{1s}, a_{2s}\}_{s \in S}$ be a set such that $A \cap L = \emptyset$ and $a_{is(1)} \neq a_{js(2)}$ whenever $(i, s(1)) \neq (j, s(2))$ $(i, j \in \{1, 2\})$. Put $L' = L \cup A$ and define the relation $p \leqq q$ for elements $p, q \in L'$ as follows:

(i) for $p, q \in L$ we put $p \leqq q$ in L' if $p \leqq q$ is valid in L;

(ii) for $p = a_{is} \in A$ $(i = 1$ or $2)$ and $q \in L$ we set $p \leqq q$ if $v_s \leqq q$;

(iii) for $p \in L$ and $q = a_{is} \in A$ $(i = 1, 2)$ we put $p \leqq q$ if $p \leqq u_s$;

(iv) for $p = a_{is(1)} \in A$ and $q = a_{js(2)} \in A$ we put $p \leqq q$ if either $p = q$ or $v_{s(1)} \leqq u_{s(2)}$.

(Cf. Fig. 2.1.) Then we have

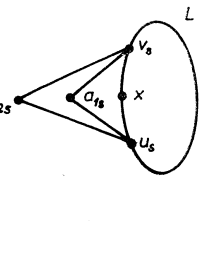

Fig. 2.1

2.1. Lemma. (i) *The relation $\leqq$ is a partial order on L'.* (ii) $\mathscr{L}' = (L'; \leqq)$ *is a lattice.* (iii) $\mathscr{L}$ *is a sublattice of* $\mathscr{L}'$. (iv) *If $[p, q]$ is a prime interval in $\mathscr{L}'$ and $p = a_{is} \in A$, then $q = v_s$.* (v) *If $[p, q]$ is a prime interval in $\mathscr{L}'$ and $q = a_{is} \in A$, then $p = u_s$.*

All the assertions (i)–(v) are immediate consequences of the above construction.

2.2. Lemma. *Let $p, q, r \in L'$ and let the relations $p \prec q \prec r$ be valid in $\mathscr{L}'$. Then the intervals $[p, q]$ and $[q, r]$ are equivalent in $\mathscr{L}'$.*

Proof. We distinguish the following cases (cf. Fig. 2.2 (i) – 2.2 (v)):

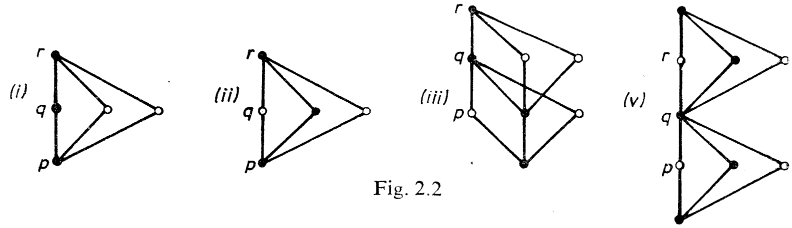

Fig. 2.2

(i) Let $p, q, r \in L$. There exist $a_1, a_2 \in A$, $a_1 \neq a_2$ with $p \prec a_i \prec r$ $(i = 1, 2)$. Thus in view of 1.2, $[p, q]$ and $[q, r]$ are equivalent.

(ii) Let $p, r \in L$, $q \in A$. In view of 2.1 (iv) and (2.1) (v) there is $s \in S$ such that $p = u_s$, $r = v_s$ and $q = a_{is}$ with $i = 1$ or $i = 2$. Let $j \in \{1, 2\}$, $j \neq i$. There exists $x \in L$ with $u_s \prec x \prec v_s$. If we consider the elements p, r, x, a_{1s}, a_{2s}, then from 1.2 we infer that $[p, q]$ and $[q, r]$ are equivalent.

(iii) Let $p \in A$, $q, r \in L$. In view of 2.1 (iv) there is $s \in S$ such that $q = v_s$ and $p = a_{is}$ with $i = 1$ or 2. Let $j \in \{1, 2\}$, $j \neq i$. There is $x \in L$ such that $u_s \prec x \prec v_s$. Hence in view of 1.2, $[p, q]$ is equivalent with $[x, q]$. Now we have $x \prec q \prec r$. There exist distinct elements b_1, b_2 of A such that $x \prec b_i \prec r$ $(i = 1, 2)$. Hence 1.2 yields that $[x, q]$ is equivalent to $[q, r]$. Therefore $[p, q]$ and $[q, r]$ are equivalent.

(iv) The case $p, q \in L$, $r \in A$ is dual to (iii).

(v) Let $q \in L$ and $p, r \in A$. Then in view of 2.1 (iv) and 2.1 (v) there are $s, t \in S$ such that $q = v_s = u_t$, $p = a_{is}$ with $i = 1$ or 2, and $r = a_{i(1)t}$ with $i(1) = 1$ or 2. Let $j \in \{1, 2\}, j \neq i$. There is $x \in L$ such that $u_s \prec x \prec v_s$. By considering the elements u_s, p, q, a_{js}, x we obtain from 1.2 that $[p, q]$ is equivalent to $[x, q]$.

Let $j(1) \in \{1, 2\}$, $j(1) \neq i(1)$. There is $y \in L$ with $u_t \prec y \prec v_t$. In view of the set $\{q, r, v_t, a_{j(1)t}, y\}$ we infer from 1.2 that $[q, r]$ is equivalent to $[q, y]$. Now because of $x \prec q \prec y$, from (i) above it follows that $[x, q]$ and $[q, y]$ are equivalent. Therefore $[p, q]$ and $[q, r]$ are equivalent.

In view of 2.1, no other cases can occur; the proof is complete.

2.3. Lemma. *Let $[p, q]$ and $[p_1, q_1]$ be prime intervals of $\mathscr{L}'$ such that either $q \leqq p_1$ or $q_1 \leqq p$. Then $[p, q]$ and $[p_1, q_1]$ are equivalent.*

Proof. This follows by induction from 2.2.

2.4. Lemma. *Any two prime intervals of the lattice $\mathscr{L}'$ are equivalent.*

Proof. By way of contradiction, suppose that there are prime intervals $[p, q]$ and $[p_1, q_1]$ in $\mathscr{L}'$ which fail to be equivalent in $\mathscr{L}'$. Hence there exists a graph isomorphism h of $\mathscr{L}'$ onto a lattice $\mathscr{L}_1 = (L_1; \leqq_1)$ such that $[p, q]$ is preserved under h and $[p_1, q_1]$ is reversed under h. Denote $u = p \wedge p_1$ and $v = q \vee q_1$. Since $[p, q]$ is preserved under h, we obtain from 2.3 that $h(u) <_1 h(v)$; similarly, because $[p_1, q_1]$ is reversed under h, Lemma 2.3 implies that $h(u) >_1 h(v)$ are valid, which is a contradiction.

From Lemma 2.4 we obtain immediately:

2.5. Theorem. *Let $\mathscr{L}$ be a lattice and let $\mathscr{L}'$ be as above. Let h be a graph isomorphism of $\mathscr{L}'$ onto lattice $\mathscr{L}_1$. Then h is either an isomorphism or a dual isomorphism of $\mathscr{L}'$ onto $\mathscr{L}_1$.*

2.6. Corollary. *$\mathscr{L}'$ belongs to C_1.*

2.6.1. Corollary. *Each lattice can be embedded into a lattice belonging to C_1.*

Let us establish some further properties of $\mathscr{L}'$. From the considerations performed in the proof of Lemma 2.2 we infer that if p, q, r are as in 2.2 then the intervals $[p, q]$ and $[q, r]$ are projective. By induction we obtain:

2.7. Lemma. *Let $[p, q]$ and $[p_1, q_1]$ be prime intervals in $\mathscr{L}$ such that $q \leqq p_1$. Then $[p, q]$ and $[p_1, q_1]$ are projective.*

A lattice is said to be *simple* if it has no nontrivial congruence relation.

2.8. Proposition. *The lattice $\mathscr{L}'$ is simple.*

Proof. Let θ be a congruence relation on $\mathscr{L}'$ distinct from the minimal congruence. Hence there exists a prime interval $[p, q]$ of $\mathscr{L}'$ such that $p \,\theta\, q$. Let $a, b \in L'$. Put $u = p \wedge a \wedge b$, $v = q \vee a \vee b$. From 2.7 it follows that $u \,\theta\, v$ is valid, and hence $a \,\theta\, b$. Therefore θ is the greatest congruence relation of $\mathscr{L}'$.

2.9. Proposition. *The following conditions are equivalent:* (i) card $L \leqq 2$; (ii) $\mathscr{L}' = \mathscr{L}$; (iii) $\mathscr{L}'$ *is distributive.*

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If there are elements $p, q, r \in L$ with $p \prec q \prec r$, then it follows from the construction of $\mathscr{L}'$ that the interval $[p, r]$ of $\mathscr{L}'$ fails to be distributive. Thus if $\mathscr{L}'$ is distributive, then card $L \leqq 2$.

Let us give some examples:

2.10.1. If $\mathscr{L}$ is linearly ordered, card $L > 2$, then $\mathscr{L}'$ is modular and each maximal antichain in $\mathscr{L}'$ has three elements.

This follows immediately from the construction of $\mathscr{L}'$.

2.10.2. If $\mathscr{L}$ is a modular lattice of length 3, then $\mathscr{L}'$ is a modular lattice of length 3 with card $L' = \text{card } L + 2$.

A sublattice $\mathscr{L}_1$ of a lattice $\mathscr{L}$ is said to be a *c-sublattice* of $\mathscr{L}$ if, whenever p, q are elements of $\mathscr{L}_1$ such that p is covered by q in $\mathscr{L}_1$, then p is covered by q in $\mathscr{L}$.

2.10.3. Let $\mathscr{L}$ be a lattice having a c-sublattice $\mathscr{L}_1$ which is isomorphic to the lattice in Fig. 2.10. Then $\mathscr{L}'$ fails to be modular. (Hence if $\mathscr{L}$ is distibutive, then $\mathscr{L}'$ need not be modular.)

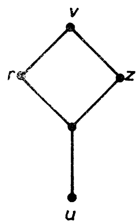

Fig. 2.10

Proof. Let the elements of $\mathscr{L}_1$ be denoted as in Fig. 2.10. There exist elements $a_1, a_2 \in A$ such that $u \prec a_1 \prec r$, $u \prec a_2 \prec z$. Thus $u = a_1 \wedge a_2$ is covered by both a_1 and a_2 in $\mathscr{L}'$. In view of the construction of $\mathscr{L}'$ we have $a_1 \vee a_2 = v$, hence $a_1 \vee a_2$ covers neither a_1 nor a_2. Therefore $\mathscr{L}'$ is not modular.

From the definition of $\mathscr{L}'$ we immediately obtain (under the denotations as above):

2.11. Proposition. (i) *Let $\mathscr{L}$ be a lattice and $a \in L'$. Then the graph $\mathscr{G}'(L' \setminus \{a\})$ is connected.* (ii) *Let $b \in L'$. The element b is the least element (greatest element) of $\mathscr{L}'$ if and only if b is the least element (or the greatest element, respectively) of $\mathscr{L}$.*

3. THE SECOND EMBEDDING

In this section we shall show that each bounded lattice can be embedded into a lattice belonging to C_0.

Let $\mathscr{L} = (L; \leqq)$ be a bounded lattice with card $L \geqq 3$. (Let us remark that if card $L \leqq 2$, then clearly $\mathscr{L} \in C_0$.) Let I be the set of all integers. For $i \in I$ we put $\mathscr{L}_i = \mathscr{L}$ if i is even and $\mathscr{L}_i = \mathscr{L}^\sim$ if i is odd (the symbol $\mathscr{L}^\sim$ denotes the lattice dual to $\mathscr{L}$).

Let u and v be the least and the greatest element of $\mathscr{L}$, respectively. We denote by u_i and v_i the least and the greatest element in $\mathscr{L}_i$,respectively. Hence if i is even, then $u_i = u$, $v_i = v$; if i is odd, then $u_i = v$ and $v_i = u$. Let L^* be the set of all pairs (i, x) where (a) $i \in I$ and $x \in L$, (b) the elements (i, v_i) and $(i + 1, u_{i+1})$ are identified for each $i \in I$.

Let (i_1, x_1) and (i_2, x_2) be distinct elements of L^*. We put $(i_1, x_1) < (i_2, x_2)$ if either $i_1 < i_2$, or $i_1 = i_2$ and $x_1 < x_2$ is valid in $\mathscr{L}_i$. Put $\mathscr{L}^* = (L^*; \leqq)$. Then $\mathscr{L}^*$ is a lattice. Let us establish some properties of $\mathscr{L}^*$.

Denote $P = \{(i, u_i) : i \in I\}$. For $(i, u_i), (j, u_j) \in P$ we set $(i, u_i)\, s\, (j, u_j)$ if $|i - j| = 1$.

In the following Lemmas 3.1–3.7 we assume that for each $z \in L \setminus \{u, v\}$ there exists $z_1 \in L$ such that z and z_1 are incomparable. From this assumption and from 1.3 we obtain:

3.1. Lemma. *Let $z \in L^*$. Then z belongs to P if and only if $z \in D(\mathscr{L}^*)$.*

Since $D(\mathscr{L}^*)$ is uniquely determined by the graph $\mathscr{G}(\mathscr{L}^*)$, we have

3.2. Corollary. *The set P is uniquely determined by the graph $\mathscr{G}(\mathscr{L}^*)$.*

Also, from the assumption on $\mathscr{L}$ and from 1.5 we infer:

3.3. Lemma. *Let z_1 and z_2 be distinct elements of P. (i) The graph $\mathscr{G}'(L^* \setminus \{z_1, z_2\})$ has three connected components Q_1, Q_2, Q_3. (ii) $z_1\, s\, z_2$ if and only if there is (a unique) $j \in \{1, 2, 3\}$ such that, for each $x \in Q_j$, the graph $\mathscr{G}'(Q_j \setminus \{x\})$ is connected.*

3.4. Corollary. *The relation s on P is uniquely determined by $\mathscr{G}(\mathscr{L}^*)$.*

Let us choose any pair z_0 and z_1 of distinct elements of P such that $z_0\, s\, z_1$. By means of z_0 and z_1 we define a binary relation $\prec_0$ on P as follows:

Let $z_2, z_3 \in P$. We put $z_2 \prec_0 z_3$ if some of the following conditions is valid:

(i) $z_2 = z_0$ and $z_3 = z_1$.

(ii) $z_3 = z_0$, $z_2\, s\, z_0$ and $z_2 \neq z_1$.

(iii) $z_2 = z_1$, $z_3\, s\, z_1$ and $z_3 \neq z_0$.

(iv) $z_2\, s\, z_3$ and there are connected components P_1 and P_2 of the graph $\mathscr{G}'(L^* \setminus \{z_2\})$ such that $P_1 \neq P_2$, $z_0, z_1 \in P_1$ and $z_3 \in P_2$.

(v) $z_2\, s\, z_3$ and there are connected components P_1 and P_2 of the graph $\mathscr{G}'(L^* \setminus \{z_3\})$ such that $P_1 \neq P_2$, $z_2 \in P_1$ and $z_0, z_1 \in P_2$.

From 3.2 and 3.4 we infer:

3.5. Lemma. *The relation $\prec_0$ is uniquely determined by the pair z_0, z_1 and by the graph $\mathscr{G}(\mathscr{L}^*)$.*

For $z_2, z_3 \in P$ we put $z_2 \leqq_0 z_3$ if either $z_2 = z_3$ or there are elements $z'_1, z'_2, \ldots, z'_n$ in P such that $z'_1 = z_2$, $z'_n = z_3$ and $z_i \prec_0 z_{i+1}$ $(i = 1, \ldots, n - 1)$. Then $\leqq_0$ is a linear order on the set P. This linear order either coincides with the linear order $\leqq$ on P, or is dual to $\leqq$ on P.

Now let $\mathscr{L}_1 = (L_1; \leqq_1)$ be a lattice and let h be a graph isomorphism of $\mathscr{L}^*$ onto $\mathscr{L}_1$. From 3.1 and 1.3 we infer that each element of $h(P)$ is comparable with each element of $\mathscr{L}_1$. In particular, $h(P)$ is a chain in $\mathscr{L}_1$. From the definition of the relation $\prec_0$ (cf. the conditions (i)–(v) above) we infer:

3.6. Lemma. *We have either (i) $z_1 \leqq_0 z_2 \Leftrightarrow h(z_1) \leqq_1 h(z_2)$ for each pair $z_1, z_2 \in P$, or (ii) $z_1 \leqq_0 z_2 \Leftrightarrow h(z_1) \geqq_1 h(z_2)$ for each pair $z_1, z_2 \in P$.*

Since the relation $\leqq_0$ on P either coincides with $\leqq$ or is dual to $\leqq$, we obtain

3.6.1. Corollary. *We have either* (i) $z_1 \leqq z_2 \Leftrightarrow h(z_1) \leqq_1 h(z_2)$ *for each pair* $z_1, z_2 \in P$, *or* (ii) $z_1 \leqq z_2 \Leftrightarrow h(z_1) \geqq_1 h(z_2)$ *for each pair* $z_1, z_2 \in P$.

In Lemmas 3.7 – 3.8 above we assume that the condition (i) from 3.6.1 is fulfilled.

3.7. Lemma. *Let* $z_2, z_3 \in P$, $z_2 \prec_0 z_3$. *Let* $x \in L^*$. *Then* $z_2 < x < z_3$ *if and only if* $h(z_2) < h(x) < h(z_3)$.

Proof. Let $z_2 < x < z_3$. The graph $\mathscr{G}'(L^* \setminus \{z_3\})$ has two connected components, and the elements z_2 and x belong to the same of these components. In view of the graph isomorphism h, the same is valid if z_2, x and z_3 are replaced by $h(z_2)$, $h(x)$ and $h(z_3)$. Because $h(z_2) <_1 h(z_3)$, we must have $h(x) <_1 h(z_3)$. By the analogous argument concerning the set $L \setminus \{z_2\}$ we obtain $h(z_2) <_1 h(x)$. Hence $h(z_2) < < h(x) < h(z_3)$. The proof of the assertion 'if' can be established in a similar way.

Now let $\mathscr{L}_0 = (L_0; \leqq)$ be a bounded lattice with card $L_0 \geqq 3$. Put $\mathscr{L} = \mathscr{L}'_0$ (in the sense introduced in § 2). If $z \in L'_0$ and z is neither the least nor the greatest element of $\mathscr{L}'_0$, then in view of the construction of $\mathscr{L}'_0$ (cf. § 2) there is $z' \in L'_0 = L$ such that z and z' are incomparable in $\mathscr{L}'_0$. Hence we can apply the above results 3.1 – 3.7 for $\mathscr{L}$ and $\mathscr{L}^*$.

3.8. Lemma. *Let* $x, y \in L^*$. *Then* $x < y$ *if and only if* $h(x) < h(y)$.

Proof. There are $i, j \in I$ and $x_1, y_1 \in L$ such that $x = (i, x_1)$, $y = (j, y_1)$. We distinguish two cases:

a) $i = j$. In view of 3.7, the mapping h reduced to the interval $[(i, u_i), (i, v_i)]$ of the lattice $\mathscr{L}^*$ maps this interval onto the interval $[h((i, u_i)), h((i, v_i))]$ of the lattice $\mathscr{L}_1$. Since $[(i, u_i), (i, v_i)]$ is either isomorphic or dually isomorphic to $\mathscr{L} = \mathscr{L}'_0$, it belongs to C_1 and thus h (reduced to this interval) is either an isomorphism or a dual isomorphism. Because of the assumption that (i) is valid, only the case of an isomorphism can occur; hence $x < y \Leftrightarrow h(x) < h(y)$.

b) $i \neq j$. First suppose that $i < j$ is valid. Put $z_1 = (i, v_i)$, $z_2 = (j, u_j)$. Then

$$x \leqq z_1 \leqq z_2 \leqq y\,.$$

In view of (i) and 3.6.1 we have $h(z_1) \leqq_1 h(z_2)$. According to 3.7, $h(x) \leqq_1 h(z_1)$ and $h(z_2) \leqq_1 h(y)$. Hence $h(x) \leqq_1 h(y)$ holds. Analogously, if $i > j$, then we obtain $x \geqq y$ and $h(x) \geqq_1 h(y)$.

3.9. Corollary. *If* (i) *holds, then h is an isomorphism of* $\mathscr{L}^*$ *onto* $\mathscr{L}_1$.

By using duality we obtain:

3.9.1. Corollary. *If* (ii) *holds, then h is a dual isomorphism of* $\mathscr{L}^*$ *onto* $\mathscr{L}_1$.

Now from 3.9 and 3.9.1 we infer that $\mathscr{L}^*$ belongs to C_1. Further, from the structure of $\mathscr{L}^*$ we easily obtain that $\mathscr{L}^*$ is self-dual; hence if (ii) holds, then $\mathscr{L}_1$ must be self-dual as well and thus $\mathscr{L}^*$ is isomorphic to $\mathscr{L}_1$. Therefore $\mathscr{L}^*$ belongs to C_0. Since $\mathscr{L}_0$ is embedded into $\mathscr{L}^*$, we conclude:

3.10. Theorem. *Each bounded lattice can be embedded into a lattice belonging to $C_0 \cap C_1$.*

4. THE CASE OF MODULAR LATTICES

Let $\mathscr{L} = (L; \leqq)$ be a bounded modular lattice with card $L \geqq 3$. By the methods described in Sections 2 and 3 above, $\mathscr{L}$ can be embedded into the lattice $(\mathscr{L}')^*$ which belongs to the class $C_0 \cap C_1$. But, as we have already shown above, the lattice $(\mathscr{L}')^*$ need not be modular (even in the case when $\mathscr{L}$ is distributive).

In this section we shall give a construction which enables one to embed a bounded modular (distributive) lattice into a modular (distributive) lattice belonging to $C_0 \cap C_1$. This construction is simpler than the construction dealt with above for general lattices (in § 2 and § 3). We shall apply the following results (A) and (B) on graph isomorphisms of modular lattices.

(A) ([3], Thm. 1.) Let h be a graph isomorphism of a modular lattice $\mathscr{L} = (L; \leqq)$ onto a modular lattice $\mathscr{L}_1 = (L_1; \leqq_1)$. Then there are lattices $\mathscr{A} = (A; \leqq)$, $\mathscr{B} = (B; \leqq)$ and direct product representations $f: \mathscr{L} \to \mathscr{A} \times \mathscr{B}$, $g: \mathscr{L}_1 \to \mathscr{A} \times \mathscr{B}^\sim$ such that $f = h \circ g$.

(In Thm. 1 of [3] it is asserted only that $\mathscr{L} \cong \mathscr{A} \times \mathscr{B}$ and $\mathscr{L}_1 \cong \mathscr{A} \times \mathscr{B}^\sim$ are valid; but, in fact, the stronger assertion (A) was proved in [3] (under different notation).)

(B) ([5], Thm. 2.) Let h be a graph isomorphism of a modular lattice $\mathscr{L}$ onto a lattice $\mathscr{L}_1$. Then $\mathscr{L}_1$ is modular as well.

4.1. Lemma. *Let $\mathscr{L}$ be a modular lattice with the least element u and the greatest element v. Let h be a graph isomorphism of $\mathscr{L}$ onto a lattice $\mathscr{L}_1 = (L_1; \leqq_1)$.* (i) *If $h(u) <_1 h(v)$, then h is an isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$.* (ii) *If $h(u) >_1 h(v)$, then h is a dual isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$.*

Proof. This is a consequence of (A) and (B).

Let us remark that the assumption of modularity in 4.1 cannot be omitted. In fact, let $\mathscr{L}$ be the lattice in Fig. 4.1.a and let $\mathscr{L}_1$ be the lattice on Fig. 4.1.b. For each $x \in L$ we put $h(x) = x'$. Then h is a graph isomorphism, but it is neither an isomorphism nor a dual isomorphism.

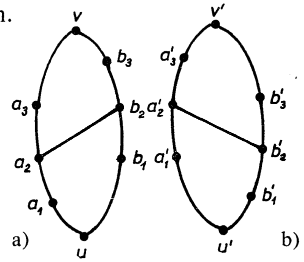

Fig. 4.1

Let L be as in 4.1 and let

$$\mathscr{L}_0 = \mathscr{L} \times \mathscr{L}^{\sim}, \quad \mathscr{L}_0 = (L_0, \leqq).$$

4.2. Lemma. *Let $\mathscr{L}_2 = (L_2; \leqq)$ be a lattice such that* (i) $L_2 = L_0 \cup \{u_1, v_1\}$, $u_1 \notin L_0$, $v_1 \notin L_0$; (ii) $\mathscr{L}_0$ *is a sublattice of* $\mathscr{L}_2$; (iii) u_1 *is the least element of* $\mathscr{L}_2$ *and* v_1 *is the greatest element of* $\mathscr{L}_2$. *Then* $\mathscr{L}_2 \in C_1 \cap C_0$.

Proof. Let h be a graph isomorphism of $\mathscr{L}_2$ onto a lattice $\mathscr{L}_3 = (L_3; \leqq)$. There exists exactly one element of L_2 (namely u_0, the least element in $\mathscr{L}_0$) such that $\{u_1, u_0\}$ is an edge in $\mathscr{G}(\mathscr{L}_2)$; hence $h(u_1)$ is either the least or the greatest element of $\mathscr{L}_3$. Similarly, $h(v_1)$ is either the least or the greatest element of $\mathscr{L}_3$.

First assume that $h(u_1)$ is the least element of $\mathscr{L}_3$. Then $h(v_1)$ is the greatest element of $\mathscr{L}_3$. Since $\mathscr{L}$ is modular, $\mathscr{L}_2$ is modular as well. Then we infer from 4.1 that h is an isomorphism of $\mathscr{L}_2$ onto $\mathscr{L}_3$. Analogously, if $h(u_1)$ is the greatest element of $\mathscr{L}_3$, then h is a dual isomorphism of $\mathscr{L}_2$ onto $\mathscr{L}_3$. Therefore $\mathscr{L}_2 \in C_1$. Next, in view of the fact that $\mathscr{L}_3$ is isomorphic to its dual, we infer that $\mathscr{L}_2$ belongs to C_0.

4.3. Theorem. *Let $\mathscr{L}$ be a bounded modular lattice. There exists a bounded modular lattice $\mathscr{L}_2$ which has the following properties*:

(i) $\mathscr{L}_2 \in C_0 \cap C_1$.

(ii) $\mathscr{L}_2$ is modular and bounded. If $\mathscr{L}$ is distributive, then $\mathscr{L}_2$ is distributive as well.

(iii) There exists a convex sublattice of $\mathscr{L}_2$ isomorphic to $\mathscr{L}$.

There exist distributive lattices which do not belong to $C_0 \cap C_1$ (e.g., the direct product of the lattice in Fig. 2.10 with its dual); hence in view of 4.3 we have

4.4. Corollary. *Neither C_0 nor C_1 are closed with respect to convex sublattices.*

Let $\mathscr{A}$ be the lattice in Fig. 2.10 and let $\mathscr{B}$ be the four-element lattice which is not linearly ordered. Then we have $\mathscr{A} \in C_1 \setminus C_0$ and $\mathscr{B} \in C_0 \setminus C_1$.

References

[1] *G. Birkhoff:* Lattice Theory, second ed., Amer. Math. Soc., Providence 1948.

[2] *G. Birkhoff:* Some applications of universal algebra. Coll. Math. Soc. J. Bolyai, 29, Universal algebra (Esztergom 1977), North Holland, Amsterdam 1982, 107—128.

[3] *Я. Якубик:* О графическом изоморфизме структур. Чехослов. мат. ж. *4* (1954), 131—141.

[4] *J. Jakubík:* On the graph isomorphism of semimodular lattices. (Slovak.) Matem. fyz. časopis *4* (1954), 162—177.

[5] *J. Jakubík:* Unoriented graphs of modular lattices. Czech. Math. J. *25* (1975), 240—246.

[6] *J. Jakubík:* On isomorphisms of graphs of lattices. (Submitted.)

[7] *J. Jakubík:* Graph isomorphisms of semimodular lattices. Mathematica Slovaca (submitted).

[8] *Я. Якубик, М. Колибиар:* О некоторых свойствах пар структур. Чехослов. мат. ж. *4* (1954), 1—27.

Author's address: 040 01 Košice, Švermova 9, ČSSR (Vysoká škola technická).