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DIRECTLY DECOMPOSABLE TOLERANCES ON DIRECT PRODUCTS OF LATTICES AND SEMILATTICES

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A *compatible tolerance* on an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ is a reflexive and symmetric binary relation on A having the Substitution Property with respect to all operations of $\mathscr{F}$ with non-zero arities. In particular, if L is a lattice and T is a reflexive and symmetric relation on the support of L, then T is a compatible tolerance on L if and only if $(a, b) \in T$, $(c, d) \in T$ always imply $(a \vee c, b \vee d) \in T$ and $(a \wedge c, b \wedge d) \in T$.

Let L_γ be lattices for $\gamma \in \Gamma$ and $L = \prod_{\gamma \in \Gamma} L_\gamma$, let T be a compatible tolerance on L. The tolerance T is called *directly decomposable* if there exist compatible tolerances T_γ on L_γ (for each $\gamma \in \Gamma$) such that $T = \prod_{\gamma \in \Gamma} T_\gamma$ (this means that $(a, b) \in T$ if and only if $(pr_\gamma\, a, pr_\gamma\, b) \in T_\gamma$ for each $\gamma \in \Gamma$, where pr_γ means the projection onto the γ-th direct factor of L).

Some conditions for the direct decomposability of compatible tolerances were investigated in [4]. The paper [3] contains the complete solution of the problem of direct decomposability of compatible tolerances on lattices in the case of Γ finite. If $\mathfrak{A}_\gamma \in \mathscr{V}$ for finite Γ and a given variety $\mathscr{V}$, the problem is completely solved in [2]. However, the case of Γ infinite hasstill remained open.

An analogous problem for congruences on (infinite) direct products of lattices was partially solved in [1] and [5]. We shall use the methods from [1] and [5] to obtain a similar result in the case of tolerances. Let us recall some notions from [5]. Suppose $L = \prod_{\gamma \in \Gamma} L_\gamma$ and $x \in L$, $y \in L$. Denote $x(\gamma) = pr_\gamma\, x$. Further denote by $f(x, y, \gamma)$ the element of L defined by

$$f(x, y, \gamma)(\gamma) = x(\gamma),$$
$$f(x, y, \gamma)(\delta) = y(\delta) \quad \text{for} \quad \delta \in \Gamma, \quad \delta \neq \gamma.$$

The following lemma is a generalization of that in [5]; the application of transitivity in that proof is avoided.

Lemma. *Let T be a compatible tolerance on the lattice $L = \prod_{\gamma \in \Gamma} L_\gamma$, let $x \in L$, $y \in L$, $(x, y) \in T$. Then $(f(x, z, \gamma), f(y, z, \gamma)) \in T$ for each $z \in L$ and each $\gamma \in \Gamma$.*

Proof. Clearly $x \wedge y \leqq f(y, x, \gamma) \leqq x \vee y$. Hence $(x, y) \in T$ implies $(x \wedge y, x \vee y) \in T$ and $(x, f(y, x, \gamma)) \in T$ in virtue of the convexity of T. By the Substitution Property we have

$$(x \wedge f(x, z, \gamma), f(y, x, \gamma) \wedge f(x, z, \gamma)) \in T.$$

Since

$$f(a_1, b_1, \gamma) \wedge f(a_2, b_2, \gamma) = f(a_1 \wedge a_2, b_1 \wedge b_2, \gamma),$$

we obtain

$$(f(x, z \wedge x, \gamma), f(x \wedge y, z \wedge x, \gamma)) \in T.$$

By using the operation $\vee$ and the pair $(f(y, z, \gamma), f(y, z, \gamma)) \in T$, we obtain $(f(x \vee y, z, \gamma), f(y, z, \gamma)) \in T$. Analogously we can prove $(f(x, z, \gamma), f(x \vee y, z, \gamma)) \in \in T$. Using the operation $\wedge$, we obtain $(f(x, z, \gamma), f(y, z, \gamma)) \in T$, which was to be proved.

Let T be a compatible tolerance on a lattice L and let $\mathfrak{m}$ be a given infinite cardinal number. The tolerance T is called *conditionally* $\vee$-$\mathfrak{m}$-*complete*, if $(a_\delta, b_\delta) \in T$ for $\delta \in \Delta$ with $|\Delta| = \mathfrak{m}$ imply $(\bigvee_{\delta\in\Delta} a_\delta, \bigvee_{\delta\in\Delta} b_\delta) \in T$ provided that both $\bigvee_{\delta\in\Delta} a_\delta$ and $\bigvee_{\delta\in\Delta} b_\delta$ exist in L. Dually we can define a *conditionally* $\wedge$-$\mathfrak{m}$-*complete tolerance.*

Theorem 1. *Each conditionally* $\vee$-$\mathfrak{m}$-*complete tolerance on the lattice* $L = \prod_{\gamma\in\Gamma} L_\gamma$ *with* $|\Gamma| = \mathfrak{m}$ *is directly decomposable. Each conditionally* $\wedge$-$\mathfrak{m}$-*complete tolerance on* L *is directly decomposable.*

Proof. Put

$$T_\gamma = \{(x_\gamma, y_\gamma) \mid x_\gamma = x(\gamma),\ y_\gamma = y(\gamma) \text{ for some } (x, y) \in T\},$$

where T is a conditionally $\vee$-$\mathfrak{m}$-complete tolerance on $L = \prod_{\gamma\in\Gamma} L_\gamma$ with $|\Gamma| = \mathfrak{m}$. Clearly T_γ is a compatible tolerance on L_γ for each $\gamma \in \Gamma$ and

$$T \subseteq \prod_{\gamma\in\Gamma} T_\gamma.$$

We prove the converse inclusion. Let $(x, y) \in \prod_{\gamma\in\Gamma} T_\gamma$. With respect to the convexity of compatible tolerances it suffices to consider only the case $x \leqq y$. Then $(x(\gamma), y(\gamma)) \in T$ for each $\gamma \in \Gamma$, i.e. there exist elements a and b of L such that $(f(x, a, \gamma), f(y, b, \gamma)) \in T$. By Lemma, this implies

$$(f(f(x, a, \gamma), x, \gamma), f(f(y, b, \gamma), x, \gamma)) \in T.$$

Since $f(f(x, a, \gamma), x, \gamma) = x$ and $f(f(y, b, \gamma), x, \gamma) = f(y, x, \gamma)$, we infer $(x, f(y, x, \gamma)) \in T$. As $x \leqq y$, we conclude $y = \bigvee_{\gamma\in\Gamma} f(y, x, \gamma)$. Since T is conditionally $\vee$-$\mathfrak{m}$-complete, we obtain $(x, y) \in T$, which was to be proved. Dually we can prove the assertion for conditionally $\wedge$-$\mathfrak{m}$-complete tolerances.

Corollary (cf. [3]). *Each compatible tolerance on the lattice* $L = L_1 \times \ldots \times L_n$ *is directly decomposable.*

Now we shall turn our attention to *semilattices*. The operation on a semilattice will be denoted by $\otimes$.

Consider a semilattice $S = \prod_{\gamma \in \Gamma} S_\gamma$, where $|\Gamma| = \mathfrak{m} \geqq \aleph_0$. A *conditionally* $\mathfrak{m}$-*complete tolerance* can be defined analogously to the above defined similar concepts for lattices.

Theorem 2. *Let* $S = \prod_{\gamma \in \Gamma} S_\gamma$, *where* S_γ *are semilattices with zero elements and* $|\Gamma| = \mathfrak{m} \geqq \aleph_0$. *Then there exists a conditionally* $\mathfrak{m}$-*complete tolerance which is not directly decomposable.*

Proof. For each $\gamma \in \Gamma$ let the zero element of the semilattice S_γ be denoted by z_γ. Let T be the tolerance on S defined so that $(a, b) \in T$ if and only if either $a = b$, or there exists an infinite subset $\Gamma(a, b)$ of Γ such that $a(\gamma) = b(\gamma) = z_\gamma$ for each $\gamma \in \Gamma(a, b)$. We shall prove that T is a conditionally $\mathfrak{m}$-complete tolerance on S. Let $(a_\delta, b_\delta) \in T$ for $\delta \in \Delta$, where $|\Delta| = \mathfrak{m}$. If $a_\delta = b_\delta$ for each $\delta \in \Delta$, then $\bigotimes_{\delta \in \Delta} a_\delta = \bigotimes_{\delta \in \Delta} b_\delta$ and $(\bigotimes_{\delta \in \Delta} a_\delta, \bigotimes_{\delta \in \Delta} b_\delta) \in T$. If there exists $\varepsilon \in \Delta$ such that $a_\varepsilon \neq b_\varepsilon$, then there exists an infinite subset $\Gamma(a_\varepsilon, b_\varepsilon)$ of Γ such that $a_\varepsilon(\gamma) = b_\varepsilon(\gamma) = z_\gamma$ for each $\gamma \in \Gamma(a_\varepsilon, b_\varepsilon)$. Now $\bigotimes_{\delta \in \Delta} a_\delta(\gamma) = a_\varepsilon(\gamma) \otimes \bigotimes_{\delta \in \Delta - \{\varepsilon\}} a_\delta(\gamma) = z_\gamma \otimes \bigotimes_{\delta \in \Delta - \{\varepsilon\}} a_\delta(\gamma) = z_\gamma$, $\bigotimes_{\delta \in \Delta} b_\delta(\gamma) = b_\varepsilon(\gamma) \otimes \bigotimes_{\delta \in \Delta - \{\varepsilon\}} b_\delta(\gamma) = z_\gamma \otimes \bigotimes_{\delta \in \Delta - \{\varepsilon\}} b_\delta(\gamma) = z_\gamma$ for each $\gamma \in \Gamma(a_\varepsilon, b_\varepsilon)$ and $(\bigotimes_{\delta \in \Delta} a_\delta, \bigotimes_{\delta \in \Delta} b_\delta) \in T$. Now define again $T_\gamma = \{(x_\gamma, y_\gamma) \mid x_\gamma = x(\gamma),\ y_\gamma = y(\gamma)$ for some $(x, y) \in T\}$. For each $\gamma \in \Gamma$ let a_γ, b_γ be two arbitrary elements of S_γ. Let c_γ, d_γ be the elements of S such that $c_\gamma(\gamma) = a_\gamma$, $d_\gamma(\gamma) = b_\gamma$, $c_\gamma(\delta) = d_\gamma(\delta) = z_\gamma$ for each $\delta \in \Gamma - \{\gamma\}$. Clearly $(c_\gamma, d_\gamma) \in T$, hence $(a_\gamma, b_\gamma) \in T_\gamma$. As a_γ, b_γ were chosen arbitrarily, T_γ is the universal binary relation on S_γ for each $\gamma \in \Gamma$. Therefore $\prod_{\gamma \in \Gamma} T_\gamma$ is the universal binary relation on S, while T is not and $T \neq \prod_{\gamma \in \Gamma} T_\gamma$.

References

[1] *Chajda, I.:* Congruence factorizations on distributive lattices. Math. Slovaca *28* (1978), 343–347.

[2] *Chajda, I.:* Varieties with directly decomposable diagonal subalgebras. Algebra Univ. (to appear).

[3] *Chajda, I. - Nieminen, J.:* Direct decomposability of tolerances on lattices, semilattices and quasilattices. Czech. Math. J. (to appear).

[4] *Chajda, I. - Zelinka, B.:* Tolerance relations on direct products. Glasnik Mat. (Zagreb) *14* (1979), 11–16.

[5] *Pócs, J.:* Congruence relations on direct products of lattices. Math. Slovaca (to appear).

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