SOME REMARKS ON TOPOLOGICALLY SEMIPRIME IDEALS

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Let S be a topological semigroup. (An algebraic semigroup is a topological semigroup with discrete topology). An ideal I of S is called *topologically semiprime* if whenever $x \notin I$, then $\Gamma(x) \cap I = \emptyset$, where $\Gamma(x)$ is the compact monothetic semigroup generated by $x \in S$, that is, $\Gamma(x) = \overline{\{x^n\}_{n=1}^{\infty}}$. An ideal Q of S is called a *group-ideal* if whenever $x \notin Q$, then $J(x)$, the principal ideal generated by x, contains a group G which misses Q. In this note, we shall study the class of topologically semiprime ideals and their relationship with group-ideals. It is shown that topologically semiprime ideals are topological generalizations of semiprime ideals and completely semiprime ideals. In particular, an ideal of a compact semigroup is topologically semiprime if and only if it is an intersection of completely open prime ideals. This result will bring together and generalize many results in the literature concerning prime ideals and their intersections.

For the definitions of prime ideals, semiprime ideals, completely prime ideals, completely semiprime ideals and other terminology used in this paper, the reader is referred to A. H. Clifford and G. B. Preston [1]. Unless otherwise stated, the word "ideal" means two sided ideal of S.

The following theorem characterizes the group-ideals in a compact semigroup.

Theorem 1. *An ideal I of a compact semigroup S is a group ideal if and only if it can be expressed as an intersection of open prime ideals of S.*

Proof. We only need to show that every group-ideal of S can be expressed as an intersection of open prime ideals of S, for it has been already shown in [10] that every intersection of open prime ideals of S is a group-ideal. Let I be a group-ideal of S. By the definition of group-ideal, there exists $e^2 = e \in S - I$. Thus $I \subset J_0(S - e)$, which is known to be an open prime ideal of S. [2]. Thus I is contained in some open prime ideals of S. Let I_1 be the intersection of all open prime ideals of S containing I. Thus $I \subset I_1$. If $I \subsetneqq I_1$, then we can pick $x \in I_1 - I$. Because I is a group-ideal, there exists $f^2 = f \in J(x) - I$. But $J(x) \subset I_1$, so $f \in I_1 - I$. Thus $I \subset \subset J_0(S - f)$, which is again an open prime of S containing I. Hence, $J_0(S - f) \supset I_1$. This is a contradiction as $f \in I_1$, $J_0(S - f) \not\supset I_1$. So $I = I_1$ as desired.

Remark 1. In [3], K. Numakura called an ideal Q of S be a *q-ideal* provided that Q can be expressed as an intersection of open prime ideals. He also defined an ideal of S to have the property ξ if every ideal of S which is not contained in I has an idempotent e such that $e \notin I$. In fact, it can be easily shown that an ideal I of a compact semigroup S has property ξ if and only if it is a group-ideal. Thus, in a compact semigroup, an ideal I is a group-ideal if and only if it is a q-ideal.

Remark 2. As topologically semiprime ideals in a compact semigroup are group ideals, so by theorem 1, all topologically semiprime ideals in a compact semigroup are q-ideals. Since open prime ideals, open completely prime ideals, open semiprime ideals, open completely prime ideals, algebraic radicals, topological radicals and nil radicals in a compact semigroup are all topologically semiprime, thus most of the results in [3] can be grouped together by introducing the term "topologically semiprime", e.g. theorem 2.2 and theorem 2.7 in [3], and also theorem 1 in [2].

Remark 3. Group-ideals need not be topologically semiprime; topologically semiprime ideals need not be open semiprime. In a compact semigroup, an ideal I is topologically semiprime $\Rightarrow I$ is a group-ideal $\Leftrightarrow I$ has property $\xi \Leftrightarrow I$ is a q-ideal.
q-ideals need not be topologically semiprime. Thus, the classes of open semiprime ideals, open completely semiprime ideals and their intersections cannot be characterized by q-ideals. However, topologically semiprime ideals can be characterized by the intersection of open completely prime ideals.

Theorem 2. *Let I be an ideal of a compact semigroup. Then I is topologically semiprime if and only if it is an intersection of open completely prime ideals.*

Proof. It was shown in the book of M. Petrich [5, p. 36] that every prime ideal p of an algebraic semigroup which is minimal relative to containing a semiprime ideal I is completely prime. Invoking his result, we can show that every open prime ideal P of a compact semigroup which is minimal relative to containing a topologically semiprime ideal I is completely prime. Also, it is apparant to see that Numakura's Theorem 3.11 [3] holds with Q open proper semiprime replaced by Q open proper q-ideal. Thus, by a simple juxtaposition of the results of Petrich and Numakura, theorem 2 is established.

Remark 4. Theorem 2 is, in fact, a topological analogue of the well known result of K. Iséki and Š. Schwarz in the literature concerning the characterization of completely semiprime ideals in algebraic semigroup, see A. H. Clifford - G. B. Preston [1, p. 126].

Remark 5. Theorem 2 gives an affirmative answer to a problem raised by Shum-Stawart in [9]: Is it true that every topologically semiprime ideal in a compact semigroup can be expressed as the intersection of open completely prime ideals?

Remark 6. The concepts of topological n-semigroup is introduced by F. Sioson in [12]. It should be noted that the result in theorem 1 and theorem 2 can be extended to topological n-semigroups.

Remark 7. The following statements are topological generalizations of semiprime and completely semiprime ideals:

(i) (Shum, [11]). Let Q be a proper topologically semiprime ideal of a compact semigroup S. Then Q is the intersection of all Q-divisors $(Q : e)$, where e runs through the set of all Q-primitive idempotents.

(ii) (Shum, [10]). If S is a compact affine semigroup, then every topologically semiprime ideal is convex and dense in S.

(iii) (Schwarz, [7]). Let S be a compact semigroup. If $\Phi(S)$, the intersection of all maximal ideals of S, is topologically semiprime, then S-$\Phi(S)$ is a disjoint union of groups.

(iv) (Numakura, [4]). If S is a compact and totally disconnected semigroup. Then every topologically semiprime ideal Q of S is closed if and only if $E \cap Q$ is closed, where E is the set of all idempotents of S.

In closing, we would like to point out that there are many other results in the literature concerning the intersection of prime ideals, maximal ideals and semiprime ideals in algebraic semigroups and rings which can be likewise generalized to topological semigroups and rings by introducing the concept of topologically semiprime ideals.

References

[1] *A. H. Clifford* and *G. B. Preston:* The algebraic theory of semigroups, vol. I, Amer. Math. Soc., Providence, R.I. (1961).

[2] *K. Numakura:* Prime ideals and idempotents in compact semigroups, Duke Math. J. *24* (1957), p. 671—680.

[3] *K. Numakura:* On q-ideals in compact semigroups, Czechoslovak Math. J. *24* (*103*), (1978), p. 312—323.

[4] *K. Numakura:* Closedness of q-ideals in a compact and totally disconnected semigroups, Proc. Japan Acad., *54*, (1978), p. 239—242.

[5] *M. Petrich:* Introduction to semigroups, Merrill Research and Lecture Notes, A. Bell & Howell Co. (1973).

[6] *Š. Schwarz:* Prime ideals and maximal ideals in Semigroups, Czechoslovak Math. J. *19* (1969), p. 72—79.

[7] *Š. Schwarz:* Semigroups containing Maximal ideals, Math. Slovaca *28* (1978), p. 157—168.

[8] *K. P. Shum* & *C. S. Hoo:* On compact N-semigroups, Czechoslovak Math. J. *24* (*99*), (1974), p. 552—562.

[9] *K. P. Shum* & *P. N. Stewart:* Completely prime ideals and idempotents in mobs, Czechoslovak Math. J. *26* (*101*), (1976), p. 211—217.

[10] *K. P. Shum:* Group ideals in a semigroup of measures, Semigroup Forum *22* (1981), p. 325—329.

[11] *K. P. Shum:* On compressed ideals in topological semigroups, Czechoslovak Math. J. *25* (*100*), (1975), p. 261—273.
[12] *F. Sioson:* Ideals in $(m + 1)$-semigroups, Annali di Math. Pura et appl., *68*, (1965), p. 161—200.

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