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REPRESENTATIONS OF GROUP AND SEMIGROUP ACTIONS

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An action of a semigroup S on a set X is a mapping from $S \times X$ to X, denoted by juxtaposition $((s, x) \to sx)$, such that $t(sx) = (ts)\,x$ for all s, t in S and all x in X. A familiar example of a semigroup acting on a set is given by the multiplicative action of the semigroup of all $n \times n$ matrices over an associative ring on the set of all $n \times 1$ matrices over the ring. Since we do not require that S has an identity element, even when S does have an identity element 1, we do not assume that $1x = x$ for x in X.

A representation of a semigroup S is a homomorphism from S to the semigroup of $n \times n$ matrices over a field. This usual notion can be extended to define a representation of a semigroup action: a representation of a semigroup action $S \times X \to X$ is a pair of mappings, a homomorphism f mapping S to the semigroup F_n of all $n \times n$ matrices over a field F and a set mapping g taking X to the n-dimensional column vectors F^n over F, such that the following diagram commutes:

$$\begin{array}{ccccc} S & \times & X & \to & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ F_n & \times & F^n & \to & F^n\,. \end{array}$$

The commutativity condition can also be written $(f(s))\,(g(x)) = g(sx)$.

Given any representation $f : S \to F_n$ of the semigroup S, and given any action $S \times X \to X$, one representation of the action is given by f together with the zero map on X (the map which takes each x in X to the zero vector in F^n). More interesting representations preserve something of the structure of X. We say that a representation of a semigroup action is S-faithful if the semigroup map is one-to-one; it is X-faithful if the set map is one-to-one; and it is faithful if both maps are one-to-one. Representations of semigroup actions arise naturally from the study of ordinary semigroup representations (see [4]).

This paper considers way of constructing representations of semigroup actions, especially various types of faithful representations.

In a representation of a semigroup action the vector space involved is finite dimensional. The finite dimensionality of F^n places some restrictions on those actions which can have X-faithful representations.

Theorem 1. *If $S \times X \to X$ is a semigroup action with an X-faithful representation in $F_n \times F^n \to F^n$, then whenever there exist elements s_i, t_i in S, x_i, y_i in X, $i = 1, \ldots, m$, with $s_i x_j = t_i y_j$ $(i < j)$, $s_i x_i \neq t_i y_i$, then $m \leqq n^2$.*

Proof. Using the X-faithful representation, we treat the elements of X as vectors in F^n. And since $sx \in X$ for s in S, x in X, we treat elements sx as vectors also.

Suppose $m > n^2$. Then $(x_1, y_1), \ldots, (x_m, y_m)$ is linearly dependent in $F^n \times F^n$, so there are coefficients $a_1, \ldots, a_m$, not all zero, with $\sum a_j(x_j, y_j) = (0, 0)$. Let i be the index of the first-zero coefficient. Then $a_i x_i = -\sum_{j>i} a_j x_j$, $a_i y_i = -\sum_{j>i} a_j y_j$, and $a_i s_i x_i = s_i(a_i x_i) = s_i(-\sum a_j x_j) = -\sum a_j s_i x_j = -\sum a_j t_i y_j = t_i(-\sum a_j y_j)$ $t_i(a_i y_i) = a_i t_i y_i$. Since $a_i \neq 0$, $s_i x_i = t_i y_i$, a contradiction. Thus $m \leqq n^2$.

There are several consequences of this theorem. Some of them can be proved more directly, and in some cases a direct proof gives a better bound. But the fact that they are consequences of the previous theorem shows its importance.

Corollary 1.1. *If $S \times X \to X$ has an X-faithful representation, then there is a number k so that for all a in S and all x, y in X, if $a^k x \neq a^k y$, then $a^m x \neq a^m y$ for all m.*

Proof. Theorem 1 allows us to verify this result with $k = n^2 + 1$. Suppose $a^{n^2+1}x \neq a^{n^2+1}y$, but $a^m x = a^m y$ for some $m > n^2 + 1$. Assume m is minimal, and in theorem 1 take $x_i = a^{i-1}x$, $y_i = a^{i-1}y$, $s_i = t_i = a^{m-i}$, $i = 1, \ldots, m - 1$. These sequences satisfy the assumptions of theorem 1, yet as $m - 1 > n^2$ we get a contradiction. Thus the corollary is established for $k = n^2 + 1$.

(A direct proof could be given with $k = n$, but it is significant that a bound can be obtained here as a consequence of theorem 1).

If $x \in X$, let $\mathrm{fix}(x) = \{s \in S \mid sx = x\}$ be the fixer in S of x (see [1], p. 54). For a set $Y \subseteq X$, let $\mathrm{fix}(Y) = \{s \in S \mid sy = y$ for all y in $Y\}$ be the fixer of Y. Theorem 1 then gives rise to the following

Corollary 1.2. *If $S \times X \to X$ has an X-faithful representation, then any chain of fixers $\ldots \subsetneqq \mathrm{fix}(z_{-1}) \subsetneqq \mathrm{fix}(z_0) \subsetneqq \mathrm{fix}(z_1) \subsetneqq \mathrm{fix}(z_2) \subsetneqq \ldots$ is finite and of bounded length. Thus S has both acc and dcc on fixers.*

Proof. Suppose the index k appears in the above chain. In theorem 1, let $x_i = y_i = z_{k-i}$, and let $s_i \in \mathrm{fix}(z_{k-i+1}) - \mathrm{fix}(z_{k-i})$, $t_i \in \mathrm{fix}(z_{k-i})$. Theorem 1 then says that the indices do not go below $k - n^2 - 1$. Thus there can be at most $n^2 + 1$ fixers in the chain.

Since $\text{fix}(\{z_a \mid a \in A\}) = \bigcap_{a\in A} \text{fix}(z_a)$, the theorem also gives us

Corollary 1.3. Suppose $S \times X \to X$ has an X-faithful representation.

Then for any set $\{z_a \mid a \in A\} \subseteq X$, *there is a finite subset* $z_1, \ldots, z_k$ *such that* $\text{fix}(\{z_a\}) = \text{fix}(\{z_1, \ldots, z_k\})$, *and* $k \leqq n^2 + 2$.

Proof. If the corollary is not true, we get an infinite chain $\text{fix}(z_1) \supsetneqq \text{fix}(\{z_1, z_2\}) \supsetneqq \supsetneqq \text{fix}(\{z_1, z_2, z_3\}) \supsetneqq \ldots$. Taking $x_i = y_i = z_{n^2-i+2}$, $s_i \in \text{fix}(\{z_1, \ldots, z_{n^2-i+1}\}) - - \text{fix}(\{z_1, \ldots, z_{n^2-i+2}\})$ $t_i \in \text{fix}(\{z_1, \ldots, z_{n^2-i+2}\})$, we get a contradiction to theorem 1. Thus the only such chains are finite, and we must be able to write $\text{fix}(\{z_a\})$ as the fixer of a finite set $z_1, \ldots, z_k$.

In actual constructions of representations the following two lemmas are quite useful.

Lemma 2. *If $S \times X \to X$ has an X-faithful representation and $S \times Y \to Y$ has a Y-faithful representation, then $S \times (X \cup Y) \to (X \cup Y)$ has an $(X \cup Y)$-faithful representation. If one of the original representations was S-faithful, so is the new one.*

Proof. Let $s \to M_s$, $x \to v_x$ and $s \to N_s$, $y \to w_y$ be the representations. The desired representation of $S \times (X \cup Y) \to (X \cup Y)$ is then $s \to \begin{pmatrix} M_s & 0 \\ 0 & N_s \end{pmatrix}$, $x \to \begin{pmatrix} v_x \\ 0 \end{pmatrix}$, $y \to \begin{pmatrix} 0 \\ w_y \end{pmatrix}$

Corollary 2.1. *If $S \times X \to X$ has an X-faithful representation and S has a faithful representation, then $S \times X \to X$ has a faithful representation.*

Proof. The construction is similar to that of lemma 2.

Lemma 3. *If $S \times X \to X$ has an X-faithful representation and $T \times Y \to Y$ has a Y-faithful representation, then $(S \times T) \times (X \times Y) \to (X \times Y)$ has an $(X \times Y)$-faithful representation.*

Proof. The construction parallels that of the previous lemma.

If we are interested in constructing X-faithful representations, then by theorem 1 we need only consider actions where all sequences s_i, t_i, x_i, y_i of the sort described in the theorem have bounded length. Two situations where such sequences will always have bounded length occur when S is finite and when X is finite. To see that this is true when S is finite, note that $S \times S$ is finite, so in any infinite sequences some pair (s_k, t_k) must repeat some previous pair (s_m, t_m). But $s_m x_k = t_m y_k$, so $s_k x_k = = t_k y_k$, and the sequences are not of the desired type. A similar argument works if X is finite. In case X is finite, it is easy to get X-faithful representations:

Theorem 4. *If X is finite, any action $S \times X \to X$ has an X-faithful representation.*

Proof. Let $X = \{x_1, \ldots, x_k\}$ and consider the map $x_i \to e_i$ taking X to the standard basis of F^k. For s in S, let N_s be the matrix with i, j entry equal to 1 if $sx_i = x_j$, 0 otherwise. It is easily verified that the map $s \to N_s$ gives a (not necessarily faithful) representation of S, which with the given map from X to F^k gives an X-faithful representation of $S \times X \to X$.

Corollary 4.1. *If S has a faithful representation and X is finite, any action $S \times \times X \to X$ has a faithful representation.*

The case where S is in finite is not so easily dealt with. The next section deals with the nicest possible case, when S is a group.

2. GROUP ACTIONS

Since groups do have an identity, and the usual definitions of and standard results on group actions assume that $1x = x$ for all x in X, our first result is concerned with allowing us to make this assumption. If we have any group action $G \times X \to X$, it is easy to verify that on the subset GX of X 1 does act as the identity transformation. We are able to make this assumption generally because of the following

Theorem 5. *Let $G \times X \to X$ be any action of a group on a set. Then $G \times X \to X$ has an X-faithful representation if and only $G \times GX \to GX$ has a GX-faithful representation.*

Proof. only if: clear.

if: Assume that $g \to N_g$, $x \to v_x$ is a GX-faithful representation of $G \times GX \to GX$. Assume the field F has cardinality greater than that of X, and let $x \to a_x$ be a set injection of $X - GX$ into $F - \{0\}$. If $x \in GX$, let a_x be 0. Then the maps

$$g \to \begin{pmatrix} 0 & 0 \\ 0 & N_g \end{pmatrix}, \quad x \to \begin{pmatrix} a_x \\ v_{1x} \end{pmatrix}$$

give an X-faithful representation of $G \times X \to X$.

In view of theorem 5, from here on we assume for group actions that 1 acts as the identity transformation on the set, and so standard results on group actions are applicable.

If G is a group then the existence of inverses gives a nice structure to X. For x in X, the orbit of x is defined to be the set $0_x = \{gx \mid g \in G\}$. The relationship $x \sim y$ if x is in the orbit of y is then an equivalence relation, so the orbits partition X. This is not the case for more general semigroups. A group action $G \times X \to X$ where X consists of just one orbit is said to be transitive.

Two group actions $G \times X \to X$ and $G \times Y \to Y$ are isomorphic if there is a bijection g from X to Y such that the diagram below commutes:

$$\begin{array}{ccccc} G & \times & X & \to & X \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g} \\ G & \times & Y & \to & Y. \end{array}$$

A general case where X-faithful representations exist is given by the following

Theorem 6. *If the group action $G \times X \to X$ has only finitely many nonisomorphic orbits, each of which is finite, then the action has an X-faithful representation.*

Proof. By lemma 2 we need only consider the case where all orbits are isomorphic. We assume our field F is chosen to have cardinality greater than that of X, and construct a representation of the action on one orbit as in theorem 4, with $x \to e_1$. If the other orbits are Gx_a, a in A, where the isomorphisms take x to the elements x_a, we extend the representation by mapping each x_a to $e_1 f_a, f_a$ in $F - \{0, 1\}, f_a \neq f_b$ for $a \neq b$. This mapping extends naturally to an X-faithful representation of the action $G \times X \to X$.

Theorem 6 covers the case where G is finite, but to show this we must look at orbit structure a bit more closely.

An important example of a group acting on a set is the following:

Example 7. Let G be a group and H a subgroup of G. Then G acts on left cosets of H by multiplication $g(g'H) = (gg')\,H$.

The universal nature of this example is given by the following well-known theorem.

Theorem 8. *If $G \times X \to X$ is any transitive group action, this action is isomorphic to an action of G on left cosets of some subgroup of G.*

Proof. See [1], p. 59.

Corollary 8.1. *If G is a finite group, any action $G \times X \to X$ has a faithful representation.*

Proof. By theorem 8, there can be only as many non-isomorphic orbits as there are subgroups of G, a finite number, and each orbit has a finite number of elements in it $(G : H$ for some subgroup H of $G)$. Thus if G is finite the conditions of theorem 6 are satisfied and any action $G \times X \to X$ has an X-faithful representation. But since G is finite G itself also has a faithful representation, so by corollary 2.1 $G \times$ $\times\ X \to X$ has a faithful representation.

A converse to the corollary is true, namely

Theorem 9. *If G is a group such that all possible actions $G \times X \to X$ have X-faithful representations, then G is finite.*

Proof. G acts on $X = \{$all left cosets of all subgroups of $G\}$. For this action every subgroup of G is a fixer (of itself). By corollary 1.2, G must have *acc* and *dcc* on subgroups. Also, since the representation of $G \times \{\text{cosets of } \{1\}\} \to \{\text{cosets of } \{1\}\}$ is X-faithful, G itself has a faithful representation. Thus G is a linear group. By [5], pp. 146, 114, any linear group with *acc* and *dcc* on subgroups must be finite.

For some other cases we can determine whether an action has an X-faithful representation. This is true of some actions of abelian groups of finite rank. Following [3], p. 49, we say an abelian group has rank n if every finitely generated subgroup has a set of n or fewer generators, and n is minimal with respect to this property. It follows from [3] that if G is an abelian group of finite rank then $G \cong \cong Z_{c_1} \oplus Z_{c_2} \oplus \ldots \oplus Z_{c_r} \oplus Z_{p_1^\infty} \oplus \ldots \oplus Z_{p_s^\infty} \oplus Q_1 \oplus \ldots \oplus Q_t$, where the Q_i's are subgroups of the rational numbers. We get

Theorem 10. *Let G be an abelian group of finite rank. If the action $G \times X \to X$ has only finitely many fixers, then the action has an X-faithful representation.*

Proof. By lemma 2 and the proof of theorem 6 it suffices to assume G has just one orbit Gx. Let H be the fixer of x. G/H is also an abelian group of finite rank, so we assume $G/H = Z_{c_1} \oplus \ldots \oplus Q_t$ (as above). The action of G on Gx is isomorphic to the action of G on cosets of H. This latter action can be identified naturally with multiplication in G/H, and G/H has a faithful representation $\bar{g} \to N_{\bar{g}}$ by [5], p. 17. The maps $g \to N_{\bar{g}}$, $x \to e_1$ then give an X-faithful representation of $G \times X \to X$.

Corollary 10.1. *Let G be an abelian group of finite rank, and assume that only finitely many subgroups of G occur as fixers in the action $G \times X \to X$. Then this action has a faithful representation.*

Proof. A faithful representation for G can be constructed by the construction indicated above. Thus, by corollary 2.1, the action has a faithful representation.

The converse of the theorem above is not true. Some abelian linear groups of finite rank have infinitely many fixers, as in the following example.

Example 11. of an abelian group of finite rank G and a group action $G \times X \to X$ with infinitely many fixers, but having a faithful representation:

$$\text{Let } G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b \text{ rational} \right\}, \text{ and let } X = \left\{ \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} \middle| \; c, d \text{ rational} \right\}.$$

Then $G \cong Q \oplus Q$, so has finite rank, and the fixer of

$$\begin{pmatrix} c \\ d \\ 1 \end{pmatrix}$$

is easily seen to be

$$\left\{\begin{pmatrix} 1 & a & -ad \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

Thus as d ranges over Q we get infinitely many distinct fixers in G.

In the last theorem the condition on the rank of G is essential. If G does not have finite rank the action of G on a single orbit may not have an X-faithful representation, as seen by the following example.

Example 12. of an abelian group of infinite rank G and a transitive action $G \times \times X \to X$ without an X-faithful representation: Let $G = \oplus Z_n$ (n in N), and take the action of G on itself by left multiplication (G acting on left cosets of the identity). It is easily verified that if the mappings $g \to M_g$, $g \to v_g$ give an X-faithful representation, then $G \cong \{M_g \mid g \text{ in } G\}$. But by [5], pp. 17–18, a torsion abelian group has a faithful representation over a field only if it is of finite rank. Thus $G \times G \to G$ has no X-faithful representations.

For general group actions we get the following two results.

Theorem 13. *Let $G \times Gx \to Gx$ be a transitive group action. Suppose H is a subgroup of G of finite index and the fixer of x is contained in H. Then $G \times \times Gx \to Gx$ has an X-faithful representation if and only if $H \times Hx \to Hx$ does.*

Proof. The only if statement is clear. For the other direction, assume $H \times Hx \to \to Hx$ has an X-faithful representation $h \to M_n$, $hx \to v_{hx}$. In the event that H is the fixer of x and $v_{hx} = 0$ for all hx, we modify our representation and take instead the mappings

$$h \to \begin{pmatrix} M_h & 0 \\ 0 & 1 \end{pmatrix}, \quad hx \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for all hx, so we assume that in our original representation $v_x \neq 0$. Suppose M_h has dimension $k \times k$. We use the given representation of H to induce a representation of G in the usual manner (c.f. [2], p. 75): if $G = \bigcup g_i H$ ($i = 1, \ldots, n$), where $g_1 = 1$, then we map g to the $kn \times kn$ matrix N_g which consists of blocks of dimension $k \times k$, the i, j block being $M_{g_j^{-1} g g_i}$ if $g_j^{-1} g g_i$ is in H, 0 otherwise. This representation of G, together with the map

$$x \to \begin{pmatrix} v_x \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

induces an X-faithful representation of $G \times Gx \to Gx$.

Let $N_G(\mathrm{fix}(x))$ denote the normalizer in G of the fixer of x, that is, $\{g \in G \mid g^{-1}\, \mathrm{fix}(x)\, g = \mathrm{fix}(x)\}$. We then get

Theorem 14. *If $N_G(\mathrm{fix}(x))$ is of finite index in G and $N_G(\mathrm{fix}(x))/\mathrm{fix}(x)$ has a faithful representation (as a group), then $G \times Gx \to Gx$ has an X-faithful representation.*

Proof. Let N denote $N_G(\mathrm{fix}(x))$. By the previous theorem we need only concern ourselves with giving an X-faithful representation of $N \times Nx \to Nx$. This action is isomorphic to the action of N on left cosets of $H = \mathrm{fix}(x)$. The action of N on left cosets of H factors through N/H, in the sense that the diagram below commutes:

$$\begin{array}{ccc} N \times \{\text{left cosets of } H\} & \to & \{\text{left cosets of } H\} \\ \big\downarrow \qquad \big\downarrow \text{id} & \text{multiplication} & \big\downarrow \text{id} \\ N/H \times \{\text{left cosets of } H\} & \to & \{\text{left cosets of } H\}\,. \end{array}$$

Let $\bar{g} \to N_{\bar{g}}$ be the faithful representation of N/H. The $k \times k$ matrices $N_{\bar{g}}$ are themselves vectors in a k^2-dimensional vector space (with basis $\{e_{ij}\}$), and the action of the matrices $N_{\bar{g}}$ on these vectors is linear. We map the matrices $N_{\bar{g}}$ to their images $v_{\bar{g}}$ as k^2-dimensional vectors, and also map the matrices $N_{\bar{g}}$ to the $k^2 \times k^2$ matrices $P_{\bar{g}}$, which permute the vectors $v_{\bar{g}}$ in the same way the matrices $N_{\bar{g}}$ act on themselves. The maps $h \to P_{\bar{h}}$, $hx \to v_{\bar{h}}$, is then an X-faithful representation of the action $N \times Nx \to$ $\to Nx$.

The last two results can be exptended in an obvious way to the case where one is interested in representations of group actions that are not necessarily X-faithful.

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