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MEAN CURVATURE FUNCTIONS FOR CODIMENSION — ONE FOLIATIONS WITH ALL LEAVES COMPACT

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1. The problem and results. The 70's have brought several results ([1], [2], [4] – [6] and the others) related to the following problem: Describe the set of all differentiable functions on a given manifold M which can occur as the curvature (of some kind) of (M, g) for some Riemannian metric g on M. The problem considered here is of this type.

Let us consider a manifold M equipped with a transversely oriented codimension-one foliation F. For any Riemannian metric g on M the mean curvature of F (with respect to the chosen orientation) can be defined as the differentiable function on M which assigns to any point x of M the mean curvature at x of the leaf L_x of F passing through x. We denote by Mean (F) the set of all functions obtained in this way for all Riemannian metrics on M. Our problem consists in describing the set Mean (F). In this note, we restrict ourselves to foliations with all leaves compact. The structure of such foliations is rather simple: From the Reeb Stability Theorem ([9], see also [7]), it follows that the holonomy groups of levaes are trivial, the space of leaves M/F carries the natural structure of an one-dimensional manifold, and the canonical projection $\pi : M \to M/F$ converts M into a fibre bundle over M/F with leaves of F as the fibres. In addition, such foliations are minimizable [10], i.e. M admits a Riemannian metric g_0 for which the mean curvature function of F vanishes identically. We shall see (Section 4) that our problem in general is more difficult.

Theorem. *If all the leaves of a transversely oriented codimension-one foliation F of a manifold M are compact and*

(i) *M is compact, then $f \in$ Mean (F) if and only if either $f \equiv 0$ or there are points x_1, x_2 of M such that $f(x_1) . f(x_2) < 0$;*

(ii) *M is non-compact, then* Mean $(F) = C^\infty(M)$.

The proof of the Theorem is given in Section 3. Section 2 contains some lemmas used in the proof. Section 4 contains some examples and remarks.

Throughout the paper, everything (manifolds, functions, foliations etc.) is assumed to be differentiable of the class C^∞. Manifolds are paracompact, without boundary.

2. Useful lemmas. Let F be a codimension-one transveresely oriented foliation of a Riemannian manifold (M, g). The *mean curvature vector* H of F is a section of the normal bundle NF of F given by the formula

$$H(x) = \sum_{i=1}^{m} B(e_i, e_i) \quad (x \in M),$$

where $m = \dim F$, $e_1, \ldots, e_m$ is an orthonormal frame of T_xF and B is the second fundamental form of F; if X and Y are sections of the bundle TF, then $B(X, Y)$ is the orthogonal to F component of $\nabla_X Y$, where ∇ is the Levi-Civita connection on (M, g). If N is the positively oriented unit section of NF, then the function

$$h = g(H, N)$$

is called the *mean curvature function* of F.

Suppose that M is endowed with another Riemannian metric g' and denote by ∇', B', N', H', and h', respectively, the Levi-Civita connection on (M, g'), the second fundamental form, the positively oriented unit normal section, the mean curvature vector, and the mean curvature function of F with respect to g'. Our first goal in this section is to establish relations between H and H' (h and h', respectively) in the following three cases: when the metrics g and g' are pointwise conformal, when our metrics agree in all directions tangent to F and the normal bundles of F with respect to them are the same, and when g' is obtained from g by the pull-back via a diffeomorphism which preserves the foliation and its transverse orientation.

Lemma 1. (i) *If* $g' = e^{2\psi} . g$, *then* $H' = e^{-\psi} . H + m \nabla(e^{-\psi})$ *and* $h' = e^{-\psi} . h + mg(N, \nabla(e^{-\psi}))$, *where* $m = \dim F$ *and* $\nabla\alpha$ *denotes the gradient (with respect to* g*) of a differentiable function* α *on* M. (ii) *If* $g' \mid TF \otimes TM = g \mid TF \otimes TM$ *and* $g'(X, Y) = e^{2\psi} . g(X, Y)$ *whenever* X *and* Y *are orthogonal to* F, *then* $H' = e^{-2\psi} . H$ *and* $h' = e^{-2\psi} . h$. (iii) *If* $g' = \Phi^* g$, *where* Φ *is a diffeomorphism of* M *which preserves* F $(\Phi^* F = F)$ *and the transverse orientation of* F, *then* $H' = \Phi_*^{-1} \circ H \circ \Phi$ *and* $h' = h \circ \Phi$.

Proof. (i) The result can be obtained by simple calculation based on the following well-known formula relating the connections ∇ and ∇':

$$\nabla'_X Y = \nabla_X Y + d\psi(X) . Y + d\psi(Y) . X - g(X, Y) . \nabla\psi .$$

(ii) From the formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g(Z, [X, Y]) + g(Y, [Z, X]) + g(X, [Z, Y])$$

and the similar formula for ∇', it follows that

$$g'(\nabla'_X X, Z) = g(\nabla_X X, Z)$$

for any vector field X tangent to F and any field Z orthogonal to F. Applying the definition of the mean curvature we complete the proof.

(iii) Trivial.

Note that the mean curvature vector can be defined for foliations of arbitrary codimension and that the transverse orientation of F plays no role in this definition. The equations relating H and H' in Lemma 1 remain valid in this general situation.

Now, consider a submersion $\pi : M \to S^1$, where M is a compact manifold. Suppose that X is a nowhere-vanishing vector field on M which is transverse to the fibres of π. Denote by (φ_t) the one-parameter group of diffeomorphisms of M generated by X. Choose a point θ_0 of S^1 and let $L = \pi^{-1}(\theta_0)$. For any point x of L there exists the smallest positive number t such that $\varphi_t(x) \in L$. Denote it by $t(x)$.

Lemma 2. *Suppose that f is a differentiable function on M such that for any point x of L there are numbers $t_1, t_2 \in (0; t(x))$ for which the inequality*

$$f(\varphi_{t_1}(x)) \,.\, f(\varphi_{t_2}(x)) < 0$$

holds. Then there exists a positive differentiable function k on M such that

(a) $\operatorname{supp}(1 - k) \subset M - L$,

(b) $\int_0^{t(x)} f(\varphi_t(x))\, k(\varphi_t(x))\, \mathrm{d}t = 0$ *for every x of L.*

Proof. Put $\alpha(x, t) = f(\varphi_{t.t(x)}(x))$ for $x \in L$, $t \in [0; 1]$. α is a differentiable function on $L \times [0; 1]$. For any point x of L there are numbers $t_1, t_2 \in (0; 1)$ such that

$$\alpha(x, t_1) \,.\, \alpha(x, t_2) < 0\,.$$

Using the partition of unity on L and coming back to M via the mapping $L \times [0; 1] \ni (x, t) \mapsto \varphi_{t.t(x)}(x)$ it is easy to see that the proof of the lemma reduces to the following: Take a point x_0 of L and find a positive differentiable function β on $U \times [0; 1]$, where U is an open neighbourhood of x_0 on L, which satisfies the conditions:

(a') There exists a number $\varepsilon > 0$ such that $\beta(x, t) = 1$ for any x of U and t of $[0; \varepsilon] \cup [1 - \varepsilon; 1]$.

(b') $\int_0^1 \alpha(x, t)\, \beta(x, t)\, \mathrm{d}t = 0$ for every point x of U.

The function β can be constructed as follows. Let $x_0 \in L$, $t_1, t_2 \in (0; 1)$, $\alpha(x_0, t_1) < 0$ and $\alpha(x_0, t_2) > 0$. Taking a small open interval $I \subset \operatorname{cl} I \subset (0; 1)$ around t_2 and a small open neighbourhood V of x_0 we can find a positive constant c such that

$$\int_0^1 \alpha(x, t)\, \gamma_0(x, t)\, \mathrm{d}t \geqq 1 \quad (x \in V)\,,$$

where $\gamma_0(x, t) = c$ when $t \in I$ and $\gamma_0(x, t) = 1$ otherwise. A sufficiently close approximation of γ_0 by differentiable functions yields a positive differentiable function γ on $V \times [0; 1]$ such that

$$\int_0^1 \alpha(x, t)\, \gamma(x, t)\, \mathrm{d}t \geqq \tfrac{1}{2} \quad (x \in V)\,.$$

Put $\alpha_1 = \alpha\,.\,\gamma$. Taking a small open interval $J = (t_1 - \delta;\ t_1 + \delta) \subset [t_1 - \delta;\ t_1 + \delta] \subset (0; 1)$ and a small open neighbourhood $U \subset V$ of x_0 we can find a positive constant C such that

$$\int_{[0;1]-J} \alpha_1(x, t)\, \mathrm{d}t + C\,.\int_J \alpha_1(x, t)\, \mathrm{d}t < 0 \quad (x \in U)\,.$$

For any s of $(0; \delta)$ put

$$\mu_s = 1 + (C - 1)(1 - \eta_s)(1 - \nu_s)\,,$$

where

$$\eta_s(u) = \int_{t_1+s}^{u} a_s(t)\, \mathrm{d}t \Big/ \int_{t_1+s}^{t_1+\delta} a_s(t)\, \mathrm{d}t \quad (u \in [0; 1])\,,$$

$$\nu_s(u) = \int_{u}^{t_1-s} b_s(t)\, \mathrm{d}t \Big/ \int_{t_1-\delta}^{t_1-s} b_s(t)\, \mathrm{d}t \quad (u \in [0; 1])\,,$$

$$a_s(t) = \begin{cases} \exp\left(\dfrac{1}{t - (t_1 + \delta)} - \dfrac{1}{t - (t_1 + s)}\right), & \text{when} \quad t \in (t_1 + s;\ t_1 + \delta)\,, \\ 0\,, \quad \text{otherwise}\,, \end{cases}$$

and

$$b_s(t) = \begin{cases} \exp\left(\dfrac{1}{t - (t_1 - s)} - \dfrac{1}{t - (t_1 - \delta)}\right), & \text{when} \quad t \in (t_1 - \delta;\ t_1 - s)\,, \\ 0\,, \quad \text{otherwise}\,. \end{cases}$$

The function μ_s depends smoothly on s; its graph is sketched in Figure 1.

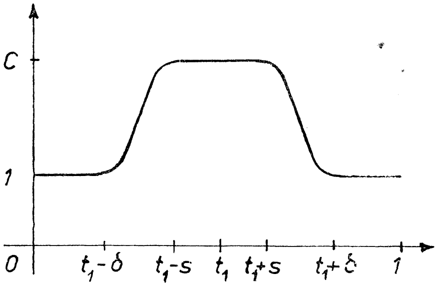

Figure 1.

If $x \in U$, then the function

$$s \mapsto K_x(s) = \int_0^1 \mu_s(t)\, \alpha_1(x, t)\, \mathrm{d}t$$

is continuous, decreasis, $\lim_{s\to 0^+} K_x(s) > 0$, and $\lim_{s\to\delta^-} K_x(s) < 0$. This implies the existence of the unique number $s(x)$ such that $0 < s(x) < \delta$ and $K_x(s(x)) = 0$. Applying the Implicit Function Theorem one can prove that the function $x \mapsto s(x)$ is differentiable. The function β defined by

$$\beta(x, t) = \mu_{s(x)}(t) \cdot \gamma(x, t)$$

satisfies conditions (a') and (b').

Lemma 3. *If D is a differentiable m-cell on a compact m-dimensional manifold L, then for any open intervals $J \subset \operatorname{cl} J \subset I$ there exists a differentiable one-parameter family $(\Lambda_t;\ t \in I)$ of diffeomorphisms of L such that* (i) $\bigcup_{t\in I} \Lambda_t(D) = L$, (ii) $\Lambda_t = \mathrm{id}$ *for every t of $I - J$.*

Proof. There exist differentiable m-cells $C_1, \ldots, C_r$ covering L. Let $J_1, \ldots, J_r$ be closed intervals such that $J_i \subset J$ and $J_i \cap J_k = \emptyset$ for $i, k = 1, \ldots, r$, $i \neq k$. According to [8], we can find differentiable one-parameter families $(\Lambda_t^i;\ t \in I)$ of diffeomorphisms of L such that $\Lambda_t^i = \mathrm{id}$ whenever $t \notin J_i$ and $\Lambda_{t_i}^i(D) = C_i$ for some t_i of J_i. Putting $\Lambda_t = \Lambda_t^i$ when $t \in J_i$ and $\Lambda_t = \mathrm{id}$ when $t \in I - \bigcup_{i=1}^{r} J_i$ we define the family (Λ_t) with the desired properties.

3. Proof of the Theorem. We will start with the proof of part (i), i.e. we suppose M to be compact. In this case, the foliation F consists of fibres of a locally trivial fibre bundle $\pi : M \to S^1$. The relation $0 \in \operatorname{Mean}(F)$ follows from the mentioned in Section 1 Rummler's result [10]. Let us choose a Riemannian metric g_0 on M for which all the leaves of F are minimal submanifolds of (M, g_0), and denote by N the positively oriented orthogonal to F unit vector field on M.

Let us take a function $f \in C^\infty(M)$ such that $f(x_1) > 0$ and $f(x_2) < 0$ for some points x_1 and x_2 of M, and fix a leaf L of F. Applying Lemma 3 we can show the existence of a diffeomorphism Φ of M which preserves F (i.e., $\pi \circ \Phi = \pi$) and its transverse orientation, and satisfies the following condition: For any segment γ of a trajectory of N which has its endpoints on L there are two points y_1 and y_2 of γ such that $(f \circ \Phi^{-1})(y_1) > 0$ and $(f \circ \Phi^{-1})(y_2) < 0$. In fact, we can assume that $x_1, x_2 \notin L$ and $\pi(x_1) \neq \pi(x_2)$, and multiply N by a positive factor to get a vector field X on M with $\pi_* \circ X = (d/d\theta) \circ \pi$, θ being the standard parameter for S^1. (Note, that trajectories of N and X are the same.) Then we can find tubular neighbourhoods U_k $(k = 1, 2)$ of the leaves L_k passing through x_k such that (i) $U_k = \pi^{-1}(A_k)$ for some open arc $A_k \subset S^1$, (ii) $U_1 \cap U_2 = \emptyset$ and $L \cap U_k = \emptyset$, and (iii) the mappings $\Psi_k : L_k \times (-\varepsilon_k; \varepsilon_k) \ni (x, t) \mapsto \psi_t(x)$, where (ψ_t) is the flow generated by X on M, map $L_k \times (-\varepsilon_k; \varepsilon_k)$ onto U_k diffeomorphically. We can also find smooth m-cells $(m = \dim F) D_k$ on L_k such that $f \mid D_1 > 0$ and $f \mid D_2 < 0$. Let us take differentiable families $(\Lambda_{k,t};\ |t| < \varepsilon_k)$ of diffeomorphisms satisfying the conditions of Lemma 3 with D_k, L_k and $(-\varepsilon_k; \varepsilon_k)$ in place of D, L and I, respectively. Put $\Phi(x) = \Psi_k(t, \Lambda_{k,t}(z))$

when $x \in U_k$ and $\Psi_k^{-1}(x) = (z, t)$, and $\Phi(x) = x$ when $x \in M - (U_1 \cup U_2)$. It is easy to verify that Φ is a diffeomorphism of M which has the required property.

Put $f_1 = f \circ \Phi^{-1}$. Lemma 2 asserts the existence of a positive function $k \in C^\infty(M)$ such that the integral of $f_1 \, . \, k$ over any segment of a trajectory of N with endpoints on L vanishes.

Put $f_2 = f_1 \, . \, k$. The formula

$$\psi(\varphi_t(x)) = -\log\left(C + \int_0^t f_2(\varphi_s(x)) \, \mathrm{d}s\right) \quad (x \in L),$$

where (φ_t) is the flow generated by N and C is a sufficiently large positive constant, defines properly a differentiable function ψ on M. From Lemma 1 (i), it follows that the mean curvature of F with respect to the Riemannian metric $\mathrm{e}^{2\psi} \, . \, g_0$ equals f_2. Parts (ii) and (iii) of Lemma 1 show how to modify this metric to obtain a metric g with respect to which the mean curvature of F equals f.

In order to complete our proof, we have to show that the mean curvature function of F with respect to an arbitrary Riemannian metric cannot be either non-negative or non-positive unless it vanishes identically. To this end, let us recall variational properties of the mean curvature [11].

Let L be an arbitrary submanifold of a Riemannian manifold (N, g) and $\lambda = (\lambda_t; |t| < \varepsilon)$ be a differentiable one-parameter family of immersions of L into M such that $\lambda_0 = \iota_L =$ the inclusion map of L. λ is called a variation of L. The formula

$$V(x) = \text{the normal component of} \quad (t \mapsto \lambda_t(x))^{\cdot}\,(0)$$

defines a differentiable section of the normal bundle NL of L. If L is compact and $v(t)$ denotes the volume of L with respect to the Riemannian metric $\lambda_t^* g$, then v is a differentiable function and

$$(*) \qquad v'(0) = -\int_L g(V, H)\, \omega_0 \,,$$

where H is the mean curvature vector of L and ω_0 — the volume form on L determined by the metric $\lambda_0^* g$. If $\operatorname{codim} L = 1$ and N is a unit section of NL, then $V = \alpha N$ for some differentiable function α on L and the formula $(*)$ can be expressed in the form

$$(**) \qquad v'(0) = -\int_L \alpha \, . \, h\omega_0 \,,$$

where h is the mean curvature function of L.

Coming back to our situation let us take a Riemannian metric g on M and define $v(\theta)$ as the volume of the fibre L_θ of π over $\theta (\theta \in S^1)$ with respect to the metric g_θ induced from g. The function $S^1 \ni \theta \mapsto v(\theta)$ is differentiable and, according to $(**)$, its derivative is given by

$$v'(\theta) = -\int_{L_\theta} \alpha \, . \, h \, . \, \omega_\theta \,,$$

where α is a non-vanishing (say, positive) differentiable function on M, h – the mean curvature function of F, and ω_θ – the volume form determined by g_θ. If the function $\theta \mapsto v(\theta)$ is constant, then $v'(\theta) = 0$ for every θ. If $v'(\theta) = 0$, then either $h \equiv 0$ on L_θ or there are two points x_1, x_2 of L_θ such that $h(x_1) > 0$ and $h(x_2) < 0$. If the function $\theta \mapsto v(\theta)$ is not constant, then there exist points θ_1, θ_2 of S^1 such that $v'(\theta_1) > 0$ and $v'(\theta_2) < 0$. The inequality $v'(\theta_1) > 0$ implies the existence of a point x_1 of L_{θ_1} such that $h(x_1) < 0$. Similarly, if $v'(\theta_2) < 0$, then there exists a point x_2 of L_{θ_2} such that $h(x_2) > 0$. This completes the proof of part (i).

The proof of part (ii) is simpler. If M is non-compact and all the leaves of F are compact, then we do not loose generality assuming that $M = L \times R$ and $F = \{L \times \{t\};\ t \in R\}$, where L is a compact manifold. In this case, we can start with the product metric $g_0 = g_L + \mathrm{d}t^2$, where g_L is an arbitrary Riemannian metric on L and $\mathrm{d}t^2$ is the standard metric on R. The mean curvature of F with respect to g_0 is equal to 0 since the leaves of F are totally geodesic (consequently, minimal) in (M, g_0). From Lemma 1 (i), it follows that the mean curvature of F with respect to the Riemannian metric $\mathrm{e}^{2\psi} . g_0 (\psi \in C^\infty(M))$ equals

$$m \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\psi},$$

where $m = \dim L$. Lemma 1 (ii) shows that the problem reduces to the following: Prove that for any function $f \in C^\infty(L \times R)$ there exists a positive function $k \in C^\infty(L \times R)$ such that the equation

$$(***) \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = f . k$$

possesses a positive solution $\varphi \in C^\infty(L \times R)$. Solutions of $(***)$ are given by

$$\varphi(x, t) = C(x) + \int_0^t f(x, s)\, k(x, s)\, \mathrm{d}s,$$

where $C \in C^\infty(L)$, and can be made positive if the integrals

$$\int_{-\infty}^{\infty} |f(x, s) . k(x, s)|\, \mathrm{d}s \quad (x \in L)$$

are bounded by a constant. The existence of a suitable factor k is evident now.

4. Examples and remarks. We intend to show that the situation is quite different when leaves of the foliation under consideration are not necessarily compact. At first, we will consider a transversely oriented one-dimensional foliation F of a torus T which contains a Reeb component C (Figure 2). Such a foliation is not geodesible [3], i.e. there are no Riemannian metrics on T with respect to which leaves of F are geodesics. In our language, $0 \notin \operatorname{Mean} F$. Let us consider an arbitrary Riemannian metric on T. If N is the positive oriented unit vector field on T which is orthogonal to F, then, according to Poincaré-Bendixon Theorem, there exists a closed limit

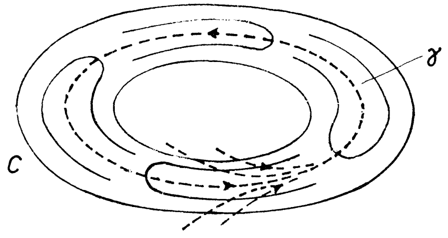

Figure 2.

trajectory γ of N contained in C. Let us choose a point $x_0 = \gamma(t_0)$ and a compact connected neighbourhood V of x_0 on the leaf L_0 of F passing through x_0. Let us apply formula $(**)$ (actually, its generalization to the case when L has a boundary [11] which reduces to $(**)$ in our situation) to the variation $(\lambda_t;\ t \in R)$ of V determined by the following conditions:

(i) $\lambda_t(V) \subset L_t$, where L_t denotes the leaf of F passing through $\gamma(t_0 + t)$,

(ii) $\lambda_t(x_0) = \gamma(t_0 + t)$ and $\lambda_t(x)$ lies on the trajectory of N passing through x $(x \in V)$.

The function $t \mapsto v(t)$, where $v(t)$ denotes the volume of V with respect to the Riemannian metric $\lambda_t^* g$, has the following property: If $\gamma(s_0 + t_0) = \gamma(t_0)$ $(s_0 > 0)$, then $v(s_0 + t) \leqq v(t)$ $(t \in R)$. From $(**)$, it follows that the mean curvature function of F cannot be negative everywhere on C. Consequently, there are differentiable functions on T which are somewhere negative and somewhere else positive and which do not belong to Mean (F). The same can be said about transversely oriented

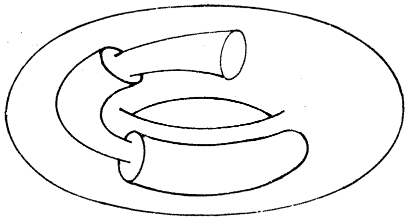

Figure 3.

2-dimensional foliations of 3-dimensional manifolds: If such a foliation F contains a Reeb component (Figure 3), then $0 \notin$ Mean (F) and there are differentiable functions f such that $f^{-1}((0; +\infty)) \neq \emptyset$, $f^{-1}((-\infty; 0)) \neq \emptyset$, and $f \notin$ Mean (F).

One dimensional foliations of a torus T which contain no Reeb components are geodesible [3]: If F is such a foliation and F has no closed leaves, then F is differentiably equivalent to the foliation determined by an irrational flow; if F has a closed leaf, then F is equivalent to the foliation F_0 described below: Let X and Y be the vector fields on $T = R^2/Z^2$ obtained from the vector fields $\partial/\partial x$ and $\partial/\partial y$ on R^2, where x and y denote the standard Euclidean coordinates on R^2, via the canonical projection $R^2 \to T$. Put $Z = Y + \alpha X$, where α is a differentiable function on T. F_0 is the foliation determined by Z (Figure 4). Leaves of F_0 are geodesics with respect to the Riemannian metric g_0 on T defined by

$$g_0(X, X) = g_0(Z, Z) = 1\,, \quad g_0(X, Z) = 0\,.$$

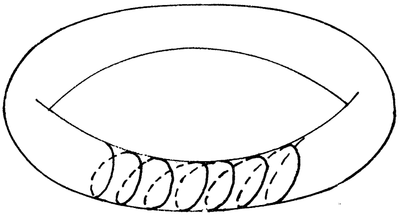

Figure 4.

If $f \in C^\infty(T)$, $f^{-1}((0; +\infty)) \neq \emptyset$ and $f^{-1}((-\infty; 0)) \neq \emptyset$, then starting with the metric g_0 described above and repeating (with slight modifications) the first part of the proof of Theorem (i) one can construct a Riemannian metric g on T such that the mean curvature function of F_0 with respect to g equals f. Therefore, $0 \in \operatorname{Mean}(F_0)$ and the class $\operatorname{Mean}(F_0)$ contains all differentiable functions on T which are somewhere negative and somewhere else positive.

Finally, let us note that a compact codimension-one submanifold L of a manifold M can be considered as a leaf of a transversely oriented foliation of an open neighbourhood $U \subset M$ of L if only the normal bundle of L is trivial. Applying this fact and our Theorem we get the following result:

If the normal bundle of a compact codimension-one submanifold L of a manifold M is trivial, then for any function $f \in C^\infty(L)$ there exists a Riemannian metric g on M such that the mean curvature of L with respect to g equals f.

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