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ATOMICITY OF TOLERANCE LATTICES OF COMMUTATIVE SEMIGROUPS

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By a *tolerance* T on an algebra $\mathscr{A} = (A, F)$ we mean a reflexive and symmetric binary relation on A satisfying with respect to each n-ary operation $f \in F$ the following condition:

(1) $$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in T$$

for $(a_i, b_i) \in T \ (i = 1, \dots, n)$. The set $\mathscr{L}(\mathscr{A})$ of all tolerances on an algebra $\mathscr{A}$ forms a complete algebraic lattice with respect to set inclusion (see [1] and [2]). In [3] I. Chajda and J. Nieminen have found some conditions for the atomicity of $\mathscr{L}(\mathscr{A})$, where $\mathscr{A}$ is a lattice or a join-semilattice. The aim of this paper is to consider the atomisity of $\mathscr{L}(\mathscr{A})$ when $\mathscr{A}$ is a commutative semigroup. The present results generalize the corresponding results in [3] for semilattices.

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Let $\mathscr{S} = (S, \cdot)$ be a commutative semigroup. From (1) it follows that for each tolerance $T \in \mathscr{L}(\mathscr{S})$ we have

(2) $$(au, bv) \in T$$

whenever $(a, b) \in T$ and $(u, v) \in T$. If $a, b \in S$, we denote by $T(a, b)$ the least tolerance on $\mathscr{S}$ containing the pair (a, b). It is clear that $T(a, b) = T(b, a)$. Denote by $\mathscr{N}$ the set of all positive integers. The notation $\mathscr{S}^1 = (S^1, \cdot)$ stands for $\mathscr{S}$ if $\mathscr{S}$ has an identity, otherwise it stands for $\mathscr{S}$ with an identity adjoined.

Lemma 1. *Let $\mathscr{S} = (S, \cdot)$ be a commutative semigroup and $a, b \in S$. Then $(x, y) \in T(a, b)$ for $x \neq y$ if and only if there exist $m \in \mathscr{N}$ and $u \in S^1$ such that either*

$$x = a^m u\,, \quad y = b^m u$$

or

$$x = b^m u\,, \quad y = a^m u\,.$$

Proof. Apply (2).

The set of all idempotents of a commutative semigroup $\mathcal{S}$, if non empty, is denoted by $E(\mathcal{S})$, and is partially ordered by: $e \leqq f$ if $ef = e$; we write $e < f$ for $e \leqq f$, $e \neq f$.

Lemma 2. *Let $\mathcal{S} = (S, \cdot)$ be a commutative semigroup, $e, f, g \in E(\mathcal{S})$, $a \in S$ and $e \neq f$, $g \neq a$. If $T(e,f) = T(g, a)$, then $g \in \{e, f\}$.*

Proof. Suppose that $T(e,f) = T(g, a)$. Then $(g, a) \in T(e,f)$ and according to Lemma 1 there exists $u \in S^1$ such that either $g = eu$, $a = fu$ or $g = fu$, $a = eu$. Without loss of generality we can suppose that $g = eu$ and $a = fu$. Then we have $g = eg$. Since $(e, f) \in T(g, a)$, there exist $m \in \mathcal{N}$ and $v \in S^1$ such that either $e = gv$, $f = a^m v$ or $e = a^m v$, $f = gv$. If $e = gv$, then $g = eg = (gv)\, g = gv = e$. If $e = a^m v$ and $f = gv$, then $g = eg = (a^m v)\, g = fa^m$ and thus we have $g = fg = (gv)\, g = = gv = f$.

Lemma 3. *Let $\mathcal{S}$ be a commutative semigroup, $e, f, g, h \in E(\mathcal{S})$ and $e \neq f$, $g \neq h$. If $T(e,f) = T(g, h)$, then $\{e, f\} = \{g, h\}$.*

Proof. Easily follows from Lemma 2.

Lemma 4. *Let $T \neq \mathrm{id}_S$ be a tolerance on a commutative semigroup $\mathcal{S} = (S, .)$. Then T is an atom of $\mathcal{L}(\mathcal{S})$ if and only if $T = T(x, y)$ for every pair $(x, y) \in T$, $x \neq y$.*

Proof. Assume that T is an atom of $\mathcal{L}(\mathcal{S})$. If $(x, y) \in T$ and $x \neq y$, then $\mathrm{id}_S \neq T(x, y) \subseteq T$. Hence we have $T = T(x, y)$.

Suppose that $T = T(x, y)$ for a pair $(x, y) \in T$, $x \neq y$. Let $\mathrm{id}_S \neq K \subseteq T$ for some tolerance K of $\mathcal{L}(\mathcal{S})$. Evidently, then there exists a pair $(a, b) \in K$ such that $a \neq b$. Hence we have $T = T(a, b) \subseteq K$. Thus T is an atom of $\mathcal{L}(\mathcal{S})$.

Define a relation $\mathcal{R}_1(\mathcal{S})$ on a commutative semigroup $\mathcal{S} = (S, \cdot)$ by $(x, y) \in \mathcal{R}_1(\mathcal{S})$ if and only if $x, y \in E(\mathcal{S})$, $x < y$ and if $xu \neq yu$ for some $u \in S$, then there exists $v \in S$ such that $y = yuv$.

Theorem 1. *Let T be a tolerance on a commutative semigroup $\mathcal{S}$. Let $(e, f) \in T$ for some $e, f \in E(\mathcal{S})$, $e \neq f$. Then T is an atom of the lattice $\mathcal{L}(\mathcal{S})$ if and only if $T = T(e, f)$ and $(e, f) \in \mathcal{R}_1(\mathcal{S}) \cup \mathcal{R}_1^{-1}(\mathcal{S})$.*

Proof. Assume that T is an atom of $\mathcal{L}(\mathcal{S})$. According to Lemma 4, we have $T = T(e, f)$ for $e, f \in E(\mathcal{S})$ and $e \neq f$. We shall show that either $e < f$ or $f < e$. By way of contradiction, we assume that $e \neq ef \neq f$. From (2) it follows that $(ef, e) \in T$ and, by Lemma 4, we have $T = T(ef, e)$. Since $ef \in E(\mathcal{S})$, from Lemma 3 it follows that $ef \in \{e, f\}$, which is a contradiction. We have the following possibilities:

Case 1. $e < f$. Then $e = ef$. If $eu \neq fu$ for some $u \in S$, then, by (2) and Lemma 4, we have $T = T(eu, fu)$. Since $(e, f) \in T$, according to Lemma 1 there exists $z \in S^1$ such that either $e = euz$, $f = fuz$ or $e = fuz$, $f = euz$. If $f = euz$, then $e = ef =$

$= e(euz) = eue = f$, a contradiction. Thus we have $f = fuz$. Put $v = z$ for $z \in S$ and $v = f$ for $z \in S^1 \setminus S$. Hence we have $f = fuv$. This means that $(e, f) \in \mathcal{R}_1(\mathcal{S})$.

Case 2. $f < e$. Using the same method as in Case 1, we obtain $(e, f) \in \mathcal{R}_1^{1-}(\mathcal{S})$.

Hence $T = T(e, f)$ and $(e, f) \in \mathcal{R}_1(\mathcal{S}) \cup \mathcal{R}_1^{-1}(\mathcal{S})$.

Conversely, suppose that $T = T(e, f)$ for $(e, f) \in \mathcal{R}_1(\mathcal{S}) \cup \mathcal{R}_1^{-1}(\mathcal{S})$. Without loss of generality we can assume that $(e, f) \in \mathcal{R}_1(\mathcal{S})$. Then $e < f$. Hence $ef = e$. We shall show that T is an atom of $\mathcal{L}(\mathcal{S})$. Let $(x, y) \in T$ and $x \neq y$. By Lemma 1, there exists $z \in S^1$ such that either $x = ez$, $y = fz$ or $x = fz$, $y = ez$. Put $u = z$ for $z \in S$ and $u = f$ for $z \in S^1 \setminus S$. Then $eu \neq fu$. Since $(e, f) \in \mathcal{R}_1(\mathcal{S})$, there exists $v \in S$ such that $f = fuv$. Hence we have $e = ef = e(fuv) = euv$. This implies that either $e = xv$, $f = yv$ or $e = yv$, $f = xv$. According to Lemma 1, we have $(e, f) \in T(x, y)$. Thus $T \subseteq T(x, y) \subseteq T$ and so $T = T(x, y)$. By Lemma 4, T is an atom of $\mathcal{L}(\mathcal{S})$.

Now, define a relation $\mathcal{R}_2(\mathcal{S})$ on a commutative semigroup $\mathcal{S} = (S, \cdot)$ by $(x, y) \in \in \mathcal{R}_2(\mathcal{S})$ if and only if

(i) $x \in E(\mathcal{S})$ and y is a periodic element of $\mathcal{S}$ such that $x \in [y]$, where by $[y]$ we denote the subsemigroup of $\mathcal{S}$ generated by y;

(ii) $[y]$ is either a cyclic subgroup of the prime order or card $[y] = 2$;

(iii) if $xu \neq yu$ for some $u \in S$, then there exists $v \in S$ such that $y = yuv$.

We shall consider tolerances T of $\mathcal{L}(\mathcal{S})$ satisfying the following implication:

$$\text{(3)} \qquad \text{if} \quad (f, g) \in T \quad \text{for} \quad f, g \in E(\mathcal{S}), \quad \text{then} \quad f = g .$$

Theorem 2. *Let T be a tolerance on a commutative semigroup $\mathcal{S} = (S, \cdot)$. Let $(e, b) \in T$ for some $e \in E(\mathcal{S})$ and $b \in S \setminus E(\mathcal{S})$. Then T is an atom of the lattice $\mathcal{L}(\mathcal{S})$ satisfying (3) if and only if $T = T(e, b)$ and $(e, b) \in \mathcal{R}_2(\mathcal{S})$.*

Proof. Suppose that T is an atom of $\mathcal{L}(\mathcal{S})$ satisfying the condition (3). Then, by Lemma 4, we have $T = T(e, b)$.

Case 1. Assume that $e \neq be$. From (2) and Lemma 4 it follows that $T = T(e, be)$. According to Lemma 1, there exist $z \in S^1$ and $m \in \mathcal{N}$ such that either $e = ez$, $b = (be)^m z$ or $e = (be)^m z$, $b = ez$. Then $be = b$.

Subcase 1a. Suppose that b is periodic. Then there exists $n \in \mathcal{N}$ such that $b^n \in \in E(\mathcal{S})$. By Lemma 1, we have $(e, b^n) \in T$. From (3) it follows that $b^n = e$ and so $[b]$ is a cyclic subgroup of $\mathcal{S}$. We shall show that $m = \text{card}\,[b]$ is prime. By way of contradiction, we assume that k/m for some $k \in \mathcal{N}$, $1 < k < m$. Then $e \neq b^k$. According to Lemma 1, we have $(e, b^k) \in T$ and, by Lemma 4, we obtain $T = T(e, b^k)$. By Lemma 1, there exist $w \in S^1$ and $r \in \mathcal{N}$ such that either $e = ew$, $b = (b^k)^r w$ or $b = ew$, $e = (b^k)^r w$. Then either $b = b^{kr}$ or $e = b^{kr}b$. This implies that either $m/(kr - 1)$ or $m/(kr + 1)$, which is a contradiction.

Suppose that $eu \neq bu$ for some $u \in S$. Then, by (2) and Lemma 4, we have $T = = T(eu, bu)$. According to Lemma 1, there exist $x \in S^1$ and $s \in \mathcal{N}$ such that either $e = (eu)^s x$, $b = (bu)^s x$ or $e = (bu)^s x$, $b = (eu)^s x$. Suppose that $b = (bu)^s x$. If

$s > 1$, then $b = buv$, where $v = (bu)^{s-1} x \in S$. If $s = 1$, then $b = bux = b(ux)^2$ and so $b = buv$, where $v = ux^2 \in S$. Assume that $e = (bu)^s x$. Then $b = be = b(bu)^s x$. Thus we have $b = buv$, where $v = b(bu)^{s-1} x \in S$ if $s > 1$ and $v = bx \in S$ if $s = 1$. This gives in both cases $(e, b) \in \mathscr{R}_2(\mathscr{S})$.

Subcase 1b. Suppose that b is not periodic. Then $e \neq b^2$. According to Lemmas 1 and 4, we have $T = T(e, b^2)$. From Lemma 1 it follows that there exist $z \in S^1$ and $m \in \mathscr{N}$ such that either $e = ez$, $b = b^{2m}z$ or $e = b^{2m}z$, $b = ez$. Then either $b = b^{2m}z = b^{2m}ez = b^{2m}e = b^{2m}$ or $e = b^{2m}z = b^{2m}ez = b^{2m+1}$, a contradiction.

Case 2. Assume that $e = be$.

Subcase 2a. Suppose that b is periodic. Then there exists $m \in \mathscr{N}$ such that $b^m \in E(\mathscr{S})$. From Lemma 1 it follows that $(e, b^m) \in T$ and so, by (3), we have $b^m = e$. Thus $b^n = e$ for all $n \geqq m$, $n \in \mathscr{N}$. Now, we shall show that $b^2 = e$. By way of contradiction, we assume that $e \neq b^2$. From Lemmas 1 and 4 it follows that $T = T(e, b^2)$ and according to Lemma 1, there exist $z \in S^1$ and $k \in \mathscr{N}$ such that either $e = ez$, $b = b^{2k}z$ or $e = b^{2k}z$, $b = ez$. Suppose that $e = ez$ and $b = b^{2k}z$. Then we can prove by induction that $e = ez^r$ and $b = b^{2kr-r+1}z^r$ for all $r \in \mathscr{N}$. It is clear that there exists $s \in \mathscr{N}$ such that $2ks - s + 1 \geqq m$. Hence we have $b = b^{2ks-s+1}z^s = ez^s = e$, which is a contradiction. If $e = b^{2k}z$ and $b = ez$, then $e = be = (ez)\, e = ez = b$, a contradiction. Therefore $e = b^2$ and so card $[b] = 2$.

Suppose that $eu \neq bu$ for some $u \in S$. From Lemmas 1 and 4 it follows that $T = T(eu, bu)$. By Lemma 1, there exist $x \in S^1$ and $r \in \mathscr{N}$ such that either $e = (eu)^r x$, $b = (bu)^r x$ or $e = (bu)^r x$, $b = (eu)^r x$. Since $e \neq b$, we have $r = 1$. If $b = eux$, then $e = be = (eux)\, e = eux = b$, a contradiction. We can suppose that $b = bux$. Then $b = buv$ for $v = ux^2 \in S$. This means that $(e, b) \in \mathscr{R}_2(\mathscr{S})$.

Subcase 2b. Suppose that b is not periodic. Then $e \neq b^2$ and so, by Lemmas 1 and 4, we have $T = T(e, b^2)$. From Lemma 1 it follows that there exist $z \in S^1$ and $m \in \mathscr{N}$ such that either $e = ez$, $b = b^{2m}z$ or $e = b^{2m}z$, $b = ez$. Assume that $e = ez$ and $b = b^{2m}z$. Put $y = b^{2m-2}z$ for $m > 1$ and $y = z$ for $m = 1$. Hence we have $b = byb$ and so $ey = e$ and $by = (by)^2 \in E(\mathscr{S})$. Lemma 1 implies that $(ey, by) \in T$ and, by (3), we have $by = e = ey$. Thus $b = byb = eb = e$, which is a contradiction. If $b = ez$, then $e = eb = e(ez) = ez = b$, a contradiction.

Conversely, suppose that $T = T(e, b)$ for $(e, b) \in \mathscr{R}_2(\mathscr{S})$. We shall show that T is an atom of $\mathscr{L}(\mathscr{S})$. Let $(x, y) \in T$ and $x \neq y$. Evidently $T(x, y) \subseteq T$. According to Lemma 1, there exist $z \in S^1$ and $m \in \mathscr{N}$ such that either $x = ez$, $y = b^m z$ or $x = b^m z$, $y = ez$. Without loss of generality we can suppose that $x = ez$ and $y = b^m z$.

Case a. $[b]$ is a cylic subgroup of the prime order p. Put $u = z$ for $z \in S$ and $u = e$ for $z \in S^1 \setminus S$. Then we have $x = eu$ and $y = b^m u$. We have $eu \neq bu$. Indeed, if $eu = bu$, then $x = eu = bu = b^2u = \ldots = b^m u = y$, which is a contradiction. Since $(e, b) \in \mathscr{R}_2(\mathscr{S})$, there exists $v \in S$ such that $b = buv$. Then $b^m = b^m uv = yv$ and $e = b^p = b^p uv = euv = xv$. Since $b^m \neq e$, there exists $n \in \mathscr{N}$ such that $(b^m)^n = b$. Then, by Lemma 1, we have $(e, b) = (e^n, (b^m)^n) = (x^n v^n, y^n v^n) \in T(x, y)$.

Case b. card $[b] = 2$ and $e = eb = b^2$. Since $x \neq y$, we have $x = ez$ and $y = bz$ for some $z \in S^1$. If $x = e$ and $y = b$, then $(e, b) = (x, y) \in T(x, y)$. If $z \in S$, then there exists $v \in S$ such that $b = bzv = yv$. Thus we have $e = eb = e(bzv) = ezv = xv$. According to Lemma 1, we obtain $(e, b) = (xv, yv) \in T(x, y)$.

This gives in both cases $T = T(e, b) \subseteq T(x, y)$ and so $T = T(x, y)$. From Lemma 4 it follows that T is an atom of $\mathscr{L}(\mathscr{S})$.

Finally, we shall prove that T satisfies (3). Let $(f, g) \in T(e, b)$ for $f, g \in E(\mathscr{S})$. Then, by Lemma 1, there exist $w \in S^1$ and $k \in \mathscr{N}$ such that either $f = ew$, $g = b^k w$ or $f = b^k w$, $g = ew$. We can suppose that $f = ew$ and $g = b^k w$. Then we have $ew = ew^2$ and $b^k w = b^{2k} w^2$. If $[b]$ is a cyclic subgroup of $\mathscr{S}$, then there exists b^{-k} such that $b^{-k} b^k = e$. Hence we have $f = ew = b^{-k} b^k w = b^{-k} b^{2k} w^2 = b^k e w^2 = b^k e w = b^k w = g$. If card $[b] = 2$ and $e = eb = b^2$, then $f = ew = ew^2 = (b^2)^k w^2 = b^k w = g$.

Define a relation $\mathscr{R}_3(\mathscr{S})$ on a commutative semigroup $\mathscr{S} = (S, \cdot)$ by $(x, y) \in \mathscr{R}_3(\mathscr{S})$ if and only if

(i) $x, y \in S \setminus E(\mathscr{S})$ and $x^2 = xy = y^2$;

(ii) $xu \in E(\mathscr{S})$ for some $u \in S$ if and only if $yu \in E(\mathscr{S})$;

(iii) if $xu \neq yu$ for some $u \in S$, then there exists $v \in S$ such that $x = xuv$ and $y = yuv$.

By induction we can prove the following implication:

(4) $$\text{if } x^2 = xy = y^2 \text{ for } x, y \in S, \text{ then } x^n = y^n \text{ for all } n \in \mathscr{N}, \ n \geqq 2.$$

Indeed, $x^{n+1} = x^n x = y^n x = y^{n-1} xy = y^{n-1} y^2 = y^{n+1}$ if $x^n = y^n$ and $n \geqq 2$.

We shall consider tolerances T of $\mathscr{L}(\mathscr{S})$ satisfying the following implication

(5) $$\text{if } (e, c) \in T \text{ and } e \in E(\mathscr{S}), \text{ then } e = c.$$

Theorem 3. *Let T be a tolerance on a commutative semigroup $\mathscr{S} = (S, \cdot)$. Let $(a, b) \in T$ for some $a, b \in S \setminus E(\mathscr{S})$, $a \neq b$. Then T is an atom of the lattice $\mathscr{L}(\mathscr{S})$ satisfying (5) if and only if $T = T(a, b)$ and $(a, b) \in \mathscr{R}_3(\mathscr{S})$.*

Proof. Suppose that T is an atom of $\mathscr{L}(\mathscr{S})$ satisfying the condition (5). Then, by Lemma 4, we have $T = T(a, b)$. We shall show that $a^2 = ab$. By way of contradiction, we assume that $a^2 \neq ab$. From Lemma 1 and 4 it follows that $T = T(a^2, ab)$. According to Lemma 1, there exist $z \in S^1$ and $m \in \mathscr{N}$ such that either $a = a^{2m} z$, $b = a^m b^m z$ or $a = a^m b^m z$, $b = a^{2m} z$. We have the following possibilities:

Case 1. $a = a^{2m} z$ and $b = a^m b^m z$. From Lemma 1 it follows that $(v, w) = (a^{2m-1} z, a^{m-1} b^m z) \in T$, where $a^0 = 1$ in $\mathscr{S}^1$ It is easy to show that $v^2 = v$ and, by (5), we have $v = w$. This implies that $a = av = aw = b$, which is a contradiction.

Case 2. $a = a^m b^m z$ and $b = a^{2m} z$. Then $a^m = a^n b^n z^m$ and $b^m = a^{2n} z^m$, where

$n = m^2$. Putting $u = a^m z^{m+1}$, we have $a = a^m(a^{2n}z^m)\, z = a^{2n}u$ and $b = a^m(a^n b^n z^m)\, z = a^n b^n u$, which is a contradiction by Case 1.

Consequently, $a^2 = ab$. Dually we obtain that $b^2 = ab$.

Let $au \in E(\mathscr{S})$ for some $u \in S$. From Lemma 1 it follows that $(au, bu) \in T$ and so, by (5), we have $bu = au \in E(\mathscr{S})$. If $bu \in E(\mathscr{S})$, then analogously we obtain $au \in E(\mathscr{S})$.

Suppose that $au \neq bu$ for some $u \in S$. From Lemmas 1 and 4 it follows that $T = T(au, bu)$. By Lemma 1, there exist $x \in S^1$ and $n \in \mathscr{N}$ such that either $a = (au)^n x$, $b = (bu)^n x$ or $a = (bu)^n x$, $b = (au)^n x$. According to (4), we have $n = 1$.

Case 1. $a = aux$ and $b = bux$. If $x \in S$, then we put $v = x$. Therefore $a = auv$ and $b = buv$. If $x \in S^1 \setminus S$, then we have $a = au$ and $b = bu$. Then $a = au^2$ and $b = bu^2$ and so $a = auv$ and $b = buv$ for $v = u$.

Case 2. $a = bux$ and $b = aux$. Putting $v = ux^2 \in S$ we have $a = bux = au^2x^2 = auv$ and $b = aux = bu^2x^2 = buv$.

Conversely, suppose that $T = T(a, b)$ for $(a, b) \in \mathscr{R}_3(\mathscr{S})$. We shall show that T is an atom of $\mathscr{L}(S)$. Let $(x, y) \in T$ and $x \neq y$. It is clear that $T(x, y) \subseteq T$. By Lemma 1 and (4), there exists $z \in S^1$ such that either $x = az$, $y = bz$ or $x = bz$, $y = az$. Without loss of generality we can suppose that $x = az$ and $y = bz$. If $z \in S^1 \setminus S$, then $(a, b) = (x, y) \in T(x, y)$. If $z \in S$, then there exists $v \in S$ such that $a = azv = xv$ and $b = bzv = yv$. According to Lemma 1, we have $(a, b) \in T(x, y)$. This gives in both cases $T = T(x, y)$ and so, by Lemma 4, T is an atom of $\mathscr{L}(\mathscr{S})$.

Finally, we shall show that T satisfies (5). Let $(e, c) \in T(a, b)$ for $e \in E(\mathscr{S})$. By way of contradiction, we assume that $e \neq c$. Then, by Lemma 1 and (4), there exists $w \in S^1$ such that either $e = aw$, $c = bw$ or $e = bw$, $c = aw$. We can suppose that $e = aw$, $c = bw$ and $w \in S$. Since $(a, b) \in \mathscr{R}_3(\mathscr{S})$, we have $bw \in E(\mathscr{S})$. Therefore $e = e^2 = a^2w^2 = b^2w^2 = bw = c$, a contradiction.

Define a relation $\mathscr{R}(\mathscr{S})$ on a commutative semigroup $\mathscr{S}$ by $\mathscr{R}(\mathscr{S}) = \mathscr{R}_1(\mathscr{S}) \cup \mathscr{R}_2(\mathscr{S}) \cup \mathscr{R}_3(\mathscr{S})$. The following result we obtain from Theorems 1, 2 and 3.

Theorem 4. *Let T be a tolerance on a commutative semigroup $\mathscr{S}$. Then T is an atom of the lattice $\mathscr{L}(\mathscr{S})$ if and only if $T = T(a, b)$ for some pair $(a, b) \in \mathscr{R}(\mathscr{S})$.*

From this and from Lemma 1 we have

Theorem 5. *The lattice $\mathscr{L}(\mathscr{S})$ of all tolerances on a commutative semigroup $\mathscr{S} = (S, \cdot)$ is atomic if and only if for any pair (a, b) of elements $a, b \in S$, $a \neq b$, there exist $m \in \mathscr{N}$ and $u \in S^1$ such that $(a^m u, b^m u) \in \mathscr{R}(\mathscr{S}) \cup \mathscr{R}^{-1}(\mathscr{S})$.*

II

In this section we shall study some consequences of Theorems 1–5 for regular commutative semigroups and semilattices. Recall that every regular commutative semigroup $\mathscr{S}$ is a semilattice fo commutative groups (see [4]). Denote by $\mathscr{Z}$ the set of all integers. An element z of $\mathcal{S}$ belongs to the maximal subgroup G_e containing an idempotent e if and only if $z^0 = e$. It is known that for elements x, y of $\mathcal{S}$ and for $k \in \mathcal{Z}$ we have

$$(xy)^k = x^k y^k . \tag{6}$$

Proposition 1. *Let $\mathcal{S}$ be a regular commutative semigroup. Then the following conditions are equivalent:*

(i) $(x, y) \in \mathcal{R}_1(\mathcal{S})$;

(ii) $x, y \in E(\mathcal{S})$, $x < y$ *and for any* $z \in E(\mathcal{S})$, $z < y$, *we have* $z \leqq x$.

(iii) $x, y \in E(\mathcal{S})$, $x < y$ *and for any* $z \in E(\mathcal{S})$, $z < y$, *we have* $zx = zy$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $(x, y) \in \mathcal{R}_1(\mathcal{S})$. Then $x, y \in E(\mathcal{S})$ and $x < y$. Let $z < y$ for some $z \in E(\mathcal{S})$. Then $zy = z$. If $xz \neq z$, then there exists $v \in S$ such that $y = yzv \leqq z$, a contradiction. Thus we have $xz = z$, which means $z \leqq x$.

(ii) $\Rightarrow$ (iii). Let $x < y$ for $x, y \in E(\mathcal{S})$. If $z < y$ for some $z \in E(\mathcal{S})$, then $z \leqq x$ and so $zx = z = zy$.

(iii) $\Rightarrow$ (i). Let $x < y$ for $x, y \in E(\mathcal{S})$. This means $x = xy$. Suppose that $xu \neq yu$ for some $u \in S$. Then $yu^0 \leqq y$. If $yu^0 < y$, then, by (iii), we have $(yu^0)\, x = (yu^0)\, y$ and so $xu^0 = xyu^0 = yu^0$. Therefore $xu = xu^0u = yu^0u = yu$, a contradiction. Hence we have $y = yu^0 = yuu^{-1}$. This implies that $(x, y) \in \mathcal{R}_1(\mathcal{S})$.

Proposition 2. *Let $\mathcal{S}$ be a regular commutative semigroup. Then the following conditions are equivalent:*

(i) $(x, y) \in \mathcal{R}_2(\mathcal{S})$;

(ii) $x \in E(\mathcal{S})$, *y is a periodic element of $\mathcal{S}$ such that* $x \in [y]$, *where* $[y]$ *is a subgroup of $\mathcal{S}$ of a prime order, and if* $xz \neq zy$ *for some* $z \in E(\mathcal{S})$, *then* $x \leqq z$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $(x, y) \in \mathcal{R}_2(\mathcal{S})$. Then $x \in E(\mathcal{S})$ and $x \in [y]$, where y is a periodic element of $\mathcal{S}$. Since $\mathcal{S}$ is a union of groups, $[y]$ is a subgroup of $\mathcal{S}$ of a primer order. Assume that $xz \neq yz$ for some $z \in E(\mathcal{S})$. Then there exists $v \in S$ such that $y = yzv$ and so, by (6), we have $x = y^0 = y^0z^0v^0 = xzv^0$. Thus we have $x \leqq z$.

(ii) $\Rightarrow$ (i). Suppose that $x \in E(\mathcal{S})$, y is a periodic element of $\mathcal{S}$ such that $x \in [y]$, where $[y]$ is a subgroup of $\mathcal{S}$ of a prime order. If $xu \neq yu$ for some $u \in S$, then $xu^0 \neq yu^0$. Thus, by hypothesis, we have $x \leqq u^0$. This means that $x = xu^0$ and therefore we obtain $y = yx = yxu^0 = yu^0 = yuu^{-1}$. Hence $(x, y) \in \mathcal{R}_2(\mathcal{S})$.

Proposition 3. *If $\mathcal{S}$ is a regular commutative semigroup, then* $\mathcal{R}_3(\mathcal{S}) = \emptyset$.

Proof. If $(x, y) \in \mathcal{R}_3(\mathcal{S})$, then $x^2 = xy = y^2$ and $x \neq y$. According to (6), we have $x^0 = (x^0)^2 = (y^0)^2 = y^0$. This implies that the elements x, y belong to the same maximal subgroup of $\mathcal{S}$ and so $x = y$, which is a contradiction.

Lemma 5. *Let $\mathcal{S}$ be a regular commutative semigroup. Then* $(x, y) \in \mathcal{R}(\mathcal{S})$ *if and only if* $(x^{-1}, y^{-1}) \in \mathcal{R}(\mathcal{S})$.

Proof. It follows from Propositions 1, 2 and 3.

Theorem 6. *The lattice $\mathscr{L}(\mathscr{S})$ of all tolerances on a regular commutative semigroup $\mathscr{S} = (S, \cdot)$ is atomic if and only if the following conditions are satisfied.*

(i) *for any pair (e, f) of idempotents $e, f \in S$, $e < f$, there exists $g \in E(\mathscr{S})$ such that $(eg, fg) \in \mathscr{R}_1(\mathscr{S})$;*

(ii) *for any pair (e, c) of elements $e \in E(\mathscr{S})$, $c \in S \setminus E(\mathscr{S})$ and $e = c^0$, there exist $m \in \mathscr{N}$ and $g \in E(\mathscr{S})$ such that $(eg, c^m g) \in \mathscr{R}_2(\mathscr{S})$.*

Proof. 1. Suppose that the lattice $\mathscr{L}(\mathscr{S})$ is atomic.

(i) Let $e, f \in E(\mathscr{S})$ and $e < f$. According to Theorem 5, there exists $u \in S^1$ such that $(eu, fu) \in \mathscr{R}(\mathscr{S}) \cup \mathscr{R}^{-1}(\mathscr{S})$. We can suppose that $u \in S$. because if $u \in S^1 \setminus S$, then we put $u = f$. Thus we have $eu \neq fu$. We shall show that $fu \in E(\mathscr{S})$. By way of contradiction, we assume that $fu \notin E(\mathscr{S})$. Then by Propositions 1, 2 and 3 we have $eu = (fu)^0 \in E(\mathscr{S})$. According to (6), we obtain $eu = eu^0 u = (eu)^0 u = (fu)^0 u = fu^0 u = fu$, a contradiction. Hence $fu \in E(\mathscr{S})$. Thus we have $eu = e(fu) \in E(\mathscr{S})$. Put $g = u^0$. Then $eg = eu^0 = (eu)^0 = eu$. Similarly we obtain $fg = fu$. Since $e < f$, we have $eg < fg$ and by Proposition 1 it follows that $(eg, fg) \in \mathscr{R}_1(\mathscr{S})$.

(ii) Let $e \in E(\mathscr{S})$, $c \in S \setminus E(\mathscr{S})$ and $e = c^0$. According to Theorem 5, there exist $m \in \mathscr{N}$ and $u \in S^1$ such that $(eu, c^m u) \in \mathscr{R}(\mathscr{S}) \cup \mathscr{R}^{-1}(\mathscr{S})$. We can suppose that $u \in S$, because if $u \in S^1 \setminus S$, then we put $u = e$. Hence by (6) we have $(eu)^0 = eu^0 = (c^m u)^0$. According to Propositions 1, 2 and 3, we obtain $(eu, c^m u) \in \mathscr{R}_2(\mathscr{S}) \cup \mathscr{R}_2^{-1}(\mathscr{S})$. Put $g = u^0$. We have the following possibilities:

Case 1. $eu \in E(\mathscr{S})$.

It follows from (6) that $eu = (eu)^0 = eu^0 = eg$. Finally, we have $c^m u = c^m eu = c^m eg = c^m g$. This implies that $(eg, c^m g) \in \mathscr{R}_2(\mathscr{S})$.

Case 2. $c^m u \in E(\mathscr{S})$.

By (6) we have $c^m u = (c^m u)^0 = eg$ and $eu = (eu)(c^m u)^{-1} = c^{-m} g$. From this follows that $(c^{-m} g, eg) \in \mathscr{R}_2^{-1}(\mathscr{S})$ and so, by Lemma 5 and (6), we have $(eg, c^m g) \in \mathscr{R}_2(\mathscr{S})$.

2. Let the conditions (i) and (ii) be satisfied. Using Theorem 5 we shall show that the lattice $\mathscr{L}(\mathscr{S})$ is atomic. Let $a, b \in S$ and $a \neq b$. Put $e = a^0$ and $f = b^0$.

Case 1. $af = be$.

Then we have $ab^{-1} = afb^{-1} = ebb^{-1} = ef$. If $e = f$, then $a = ae = af = be = bf = b$, which is a contradiction. Thus we have $e \neq f$. This implies that either $e \neq ef$ or $ef \neq f$. Without loss of generality we can assume that $ef \neq f$ and so $ef < f$. From (i) it follows that there exists $g \in E(\mathscr{S})$ such that $(efg, fg) \in \mathscr{R}_1(\mathscr{S})$. Put $u = b^{-1}g$. Hence we have $au = ab^{-1}g = efg$ and $bu = bb^{-1}g = fg$. Therefore $(au, bu) \in \mathscr{R}_1(\mathscr{S})$.

Case 2. $af \neq be$.

If $ef = a^{-1}b$, then $af = aef = aa^{-1}b = be$, a contradiction. Thus we have $ef \neq a^{-1}b$. Put $c = a^{-1}b$. By (6) we obtain $c^0 = ef$. According to (ii), there exist

$m \in \mathcal{N}$ and $g \in E(\mathcal{S})$ such that $(efg, c^m g) \in \mathcal{R}_2(\mathcal{S})$. Putting $u = a^{-m}fg$ we obtain $a^m u = a^m a^{-m} fg = efg$ and $b^m u = b^m a^{-m} fg = (a^{-1}b)^m g = c^m g$. Therefore $(a^m u, b^m u) \in \mathcal{R}_2(\mathcal{S})$.

Note 1. Let $\mathcal{S}$ be a regular commutative semigroup. The condition (i) of Theorem 6 is satisfied if and only if for any pair (e, f) of idempotents of $\mathcal{S}$, $e < f$, there exists $t \in E(\mathcal{S})$ such that $et < ft$ and $eh = fh$ for all $h \in E(\mathcal{S})$, where $h < t$.

Proof. Suppose that $(eg, fg) \in \mathcal{R}_1(\mathcal{S})$ for $e, f, g \in E(\mathcal{S})$ with $e < f$. Put $t = fg$. If $h < t$, $h \in E(\mathcal{S})$, then $h < g$. According to Proposition 1, we have $h(eg) = h(fg)$ and so $eh = fh$.

Conversely, let $e, f \in E(\mathcal{S})$ and assume that $et < ft$ for some $t \in E(\mathcal{S})$ and $eh = fh$ for all $h \in E(\mathcal{S})$, where $h < t$. We shall show that $(et, ft) \in \mathcal{R}_1(\mathcal{S})$. If $z < ft$ for some $z \in E(\mathcal{S})$, then $z < t$ and so $(et)\, z = ez = fz = (ft)\, z$. According to Proposition 1, we have $(et, ft) \in \mathcal{R}_1(\mathcal{S})$.

Corollary 1. *The lattice $\mathcal{L}(\mathcal{S})$ of all tolerances on a semilattice $\mathcal{S} = (S, \cdot)$ is atomic if and only if for any pair (e, f) of elements $e, f \in S$, $e < f$, there exists $t \in S$ such that $et < ft$ and $eh = fh$ for all $h < t$, $h \in S$.*

Proof. It follows from Theorem 6 and Note 1.

Note 2. See the dual of Theorem 4 of [3].

III

A regular commutative semigroup $\mathcal{S} = (S, \cdot)$ can be found to be an algebra $\mathcal{S}^* = (S, \cdot, ^{-1})$. From (1) it follows that a tolerance T on $\mathcal{S}^*$ is a tolerance on $\mathcal{S}$ satisfying the following implication:

(7) $$\text{If} \quad (a, b) \in T, \quad \text{then} \quad (a^{-1}, b^{-1}) \in T.$$

Let T be a tolerance of $\mathcal{L}(\mathcal{S})$. By T^* we denote the relation on S defined by

(8) $$(a, b) \in T^* \quad \text{if and only if} \quad (a^{-1}, b^{-1}) \in T.$$

Using (6) we can easily show that T^* is a tolerance on $\mathcal{S}$. Further, we can prove that $*$ is an involutional order-automorphism on $\mathcal{L}(\mathcal{S})$. This means that for $T, U \in \mathcal{L}(\mathcal{S})$ we have

(9) $$T \subseteq U \Rightarrow T^* \subseteq U^*$$

and

(10) $$(T^*)^* = T.$$

From (7) and (8) it follows that

$$\mathcal{L}(\mathcal{S}^*) = \{T \in \mathcal{L}(\mathcal{S});\ T = T^*\}.$$

If $a, b \in S$, we denote by $I(a, b)$ the least tolerance on $\mathscr{S}^* = (S, \cdot, {}^{-1})$ containing the pair (a, b). It is clear that

$$T(a, b) \subseteq I(a, b)\,. \tag{11}$$

Lemma 6. *Let $\mathscr{S} = (S, \cdot)$ be a regular commutative semigroup and $a, b \in S$. Then $(x, y) \in I(a, b)$ for $x \neq y$ if and only if there exist $k \in \mathscr{Z}$ and $u \in S^1$ such that either*

$$x = a^k u\,, \quad y = b^k u$$

or

$$x = b^k u\,, \quad y = a^k u\,.$$

Proof. Apply (2) and (7).

Theorem 7. *Let $\mathscr{S}$ be a regular commutative semigroup. $\mathscr{L}(\mathscr{S}^*)$ is a complete sublattice of $\mathscr{L}(\mathscr{S})$. Moreover the following conditions are equivalent:*

(i) $\mathscr{L}(\mathscr{S}^*) = \mathscr{L}(\mathscr{S})$;

(ii) *$I(a, b) = T(a, b)$ for all elements a, b of $\mathscr{S}$, $a \neq b$;*

(iii) *$\mathscr{S}$ is either periodic or $E(\mathscr{S})$ contains the greatest element f and the maximal subgroup G_e of $\mathscr{S}$ is periodic for each $e < f$.*

Proof. (i) $\Rightarrow$ (ii). Evident.

(ii) $\Rightarrow$ (iii). Suppose the condition (ii) is satisfied. We shall prove the following implication:

(12) If an element x of $\mathscr{S}$ is not periodic, then x^0 is the greatest element in $E(\mathscr{S})$.

Let $x \in S$ and $\mathscr{S} = (S, \cdot)$. Suppose that x is not periodic. For an arbitrary idempotent e of $\mathscr{S}$, by Lemma 6, we have $(e, x^{-1}) \in I(e, x) = T(e, x)$. According to Lemma 1, there exist $u \in S^1$ and $m \in \mathscr{N}$ such that either

$$e = eu \quad \text{and} \quad x^{-1} = x^m u \tag{13}$$

or

$$e = x^m u \quad \text{and} \quad x^{-1} = eu\,. \tag{14}$$

First, we shall show that $f = x^0$ is a maximal element in $E(\mathscr{S})$. Suppose that $f \leqq e$ for some $e \in E(\mathscr{S})$. If (13) is satisfied, then $x^{-1} = x^m u = x^m f u = x^m f e u = x^m f e = x^m f = x^m$ and so x is a periodic element, which is a contradiction. There holds (14) and so $f = fe = f x^m u = x^m u = e$.

Now, we shall prove that f is the greatest element in $E(\mathscr{S})$. Let e be an arbitrary idempotent of $\mathscr{S}$. Suppose that (13) holds. It is clear that $u \in S$. Put $g = u^0$. Then, by (6), we have $e = (eu)^0 = eg$ and $f = (x^{-1})^0 = (x^m u)^0 = fg$. This means that $e \leqq g$ and $f \leqq g$. The idempotent f is maximal in $E(\mathscr{S})$, hence $f = g$. Thus we have $e \leqq f$. If (14) is satisfied, then $ef = x^m u f = x^m u = e$. Therefore we have $e \leqq f$.

The rest of the proof follows immediately from (12).

(iii) $\Rightarrow$ (i). Suppose that $\mathscr{S}$ satisfies the condition (iii). To prove (i) it suffices to show $\mathscr{L}(\mathscr{S}) \subseteq \mathscr{L}(\mathscr{S}^*)$. This means that every tolerance $T \in \mathscr{L}(\mathscr{S})$ fulfils the implication (7). Let $(x, y) \in T$. Put $e = x^0$ and $g = y^0$.

Case 1. Let $e = g$. Then $(x, y) \in T$ implies $(y^{-1}xx^{-1}, y^{-1}yx^{-1}) \in T$. Thus we have $(y^{-1}, x^{-1}) \in T$, hence $(x^{-1}, y^{-1}) \in T$.

Case 2. Let $e \neq g$.

Subcase 2a. The elements x, y are periodic. Then there exists $m \in \mathscr{N}$, $m > 1$, such that $x^m = e$ and $y^m = g$. According to (2), we have $(x^{-1}, y^{-1}) = (x^{m-1}, y^{m-1}) \in T$.

Subcase 2b. One of the elements x, y is not periodic. Without loss of generality we can suppose that x is not periodic. Then $g < e$ and so $ge = g$. This implies that yx^{-1} belongs to the maximal subgroup G_g. Hence yx^{-1} is a periodic element. There exists $m \in \mathscr{N}$ such that $xy^{-1} = (yx^{-1})^{-1} = (yx^{-1})^m$. Then, by (2), $(x, y) \in T$ implies that $(e, yx^{-1}) = (xx^{-1}, yx^{-1}) \in T$ and so $(e, xy^{-1}) = (e^m, (yx^{-1})^m) \in T$. Since $eg = g$, we have $y^{-1} = gy^{-1} = egy^{-1} = ey^{-1} = x^{-1}(xy^{-1})$ and thus, by (2), we obtain $(x^{-1}, y^{-1}) = (x^{-1}e, x^{-1}(xy^{-1})) \in T$.

Using the same method of proof as in Lemma 4, we obtain:

Lemma 7. *Let $I \neq \mathrm{id}_S$ be a tolerance on a regular commutative semigroup $\mathscr{S}^* = (\mathscr{S}, \cdot, ^{-1})$. Then I is an atom of $\mathscr{L}(\mathscr{S}^*)$ if and only if $I = I(x, y)$ for any pair $(x, y) \in I$, $x \neq y$.*

Theorem 8. *Let $\mathscr{S}$ be a regular commutative semigroup. Then the atoms of $\mathscr{L}(\mathscr{S})$ and of $\mathscr{L}(\mathscr{S}^*)$ coincide.*

Proof. Let $\mathscr{S} = (S, \cdot)$ be a regular commutative semigroup. Let T be an atom in $\mathscr{L}(\mathscr{S})$. From Theorem 4 it follows that $T = T(a, b)$ for some pair $(a, b) \in \mathscr{R}(\mathscr{S})$. According to Propositions 1, 2 and 3, the elements a, b are periodic. By Lemma 1 and Lemma 6, we have $T(a, b) = I(a, b)$. Therefore $T \in \mathscr{L}(\mathscr{S}^*)$. From Theorem 7 it follows that T is an atom in $\mathscr{L}(\mathscr{S}^*)$.

Let I be an atom in $\mathscr{L}(\mathscr{S}^*)$. We shall show that I is an atom in $\mathscr{L}(\mathscr{S})$.

Case 1. Suppose that there exist $e, f \in E(\mathscr{S})$ such that $(e, f) \in I$ and $e < f$. Since I is an atom in $\mathscr{L}(\mathscr{S}^*)$, by Lemma 7 we have $I = I(e, f)$. Let $(x, y) \in I$ for some $x, y \in S$ and $x \neq y$. According to Lemma 7, we obtain $I(x, y) = I(e, f)$. Without loss of generality, by Lemma 6, we can suppose that $e = x^k u$, $f = y^k u$ for some $k \in \mathscr{Z}$ and some $u \in S^1$. We shall show that $(e, f) \in T(x, y)$. If $k > 0$, then, by Lemma 1, we have $(e, f) \in T(x, y)$. If $k < 0$, then according to (6), we obtain $e = x^{-k}u^{-1}$, $f = y^{-k}u^{-1}$ and so $(e, f) \in T(x, y)$. Suppose that $k = 0$. Then $e = x^0 u$ and $f = y^0 u$. By (6) we have $e = x^0 u^0$, $f = y^0 u^0$ and so $e \leqq x^0$, $f \leqq y^0$. Since $(x, y) \in I(e, f)$, according to Lemma 6, we have either $x = ev$, $y = fv$ or $x = fv$, $y = ev$ for some $v \in S^1$. If $y = ev$, then, by (6), we have $y^0 = ev^0$ and so $y^0 \leqq e$. This implies that $f \leqq e$, which is a contradiction. Hence we have $x = ev$ and $y = fv$. From (6) it follows that $x^0 = ev^0$, $y^0 = fv^0$ and so $x^0 \leqq e$, $y^0 \leqq f$. Therefore $x^0 = e$ and $y^0 = f$. Further, we have $xv^{-1} = evv^{-1} = ev^0 = x^0 = e$ and $yv^{-1} = fvv^{-1} = fv^0 =$

$= y^0 = f$. According to Lemma 1, we obtain $(e, f) = (xv^{-1}, yv^{-1}) \in T(x, y)$. Using Lemma 1 and (11), we get $I = T(e, f) \subseteq T(x, y) \subseteq I(x, y) = I$ and so $I = T(x, y)$ for every pair $(x, y) \in I$, $x \neq y$. From Lemma 4 it follows that I is an atom in $\mathscr{L}(\mathscr{S})$.

Case 2. Suppose that the following implication is true:

(15) If $(e, f) \in I$, $e, f \in E(\mathscr{S})$ and $e \leqq f$, then $e = f$.

Since $\mathrm{id}_S \neq I$, there exist $a, b \in S$ such that $(a, b) \in I$ and $a \neq b$. Put $a^0 = e$ and $b^0 = f$. From (7) and (2) it follows that $(e, f) = (aa^{-1}, bb^{-1}) \in I$ and so $(e, ef) \in I$ and $(ef, f) \in I$. According to (15), we have $e = ef = f$. Put $c = ba^{-1}$. Since $a \neq b$, we have $e \neq c$. By (2), we obtain $(e, c) = (aa^{-1}, ba^{-1}) \in I$. We shall show that c is periodic. By way of contradiction, we assume that c is not periodic. Then $e \neq c^2$. From (2) it follows that $(e, c^2) \in I$. Since I is an atom in $\mathscr{L}(\mathscr{S}^*)$, we have $I(e, c^2) = I = I(e, c)$. According to Lemma 6, there exist $u \in S^1$ and $k \in \mathscr{Z}$ such that either $e = eu$, $c = c^{2k}u$ or $e = c^{2k}u$, $c = eu$. Hence we have either $c = c^{2k}$ or $e = c^{2k+1}$, which is a contradiction in both cases. Therefore the element c is periodic.

Now, we shall prove that $I = I(e, c)$ is an atom in $\mathscr{L}(\mathscr{S})$. Let $(x, y) \in I$ for some $x, y \in S$ and $x \neq y$. Using the same method of proof as at the beginning of Case 2 we obtain $x^0 = y^0$. Since I is an atom in $\mathscr{L}(\mathscr{S}^*)$, according to Lemma 7, we have $I(e, c) = I = I(x, y)$. Lemma 6 implies that there exist $k \in \mathscr{Z}$ and $u \in S^1$ such that either $e = x^k u$, $c = y^k u$ or $e = y^k u$, $c = x^k u$. Without loss of generality we can suppose that $e = x^k u$, $c = y^k u$. Since $e \neq c$, we have $k \neq 0$. If $k > 0$, then from Lemma 1 it follows that $(e, c) \in T(x, y)$. If $k < 0$, then, by (6), we obtain $e = x^{-k}u^{-1}$, $c^{-1} = y^{-k}u^{-1}$ and so $(e, c^{-1}) \in T(x, y)$. According to Lemma 1, Lemma 6 and (11), we have $I = I(e, c) \subseteq T(e, c) \cap T(e, c^{-1}) \subseteq T(x, y) \subseteq I(x, y) = I$, because c is a periodic element of $\mathscr{S}$. Hence $I = T(x, y)$ for every pair $(x, y) \in I$, $x \neq y$. Lemma 4 implies that I in an atom is $\mathscr{L}(\mathscr{S})$.

Theorem 9. *Let $\mathscr{S}$ be a regular commutative semigroup. Then the lattice $\mathscr{L}(\mathscr{S})$ is atomic if and only if the lattice $\mathscr{L}(\mathscr{S}^*)$ is atomic.*

Proof. Let $\mathscr{S} = (S, \cdot)$ be a regular commutative semigroup. Suppose that the lattice $\mathscr{L}(\mathscr{S})$ is atomic. From Theorem 7 and Theorem 8 it follows that the lattice $\mathscr{L}(\mathscr{S}^*)$ is atomic.

Now, assume that the lattice $\mathscr{L}(\mathscr{S}^*)$ is atomic. Using Theorem 6 we shall show that the lattice $\mathscr{L}(\mathscr{S})$ is atomic.

Let $e, f \in E(\mathscr{S})$, $e < f$. According to Theorem 8 and Theorem 4, there exists a pair $(a, b) \in \mathscr{R}(\mathscr{S})$ such that $(a, b) \in I(e, f)$. Without loss of generality we can suppose (by Lemma 6) that $a = eu$, $b = fu$ for some $u \in S^1$. If $eu^0 = fu^0$, then $a = eu^0u = fu^0u = b$, which is a contradiction. Thus, by (6), we have $a^0 = eu^0 \neq fu^0 = b^0$ and so according to Propositions 1, 2 and 3, we obtain that $(a, b) \in \mathscr{R}_1(\mathscr{S})$. Therefore $(eu^0, fu^0) = (a^0, b^0) = (a, b) \in \mathscr{R}_1(\mathscr{S})$.

Let $e \in E(\mathscr{S})$, $c \in S \setminus E(\mathscr{S})$ and $c^0 = e$. According to Theorem 8 and Theorem 4, there exists a pair $(a, b) \in \mathscr{R}(\mathscr{S})$ such that $(a, b) \in I(e, c)$. From Lemma 6 it follows that either $a = eu$, $b = c^k u$ or $a = c^k u$, $b = eu$ for some $u \in S^1$ and some $k \in \mathscr{Z}$, $k \neq 0$. Using (6) we have $a^0 = eu^0 = b^0$ and so, by Propositions 1, 2 and 3 we obtain that $(a, b) \in \mathscr{R}_2(\mathscr{S})$. Thus we have $a = a^0$ and $ab = b$.

Case 1. Suppose that $a = eu$ and $b = c^k u$. Then $a = e(eu) = ea$ and $b = ab = (ea)(c^k u) = c^k a^2 = c^k a$. If $k > 0$, then $(ea, c^k a) = (a, b) \in \mathscr{R}_2(\mathscr{S})$. If $k < 0$, then, by Lemma 5 and (6), we have $(ea, c^{-k} a) = (a, b^{-1}) \in \mathscr{R}_2(\mathscr{S})$.

Case 2. Suppose that $a = c^k u$ and $b = eu$. Then $a = (c^k u) a = (c^k u)(eu^0) = e(c^k u) = ea$ and by (6) we have $b = ab = a^{-1} b = (c^{-k} u^{-1})(eu) = c^{-k} eu^0 = c^{-k} a$. If $k > 0$, then according to Lemma 5 and (6), we have $(ea, c^k a) = (a, b^{-1}) \in \mathscr{R}_2(\mathscr{S})$. If $k < 0$, then $(ea, c^{-k} a) = (a, b) \in \mathscr{R}_2(\mathscr{S})$.

Recall that a tolerance T on an algebra $\mathscr{A}$ is a congruence on $\mathscr{A}$ if and only if it is transitive. By $\mathscr{C}(\mathscr{A})$ we denote the lattice of all congruences on $\mathscr{A}$. It is well known (see Theorem 7.36 of [4]) that $\mathscr{C}(\mathscr{S}) = \mathscr{C}(\mathscr{S}^*)$ for every regular commutative semigroup. This implies that $\mathscr{C}(\mathscr{S}) \subseteq \mathscr{L}(\mathscr{S}^*) \subseteq \mathscr{L}(\mathscr{S})$ for every regular commutative semigroup. Further, it is known (see [5] and [6]) that for a commutative semigroup $\mathscr{S}$ with at least three elements we have $\mathscr{C}(\mathscr{S}) = \mathscr{L}(\mathscr{S})$ if and only if $\mathscr{S}$ is a group. Now, we shall consider the case when $\mathscr{C}(\mathscr{S}) = \mathscr{L}(\mathscr{S}^*)$.

Theorem 10. *Let $\mathscr{S}$ be a regular commutative semigroup with at least there elements. Then the following conditions on $\mathscr{S}$ are equivalent:*

(i) $\mathscr{C}(\mathscr{S}) = \mathscr{L}(\mathscr{S})$;
(ii) $\mathscr{C}(\mathscr{S}) = \mathscr{L}(\mathscr{S}^*)$;
(iii) *$\mathscr{S}$ is a group.*

Proof. (i) $\Rightarrow$ (ii). Evident.

(ii) $\Rightarrow$)(iii). Let $\mathscr{S} = (S, \cdot)$ be a regular commutative semigroup. First, we shall prove the following implication:

(16) If in $\mathscr{S}$ there exist elements e, a, b such that $e \in E(\mathscr{S})$, $e < a^0$, $e < b^0$ and $a \neq b$, then $\mathscr{C}(\mathscr{S}) \neq \mathscr{L}(\mathscr{S}^*)$.

Indeed, we define a relation I on S as follows: $(x, y) \in I$ if and only if either $x = y$ or $x^0 \leqq e$ or $y^0 \leqq e$. It is easy to show that I is a tolerance on $\mathscr{S}^*$. Clearly $(a, e) \in I$, $(e, b) \in I$ and $(a, b) \notin I$. Hence I is not transitive and so $I \in \mathscr{L}(\mathscr{S}^*) \setminus \mathscr{C}(\mathscr{S})$.

Now, we can prove the implication (ii) $\Rightarrow$ (iii). By way of contradiction, we assume that a regular commutative semigroup $\mathscr{S}$ (with card $S \geqq 3$) is no group and satisfies the condition $\mathscr{C}(\mathscr{S}) = \mathscr{L}(S^*)$.

Case 1. card $E(\mathscr{S}) \geqq 3$. It can be shown that there exist $e, f, g \in E(\mathscr{S})$ such that $e < f$, $e < g$ and $f \neq g$. From (16) it follows that $\mathscr{C}(\mathscr{S}) \neq \mathscr{L}(\mathscr{S}^*)$, which is a contradiction.

Case 2. card $E(\mathscr{S}) = 2$. Then $E(\mathscr{S}) = \{e, f\}$, where $e < f$.

Subcase 2a. There exists $c \in S$ such that $c \neq f$ and $c^0 = f$. According to (16), we have $\mathscr{C}(\mathscr{S}) \neq \mathscr{L}(\mathscr{S}^*)$, a contradiction.

Subcase 2b. For every $z \in S$, $z^0 = f$, we have $z = f$. Define a relation I on $\mathscr{S}$ as follows: $(x, y) \in I$ if and only if either $x^0 = y^0$ or $(x, y) = (e, f)$ or $(x, y) = (f, e)$. It is easy to show that I is a tolerance on $\mathscr{S}^*$ and so, by hypothesis, I is a congruence on $\mathscr{S}$. Since card $S \geqq 3$, there exists $c \in S$, $c^0 = e$ and $c \neq e$. We have $(c, e) \in I$, $(e, f) \in I$ and $(c, f) \notin I$. Hence I is not transitive, which is a contradiction.

(iii) $\Rightarrow$ (i). This is well known (see [5]).

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