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TOLERANCES ON MONOUNARY ALGEBRAS

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Congruence relations on monounary algebras were studied by J. Berman [1], L. A. Skornjakov and D. P. Jegorova [2], D. Jakubíková-Studenovská [3], [4], D. P. Jegorova [5], [6] and G. Č. Kurinnoj [7], [8]. The aim of this paper is to investigate, for the case of tolerances, some questions which were dealt with by D. Jakubíková-Studenovská [3] for congruence relations. A monounary algebra is an ordered pair (A, f), where A is a non-empty set and f is a unary operation on A, i.e. a mapping of A into A.

For every non-negative integer n and for each $a \in A$ we may define $f^n(x)$ recurrently so that $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for $n \geqq 1$.

A tolerance on a unary algebra (A, f) is a reflexive and symmetric binary relation T on A with the property that $(x, y) \in T$ implies $(f(x), f(y)) \in T$ for any $x \in A$, $y \in A$. If a tolerance on (A, f) is transitive, it is a congruence on (A, f).

In our considerations we shall often use graphs.

The graph of the algebra (A, f) is the directed graph $G(A, f)$ whose vertex set is A and in which there exists an edge going from x to y if and only if $f(x) = y$ (such an edge may be a loop).

The graph of a tolerance T on (A, f) is the undirected graph $G(T)$ whose vertex set is A and in which two vertices x, y are adjacent if and only if $x \neq y$ and $(x, y) \in T$. By $G_0(T)$ we denote the graph obtained from $G(T)$ by deleting all isolated vertices.

The vertex set of a connected component of $G(A, f)$ is the support of a subalgebra of (A, f); this subalgebra is called a connected component of (A, f). If $G(A, f)$ is connected (i.e. it has exactly one connected component), then (A, f) is also called connected. Each connected component of $G(A, f)$ contains at most one cycle (a vertex with a loop or two vertices joined by two oppositely directed edges are also considered cycles). The vertex set of such a cycle is called a cycle in (A, f). These concepts may be obviously defined without using graphs; see e.g. [3].

Proposition 1. *All tolerances on a monounary algebra (A, f) form a lattice $LT(A, f)$ with respect to the set inclusion. The meet of two tolerances in $LT(A, f)$ is their intersection, their join is their union. The least element of $LT(A, f)$ is the identity relation Δ on A, the greatest element is the universal relation $A \times A$.*

The proof is left to the reader.

Corollary 1. *The lattice $LT(A,f)$ is distributive.*

This follows from the fact that, by Proposition 1, $LT(A,f)$ is a sublattice of the lattice of all subsets of $A \times A$.

Now let x, y be two distinct elements of A. The symbol $T(x, y)$ will denote the least tolerance on (A,f) which contains the pair (x, y), i.e. the intersection of all tolerances on (A,f) which contain (x, y). The set of all tolerances $T(x, y)$ for $x \in A$, $y \in A$, $x \neq y$ will be denoted by $M(A,f)$.

Proposition 2. *Let $T_0 \in LT(A,f)$. The following two conditions are equivalent:*
(i) *The set $J(T_0) = \{T \in LT(A,f) \mid \Delta \neq T \subseteq T_0\}$ is a nowhere dense chain.*
(ii) *$T_0 \in M(A,f)$.*

Proof. (ii) $\Rightarrow$ (i). If $T_0 \in M(A,f)$, then $T_0 = T(x, y)$ for some elements x, y. Evidently T_0 consists of all pairs from Δ and of all pairs $(f^n(x), f^n(y))$ and $(f^n(y), f^n(x))$ for all non-negative integers n. Let $T \subseteq T_0$, $T \in LT(A,f)$, $T \neq \Delta$. Then there exists the least non-negative integer k with the property that $(f^k(x), f^k(y)) \in T$. Then $T(f^k(x), f^k(y)) \subseteq T$, because $T(f^k(x), f^k(y))$ is the least tolerance on (A,f) which contains the pair $(f^k(x), f^k(y))$. On the other hand, $T(f^k(x), f^k(y))$ consists of all pairs from Δ and of all pairs $(f^n(x), f^n(y))$ and $(f^n(y), f^n(x))$ for $n \geqq k$. Then the minimality of k implies $T \subseteq T(f^k(x), f^k(y))$ and hence $T = T(f^k(x), f^k(y))$. The set $J(T_0)$ consists exactly of all tolerances $T(f^k(x), f^k(y))$ for all non-negative integers k; evidently $T(f^k(x), f^k(y)) \subset T(f^l(x), f^l(y))$ for $l < k$ and therefore these tolerances form a nowhere dense chain.

(i) $\Rightarrow$ (ii). Suppose that $J(T_0)$ is a nowhere dense chain. If $J(T_0) = \{T_0\}$, then evidently T_0 is an atom of $LT(A,f)$ and thus $T_0 \in M(A,f)$. Otherwise $J(T_0) - \{T_0\} \neq \emptyset$; as $J(T_0)$ is nowhere dense, there exists the greatest element T_1 of $J(T_0) - \{T_0\}$; then T_0 covers T_1 in $J(T_0)$. We have $T_0 - T_1 \neq \emptyset$; thus let $(x, y) \in T_0 - T_1$. Evidently $T(x, y) \subseteq T_0$, $T(x, y) \nsubseteq T_1$, $T(x, y) \in J(T_0)$. As $J(T_0)$ is a chain, we have $T_1 \subset T(x, y) \subseteq T_0$ and, as T_0 covers T_1, we obtain $T_0 = T(x, y)$.

If A and the lattice $LT(A,f)$ on A is given, according to Proposition 2 we can distinguish the tolerances belonging to $M(A,f)$ from the other ones.

Now consider again a monounary algebra (A,f). The union of all cycles of (A,f) will be denoted by $C_0(A,f)$. By $C_1(A,f)$ we denote the set of all elements $x \in A$ such that $x \notin C_0(A,f)$ and $f(x) \in C_0(A,f)$. By $C_2(A,f)$ we denote the set of all elements $x \in A$ such that $f(x) \in C_1(A,f)$. Finally, by $C_3(A,f)$ we denote the set $A - (C_0(A,f) \cup C_1(A,f) \cup C_2(A,f))$.

For each $x \in A$, by $C_0[x]$, $C_1[x]$, $C_2[x]$, $C_3[x]$ we shall denote the intersections of the connected component of (A,f) containing x with the sets $C_0(A,f)$, $C_1(A,f)$, $C_2(A,f)$, $C_3(A,f)$, respectively.

Proposition 3. *Let (A,f) be a monounary algebra, $a \in A$. Let $C_3(A,f) \neq \emptyset$. Then the following two assertions are equivalent:*
(i) $f(a) = a$;

(ii) *for each* $x \in A - \{a\}$ *the graph* $G_0(T(a, x))$ *either has only one edge, or is a star wtth the centre* a.

Proof. (i) $\Rightarrow$ (ii). Let $f(a) = a$. Then $T(a, x)$ consists of all pairs from Δ and of the pairs $(a, f^n(x)), (f^n(x), a)$ for all non-negative integers n. This implies (ii).

(ii) $\Rightarrow$ (i). Suppose that $f(a) = b \neq a$. Let $c \in C_3(A, f)$ and consider $T(a, c)$. This tolerance contains the pair $(b, f(c))$. We have $b \neq a$. If $f(c) \neq a$, then in $G_0(T(a, c)$ there exists an edge joining b and $f(c)$ and this edge is not incident with a, hence $G_0(T(a, c))$ is not a star with the centre a. Suppose $f(c) = a$; then $a \in C_2(A, f) \cup \cup C_3(A, f)$. The tolerance $T(a, c)$ contains the pair $(f(b), f^2(c)) = (f(b), b)$. We have $b \neq a$. The equality $f(b) = a$ would imply $a = f^2(a) \in C_0(A, f)$, which would contradict $a \in C_2(A, f) \cup C_3(A, f)$; thus $f(b) \neq a$. Again the graph $G_0(T(a, c)$ contains the edge joining $f(b)$ and b, none of whose end vertices is a. Hence (ii) does not hold.

Now we shall define a certain set of tolerances on the algebra (A, f). The set $P(A, f)$ consists of all tolerances T such that $T \in LT(A, f)$ and the graph $G_0(T)$ has one of the following properties:

(a) $G_0(T)$ is a one-way infinite path;

(b) $G_0(T)$ is a finite path of the length at least 3 and one of its vertices forms a one-element cycle in (A, f);

(c) $G_0(T)$ is a finite path of the length at least 3, none of its terminal vertices forms a one-element cycle in (A, f), but one of its terminal edges has the property that there exists $T_0 \in LT(A, f)$ such that this edge is the unique edge of $G(T_0)$, while the edge adjacent to it in $G_0(T)$ has not this property.

(d) $G_0(T)$ is obtained from a finite path of the length at least 2 and a circuit by identifying a terminal vertex of the path with a vertex of the circuit.

Proposition 4. *Let* (A, f) *be a monounary algebra, let* $x \in A$, $y \in A$, *let* $C_0[x] = \emptyset$. *If* $f(x) = y$ *or* $f(y) = x$, *then* $T(x, y) \in P(A, f)$ *and the case* (a) *from the definition pf* $P(A, f)$ *occurs. Otherwise* $T(x, y) \notin P(A, f)$.

Proof. Let $f(x) = y$; then evidently $C_0[y] = C_0[x] = \emptyset$. The tolerance $T(x, y)$ consists of all pairs from Δ and of the pairs $(f^n(x), f^{n+1}(x)), (f^{n+1}(x), f^n(x))$ for all non-negative integers n. As $C_0[x] = \emptyset$, the elements $x, f(x), f^2(x), \dots$ form an infinite sequence of pairwise distinct elements and $G_0(T(x, y))$ is a one-way infinite path. Analogously if $f(y) = x$.

Now suppose that $f(x) \neq y$ and $f(y) \neq x$. If $x = y$, then $T(x, y) = \Delta \notin P(A, f)$. If $x \neq y$ and if there exists $k \geqq 2$ such that $y = f^k(x)$, then $T(x, y)$ consists of all pairs from Δ and the pairs $(f^n(x), f^{n+k}(x))$ for all non-negative integers n. As $C_0[x] = \emptyset$, we have also $C_0[f^n(x)] = \emptyset$ for every n. Thus in this case $G_0(T(x, y))$ consists of k disjoint one-way infinite paths. Analogously if $x = f^k(y)$. Thus we may suppose that x is no power of y and y is no power of x. If $C_0[y] = \emptyset$, then $G_0(T(x, y))$ consists of infinitely many pairwise disjoint paths of the length 1. If $C_0[y] \neq \emptyset$, then each power of y belonging to a cycle in (A, f) has the infinite degree in $G_0(T(x, y))$. Thus $T(x, y) \notin P(A, f)$.

Proposition 5. *Let (A, f) be a monounary algebra, let $x \in A$, $y \in A$, let $|C_0[x]| = 1$. If $x \in C_3[x]$ and $f(x) = y$ or $y \in C_3[y]$ and $f(y) = x$, then $T(x, y) \in P(A, f)$ and the case (b) from the definition of $P(A, f)$ occurs. Otherwise $T(x, y) \notin P(A, f)$.*

Proof. Let $x \in C_3[x]$ and let $f(x) = y$; then evidently $C_0[y] = C_0[x]$ and $|C_0[y]| = 1$. There exists a positive integer h such that $C_0[x] = \{f^h(x)\} = \{f^{h-1}(y)\}$; suppose that h is the least positive integer with this property. The tolerance $T(x, y)$ consists of all pairs from Δ and of the pairs $(f^n(x), f^{n+1}(x))$, $(f^{n+1}(x), f^n(x))$ for $n = 0, 1, \ldots, h - 1$, and $G_0(T(x, y))$ is a finite path. As $x \in C_3[x]$, we have $h \geqq 3$ and (b) occurs. Analogously for $y \in C_3[y]$ and $f(y) = x$. If $f(x) = y$ and $x \notin C_3[x]$ or $f(y) = x$ and $y \notin C_3[y]$, then $h \leqq 2$ and the path has the length smaller than 3.

Now suppose that $f(x) \neq y$ and $f(y) \neq x$. If $x = y$, then again $T(x, y) = \Delta \notin P(A, f)$. Thus suppose that $x \neq y$ and there exists an integer $k \geqq 2$ such that $f^k(x) = y$. Then $T(x, y)$ consists of all pairs from Δ and the pairs $(f^n(x), f^{n+k}(x))$, $(f^{n+k}(x), f^n(x))$ for all non-negative integers n. In particular, $T(x, y)$ contains the pairs $(f^{h-k+i}(x), f^h(x))$ for $i = 0, 1, \ldots, k - 1$. If $k \geqq 3$, then the vertex $f^h(x)$ has a degree at least 3 in $G_0(T(x, y))$ and this graph is not a path. If $k = 2$, then the element forming $C_0[x]$ is not a terminal vertex of $G_0(T(x, y))$ and (b) does not occur. The terminal edges of the path $G_0(T(x, y))$ are the edge joining x with $f^2(x)$ and the edge joining $f(x)$ with $f^3(x)$. The case (c) can occur only if $h = 2$ or $h = 3$. Neither (a) nor (d) can occur. In the case $h = 2$ we should not have a path of the length at least 3; thus suppose $h = 3$. Then $G_0(T(x, y))$ has the length 3 and its inner edge is adjacent to both terminal ones. But this edge joins $f^2(x)$ with $f^3(x)$; it is the unique edge of $G_0(T(f^2(x), f^3(x)))$ and (c) does not occur. Analogously for $x = f^k(y)$. The case when x is no power of y and y is no power of x is analogous to the corresponding case in the proof of Proposition 4.

Proposition 6. *Let (A, f) be a monounary algebra, let $x \in A$, $y \in A$, let $|C_0[x]| = 2$. If $x \in C_2[x] \cup C_3[x]$ and $f(x) = y$ or $f(y) = x$, then $T(x, y) \in P(A, f)$ and the case (c) from the definition of $P(A, f)$ occurs. Otherwise $T(x, y) \notin P(A, f)$.*

Proof. Let $x \in C_2[x] \cup C_3[x]$ and $f(x) = y$; then $C_0[y] = C_0[x]$ and $|C_0[y]| = 2$. There exists a positive integer h such that $C_0[x] = \{f^{h-1}(x), f^h(x)\} = \{f^{h-2}(y), f^{h-1}(y)\}$; suppose that h is the least positive integer with this property. The tolerance $T(x, y)$ consists of all pairs from Δ and of the pairs $(f^n(x), f^{n+1}(x))$, $(f^{n+1}(x), f^n(x))$ for $n = 0, 1, \ldots, h - 1$, and $G_0(T(x, y))$ is a finite path. As $x \in C_2[x] \cup C_3[x]$, we have $h \geqq 3$. The edge joining $f^{h-1}(x)$ with $f^h(x)$ has the property that it is the unique edge of the tolerance $T(f^{h-1}(x), f^h(x))$; the edge joining $f^{h-2}(x)$ with $f^{h-1}(x)$ evidently has not this property. Hence (c) occurs. Analogously for $x \in C_2[x] \cup C_3[x]$, $f(y) = x$.

If $x \notin C_2[x] \cup C_3[x]$, evidently $G_0(T(x, y))$ cannot be a path of a length at least 3 and for $G_0(T(x, y))$ the condition (d) is not satisfied. Thus we can assume that $x \in C_2[x] \cup C_3[x]$. Now suppose $f(x) \neq y$ and $f(y) \neq x$. First suppose that there exists an integer $k \geqq 2$ such that $f^k(x) = y$. Then $T(x, y)$ consists of all pairs from Δ

and of the pairs $(f^n(x), f^{n+k}(x))$, $(f^{n+k}(x), f^n(x))$ for all non-negative integers n. There are at least two vertices of degree 1 in $G_0(T(x, y))$, namely x and $f(x)$; none of them forms a one-element cycle in (A, f). The edge joining x and $f^k(x)$ evindently has not the property from the condition (c). The edge joining $f(x)$ with $f^{k+1}(x)$ can have this property only if $(f^2(x), f^{k+2}(x)) \in \Delta$, i.e. $f^{k+2}(x) = f^2(x)$. This is possible only if $f^2(x) \in C_0[x]$ and this implies $f^2(x) = f^h(x)$ or $f^2(x) = f^{h-1}(x)$. The first case is excluded by the assumption that $x \in C_2[x] \cup C_3[x]$ and $f^{h-1}(x) \in C_0[x]$. Thus let $f^2(x) = f^{h-1}(x)$; the minimality of h implies that $h = 3$. As $f^{k+2}(x) = f^2(x)$, the number k is even and $f^k(x) = f^2(x)$, $f^{k+1}(x) = f^3(x)$; the graph $G_0(T(x, y))$ consists of two connected components, each of which has only one edge. Thus (c) does not occur and evidently also none of the cases (a), (b), (d) occurs. Analogously for $x = f^k(y)$. The case when x is no power of y and y is no power of x is analogous to the corresponding case in the proof of Proposition 4.

Proposition 7. *Let (A, f) be a monounary algebra, let $x \in A$, $y \in A$, let $|C_0[x]| \geqq 3$. If $x \in C_2[x] \cup C_3[x]$ and $f(x) = y$ or $f(y) = x$, then $T(x, y) \in P(A, f)$ and the case* (d) *from the definition of $P(A, f)$ occurs. Otherwise $T(x, y) \notin P(A, f)$.*

Proof is analogous to the proofs of Propositions 4, 5, 6.
Now we are ready to prove the following theorem.

Theorem. *Let (A, f) be a monounary algebra with the property that for each $x \in A$ either $|C_0[x]| = 1$ and $C_3[x] \neq \emptyset$, or $C_2[x] \cup C_3[x] \neq \emptyset$. Let the set A and the lattice $LT(A, f)$ on A be given. Then the algebra (A, f) can be uniquely reconstructed.*

Proof. Let A and $LT(A, f)$ be given. According to Proposition 2 we are able to distinguish all tolerances belonging to $M(A, f)$. Among them we distinguish all tolerances from $P(A, f)$ with help of Propositions 4, 5, 6, 7. For each tolerance $T \in P(A, f)$ we construct its graph $G_0(T)$ and introduce an orientation on it. If the case (a) from the definition of $P(A, f)$ occurs, we direct each edge so that its initial vertex is that nearer to the initial vertex of the one-way infinite path. In the case (b) we direct each edge so that its terminal vertex will be its vertex nearer to the vertex forming a one-element cycle in (A, f). At the vertex forming a one-element cycle we add a directed loop. In the case (c) the mentioned terminal edge is replaced by two oppositely directed edges. Each other edge is directed so that its terminal vertex is its vertex nearer to this pair of edges. In the case (d) each edge not belonging to the circuit is directed so that its terminal vertex is its vertex nearer to the circuit. Now let a be the common vertex of the path and the circuit, let b be its adjacent vertex on the path and let c_1, c_2 be its adjacent vertices on the circuit. Evidently $f(b) = a$ and further, either $f(a) = c_1$, or $f(a) = c_2$. Thus we examine $T(b, c_1)$. If $f(a) = c_2$, then $f(c_1) = a = f(b)$ and $T(b, c_1) = \Delta \cup \{(b, c_1), (c_1, b)\}$. If $f(a) = c_1$, then $T(b, c_1)$ contains, moreover, also the pair $(a, f(c_1))$ which is evidently distinct from all pairs from Δ and from the pairs (b, c_1) and (c_1, b). Therefore in the first case the edge joining a with c_2 will be directed from a to c_2, in the second case it will be directed from c_2 to a. All the other edges of the circuit will be directed so that this circuit becomes a cycle.

The union H of the graphs thus obtained is evidently a subgraph of $G(A, f)$ and an edge of $G(A, f)$ with the initial vertex x and the terminal vertex y does not belong to this subgraph if and only if $f(x) = y$ and one of the following cases occurs:

(i) $y \in C_0[x]$, x is an image of no element of A in the mapping f;

(ii) $|C_0[x]| = 1$, $y \in C_1[x]$ and x is an image of no element of A in the mapping f;

(iii) $|C_0[x]| = 1$, $y \in C_0[x]$ and x is an image of no element of A in f^2.

Consider the case (i). If $|C_0[y]| = 1$, then such an edge belongs to $G(A, f)$ if and only if $T(x, y) = \Delta \cup \{(x, y), (y, x)\}$ and x does not form a one-element cycle in (A, f), while y does (this can be recognized with help of Proposition 2). If $|C_0[y]| \geqq$ $\geqq 3$, then it belongs to $G(A, f)$ if and only if $G_0(T(x, y))$ consists of a circuit which forms a cycle in H and of the edge joining the vertex x outside this circuit with the vertex y belonging to the circuit. If $|C_0[y]| = 2$, all is analogous to the preceding, only instead of speaking of a circuit we speak of an edge in $G_0(T(x, y))$ which forms a (two-edge) cycle in H. In the case (ii) such an edge belongs to $G(A, f)$ if and only if $T(x, y) = \Delta \cup \{(x, y), (y, x), (y, f(y)), (f(y), y)\}$ and $f(y)$ forms a one-element cycle, while neither x nor y does. The case (iii) is analogous to the case (i). In this way the graph $G(A, f)$ and thus also the algebra (A, f) is reconstructed. In the end we shall show two counterexamples for the cases when the conditions of Theorem are not satisfied.

Let A be an arbitrary set of cardinality at least 3. Let f be a mapping on A such that $f(x) = x$ for each $x \in A$. Let g be a mapping on A such that there exists an element $z \in A$ with the property that $g(x) = z$ for each $x \in A$. Then $LT(A, f) = LT(A, g)$ and this is the lattice of all reflexive and symmetric binary relations on A, while the algebras (A, f) and (A, g) are evidently different.

Another example is the following. Let $A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$ and let f, g be mappings of A into A such that $f(a_0) = a_1$, $f(a_1) = a_2$, $f(a_2) = a_3$, $f(a_4) = a_5$, $f(a_5) = a_1$ and $g(a_0) = a_2$, $g(a_1) = a_3$, $g(a_2) = a_4$, $g(a_3) = a_5$, $g(a_4) = a_1$, $g(a_5) = a_2$. Then again $LT(A, f) = LT(A, g)$, because the set of all powers of f is equal to the set of all powers of g.

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