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ON CONGRUENCE RELATIONS OF MONOUNARY ALGEBRAS II

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This paper is a continuation of [6] and [7]. Congruence relations of monounary algebras were investigated in [1]–[4], [9], [10]; for references and basic notions cf. also [8], [11] and [12].

We shall use the denotations introduced in [7] (cf. mainly § 1 of [7]). Let us recall the following ones: For a nonempty set A we denote by $F(A)$ and $F_p(A)$ the set of all unary or of all partial unary operations on A, respectively. Let $E(A)$ be the system of all equivalence relations on A. For $f \in F_p(A)$ let $\mathrm{Con}\,(A, f)$ be the set of all congruence relations of (A, f).

Let $f \in F(A)$, $f_1 \in F_p(A)$. We put

$$R(f) = \{g \in F(A): \mathrm{Con}\,(A, f) = \mathrm{Con}\,(A, g)\},$$
$$R_p(f_1) = \{g \in F_p(A): \mathrm{Con}\,(A, f_1) = \mathrm{Con}\,(A, g)\}.$$

In paragraphs 5–7 of the present paper we assume that we are given a mapping $f \in F(A)$ having a cycle C with $\mathrm{card}\, C > 2$. There are investigated those properties of mappings $g \in R(f)$ which depend on the set $\mathrm{Con}\,(A, f)$ only. (An exception is Propos. 7.6.)

In § 7 and § 9 the following estimate is established (cf. 7.9 and 9.2):

(i) *Let* $f \in F(A)$, $f_1 \in F_p(A)$. *Suppose that* $\mathrm{Con}\,(A, f) \neq E(A) \neq \mathrm{Con}\,(A, f_1)$. *Then* $\mathrm{card}\, R(f) \leqq c$ *and* $\mathrm{card}\, R_p(f_1) \leqq c$ (*independently of the cardinality of the set* A).

(ii) *For each infinite set A there exists* $f \in F(A)$ *with* $\mathrm{card}\, R(f) = c$ (*i.e., the estimate given in* (i) *cannot be sharpened*).

5. THE CASE OF A SINGLE CYCLE

The numeration of paragraphs, lemmas and theorems continues that applied in [7]. The assertions from [7] are quoted by their numbers; e.g., Lemma 3.1 means Lemma 3.1 of [7].

Let (A, f) be a monounary algebra. As we already remarked, in the paragraphs 5–7 we shall assume that there exists a subset $C \subseteq A$ such that $\mathrm{card}\, C > 2$ and that C

is a cycle of the algebra (A, f) (from Lemma 3.1 it follows that C can be determined by means of congruence relations of (A, f)). We take a fixed cycle C. Further we assume that each connected component of (A, f) has a cycle (according to the results from § 2, this case can be described by means of $\operatorname{Con}(A, f)$).

Let us recall that a cycle C' of (A, f) is said to be large (small), if $\operatorname{card} C' > 2$ ($\operatorname{card} C' \leqq 2$). For a unary operation g on A with $\operatorname{Con}(A, g) = \operatorname{Con}(A, f)$ we shall write $g \in R(f)$ or $g \sim f$. If (B, f) is a subalgebra of (A, f), (B, g) is a subalgebra of (A, g) and $\operatorname{Con}(B, g) = \operatorname{Con}(B, f)$, then we denote this situation by writing $g \sim f$ on B.

In this paragraph we suppose that $A = C$, $\operatorname{card} C = n = p_1^{\alpha_1} \dots p_k^{\alpha_k} > 2$ $(k \in N$, $p_1, \dots, p_k$ being distinct primes, $\alpha_1, \dots, \alpha_k \in N)$. We shall use the following result from [1]:

(T) *Let* card $A = n$, $x \in A$ *and suppose that* A *is a cycle with respect to an operation* f. *Then*

$\operatorname{Con}(A, f) = \{\Theta_d^f = \Theta^f(x, f^d(x))$: d divides $n\}$.

5.0. *Let us remark that for each* $x, y \in A$ *we have* $\Theta^f(x, f^d(x)) = \Theta^f(y, f^d(y))$, *hence the symbol* Θ_d^f *in* (T) *does not depend of the particular choice of* x. *Further, if* $\operatorname{Con}(A, f) = \operatorname{Con}(A, g)$, *then according to the number of classes in the partitions corresponding to the congruence relations it follows that* $\Theta_d^f = \Theta_d^g$ *for each* $d \in N$ *dividing* n.

We shall often use the fact that if $x \in A$, $m, m' \in N$ and if d is a divisor of $n = \operatorname{card} A$, then

$$f^m(x)\, \Theta_d^f\, f^{m'}(x) \Leftrightarrow m \equiv m' \pmod d .$$

In 5.1 we shall construct (by using the system $\operatorname{Con}(A, f)$, but without using explicitly the operation f) a new operation f_0 on A such that $\operatorname{Con}(A, f) = \operatorname{Con}(A, f_0)$.

5.1. Construction. According to (T) we get that $\operatorname{Con}(A, f) = \{\Theta_{p_1^{\beta_1} \dots p_k^{\beta_k}} : 0 \leqq \beta_1 \leqq \alpha_1, \dots, 0 \leqq \beta_k \leqq \alpha_k\}$. Let $i \in \{1, \dots, k\}$. The partition of the set A corresponding to the congruence relation θ_{p_i} consists of p_i classes; denote these classes by the symbols $T(i; 0)$, $T(i; 1)$, …, $T(i; p_i - 1)$. Using this notation we do not take into consideration the operation f, i.e., there is taken an arbitrary bijection of the set $\{0, 1, \dots, p_i - 1\}$ onto the system of classes corresponding to Θ_{p_i}. An analogous situation occurs also in further steps of the construction. Our aim is to verify that distinct bijections give all elements of $R(f)$.

Now consider the congruence relation $\Theta_{p_i^2}$. Each of the classes $T(i; j)$ $(j \in \{0, \dots, p_i - 1\})$ is a union of p_i classes of the partition corresponding to $\Theta_{p_i^2}$; denote them $T(i; j, 0), \dots, T(i; j, p_i - 1)$. We proceed analogously for the congruence relations $\Theta_{p_i^3}, \dots, \Theta_{p_i^{\alpha_i}}$. Hence for each $i \in \{1, \dots, k\}$ and each $\beta_i \in \{1, \dots, \alpha_i\}$ there are classes $T(i; j_{i1}, j_{i2}, \dots, j_{i\beta_i})$ where each of the symbols $j_{i1}, \dots, j_{i\beta_i}$ runs over the set $\{0, \dots, p_i - 1\}$.

Let $x \in A$. For each $i \in \{1, \ldots, k\}$ there are uniquely determined numbers $j_{i1}, \ldots, j_{i\alpha_i} \in \{0, \ldots, p_i - 1\}$ such that

$$\{x\} = \bigcap_{i=1}^{k} T(i; j_{i1}, \ldots, j_{i\alpha_i}) .$$

Further, for each $i \in \{1, \ldots, k\}$ there exist uniquely determined numbers $s_{i1}, \ldots, s_{i\alpha_i} \in \{0, \ldots, p_i - 1\}$ and $y \in A$ such that

$$\{y\} = \bigcap_{i=1}^{k} T(i; s_{i1}, \ldots, s_{i\alpha_i})$$

and

$$j_{i1} + j_{i2}p_i + \ldots + j_{i\alpha_i}p_i^{\alpha_i - 1} + 1 \equiv s_{i1} + s_{i2}p_i + \ldots + s_{i\alpha_i}p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}}.$$

Now define an operation f_0 on A by putting $f_0(x) = y$. The operation f_0 was defined only by means of the congruence relations of (A, f), without using explicitly the operation f.

5.2. Lemma. *A is a cycle with respect to the operation f_0.*

Proof. From the definition of f_0 it follows that $f_0^n(x) = x$ for each $x \in A$, since

$$j_{i1} + j_{i2}p_i + \ldots + j_{i\alpha_i}p_i^{\alpha_i - 1} + n =$$
$$= j_{i1} + j_{i2}p_i + \ldots + j_{i\alpha_i}p_i^{\alpha_i - 1} + p_1^{\alpha_1} \ldots p_k^{\alpha_k} \equiv$$
$$\equiv j_{i1} + j_{i2}p_i + \ldots + j_{i\alpha_i}p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}} .$$

If $m \in N$, $m < n$, then for each $i \in \{1, \ldots, k\}$ there are uniquely determined numbers $m_{i1}, \ldots, m_{i\alpha_i} \in \{0, \ldots, p_i - 1\}$ and $m_i \in N \cup \{0\}$ such that

$$m = m_{i1} + m_{i2}p_i + \ldots + m_{i\alpha_i}p_i^{\alpha_i - 1} + m_i p_i^{\alpha_i}$$

and we obtain

$$j_{i1} + j_{i2}p_i + \ldots + j_{i\alpha_i}p_i^{\alpha_i - 1} + m \equiv$$
$$\equiv (j_{i1} + m_{i1}) + (j_{i2} + m_{i2})\, p_i + \ldots + (j_{i\alpha_i} + m_{i\alpha_i})\, p_i^{\alpha_i - 1} \equiv$$
$$\equiv s_{i1} + s_{i2}p_i + \ldots + s_{i\alpha_i}p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}} ,$$

where $s_{i1}, \ldots, s_{i\alpha_i} \in \{0, \ldots, p_i - 1\}$ are uniquely determined. Suppose that $f_0^m(x) = x$. Then $m_{i1} = m_{i2} = \ldots = m_{i\alpha_i} = 0$, hence $m \equiv 0 \pmod{p_i^{\alpha_i}}$ for each $i \in \{1, \ldots, k\}$, which is a contradiction with the relation $0 < m < n$. Thus A is a cycle with respect to f_0.

5.3. Lemma. Con (A, f) = Con (A, f_0).

Proof. We shall prove that $\Theta_d^f = \Theta_d^{f_0}$ for each $d \in N$ such that d/n. Let $d = p_1^{\beta_1} \ldots p_k^{\beta_k}$, $0 \leqq \beta_1 \leqq \alpha_1, \ldots, 0 \leqq \beta_k \leqq \alpha_k$ and let $x, z \in A$. Then, for each $i \in \{1, \ldots, k\}$ there are uniquely determined numbers $j_{i1}, \ldots, j_{i\alpha_i}, s_{i1}, \ldots, s_{i\alpha_i} \in$

$\in \{0, \ldots, p_i - 1\}$ such that

$$\{x\} = \bigcap_{i=1}^{k} T(i; j_{i1}, \ldots, j_{i\alpha_i}), \quad \{z\} = \bigcap_{i=1}^{k} T(i; s_{i1}, \ldots, s_{i\alpha_i}).$$

First suppose that $x\, \Theta_d^{f_0}\, z$. Then there is $m \in N \cup \{0\}$, $0 \leqq m < n/d$, with $z = f_0^{md}(x)$. For each $i \in \{1, \ldots, k\}$ there exist uniquely determined numbers $m_{i1}, \ldots \ldots, m_{i(\alpha_i - \beta_i)} \in \{0, \ldots, p_i - 1\}$ and $m_i \in N \cup \{0\}$ such that

$$m = m_{i1} + m_{i2} p_i + \ldots + m_{i(\alpha_i - \beta_i)} p_i^{\alpha_i - \beta_i - 1} + m_i p_i^{\alpha_i - \beta_i}. \tag{1}$$

Let $i = \{1, \ldots, k\}$. Hence from the definition of the operation f_0 we get

$$j_{i1} + j_{i2} p_i + \ldots + j_{i\alpha_i} p_i^{\alpha_i - 1} + (m_{i1} + m_{i2} p_i + \ldots$$
$$\ldots + m_{i(\alpha_i - \beta_i)} p_i^{\alpha_i - \beta_i - 1} + m_i p_i^{\alpha_i - \beta_i})\, p_1^{\beta_1} \ldots p_k^{\beta_k} \equiv$$
$$\equiv s_{i1} + s_{i2} p_i + \ldots + s_{i\alpha_i} p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}},$$

i.e.,

$$j_{i1} + \ldots + j_{i\beta_i} p_i^{\beta_i - 1} + j_{i(\beta_i + 1)} + \frac{m_{i1} d}{p_i^{\beta_i}} p_i^{\beta_i} + \ldots + j_{i\alpha_i} + \tag{2}$$

$$+ \frac{m_{i(\alpha_i - \beta_i)} d}{p_i^{\beta_i}} p_i^{\alpha_i - 1} \equiv s_{i1} + \ldots + s_{i\beta_i} p_i^{\beta_i - 1} + \ldots + s_{i\alpha_i} p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}},$$

which implies $j_{i1} = s_{i1}, \ldots, j_{i\beta_i} = s_{i\beta_i}$. Since $\Theta_d^f = \bigcap_{i=1}^{k} \Theta_{p_i \beta_i}^f$ and the partition of $\Theta_{p_i \beta_i}^f$ possesses classes $T(i; r_{i1}, \ldots, r_{i\beta_i})$ $(r_{i1}, \ldots, r_{i\beta_i} \in \{0, \ldots, p_i - 1\}$ and since

$$\{x\} = \bigcap_{i=1}^{k} T(i; j_{i1}, \ldots, j_{i\alpha_i}) \subseteq \bigcap_{i=1}^{k} T(i; j_{i1}, \ldots, j_{i\beta_i}),$$

$$\{z\} = \bigcap_{i=1}^{k} T(i; s_{i1}, \ldots, s_{i\alpha_i}) \subseteq \bigcap_{i=1}^{k} T(i; j_{i1}, \ldots, j_{i\beta_i}),$$

we have $x\, \Theta_{p_i \beta_i}^f\, z$ for each $i \in \{1, \ldots, k\}$, i.e., $x\, \Theta_d^f\, z$.

Now let $x\, \Theta_d^f\, z$. Then $x\, \Theta_{p_i \beta_i}^f\, z$ for each $i \in \{1, \ldots, k\}$ and hence $j_{i1} = s_{i1}, \ldots \ldots, j_{i\beta_i} = s_{i\beta_i}$ for each $i \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$ there exist numbers $m_{i1}, \ldots, m_{i(\alpha_i - \beta_i)} \in \{0, \ldots, p_i - 1\}$ such that the relation (2) is valid. Further there exist $m_1, \ldots, m_k \in N \cup \{0\}$ and $m \in N$ such that (1) holds. Then $z = f_0^{md}(x)$, i.e., $x\, \Theta_d^{f_0}\, z$.

5.4. Theorem. *Let A be a cycle with* card $A = n = p_1^{\alpha_1} \ldots p_k^{\alpha_k} > 2$, *where* k, $\alpha_1, \ldots, \alpha_k \in N$, $p_1, \ldots, p_k$ *being distinct primes. Further let g be a unary operation defined on A such that A is a cycle with respect to g. The following conditions are equivalent*:

(1) Con (A, f) = Con (A, g).

(2) *The operation g is constructed by means of the construction* 5.1.

Proof. From 5.3 it follows that (2) $\Rightarrow$ (1). Let (1) be valid and let $a \in A$. We have $\mathrm{Con}(A, f) = \mathrm{Con}(A, g) = \{\Theta_{p_1^{\beta_1} \dots p_k^{\beta_k}} : 0 \leqq \beta_1 \leqq \alpha_1, \dots, 0 \leqq \beta_k \leqq \alpha_k\}$. (According to 5.0 it is not necessary to write $\Theta^f_{p_1^{\beta_1} \dots p_k^{\beta_k}}$ or $\Theta^g_{p_1^{\beta_1} \dots p_k^{\beta_k}}$.) Let $i \in \{1, \dots, k\}$. The elements $a, g(a), \dots, g^{p_i - 1}(a)$ belong to distinct classes of the partition corresponding to the congruence relation Θ_{p_i}; these classes will be denoted $T(i; 0), \dots, T(i; p_i - 1)$. Let $j \in \{0, \dots, p_i - 1\}$. The elemetns $g^j(a)$, $g^{j+p_i}(a), \dots, g^{j+(p_i-1)p_i}(a)$ belong to $T(i; j)$ and to distinct classes of the partition corresponding to $\Theta_{p_i^2}$; denote these classes $T(i; j, 0), \dots, T(i; j, p_i - 1)$. Let $1 < \beta_i \leqq \alpha_i$ and suppose that we have already defined classes $T(i; j_{i1}, \dots, j_{i\beta_i})$ $(j_{i1}, \dots, j_{i\beta_i} \in \{0, \dots, p_i - 1\})$. If $j_{i1}, \dots, j_{i\beta_i} \in \in \{0, \dots, p_{i-1}\}$ and if we denote $l = j_{i1} + j_{i2}p_i + \dots + j_{i\beta_i}p_i^{\beta_i - 1}$, then the elements $g^l(a), g^{l+p_i^{\beta_i}}(a), g^{l+2p_i^{\beta_i}}(a), \dots, g^{l+(p_i-1)p_i^{\beta_i}}(a)$ belong to $T(i; j_{i1}, \dots, j_{i\beta_i})$ and they belong to distinct classes of the partition corresponding to $\Theta_{p_i^{\beta_i}}$; denote these classes $T(i; j_{i1}, \dots, j_{i\beta_i}, 0), \dots, T(i; j_{i1}, \dots, j_{i\beta_i}, p_i - 1)$.

Let $x \in A$. Then $x = g^j(a)$ for some $j \in N$, $0 \leqq j < n$. Then for each $i \in \{1, \dots, k\}$ there exist uniquely determined numbers $j_{i1}, \dots, j_{i\alpha_i} \in \{0, \dots, p_i - 1\}$ and $j_i \in N \cup \cup \{0\}$ such that

$$j = j_{i1} + j_{i2}p_i + \dots + j_{i\alpha_i}p_i^{\alpha_i - 1} + j_i p_i^{\alpha_i}.$$

From the definition of the corresponding classes we obtain that

$$x \in T(i; j_{i1}, \dots, j_{i\alpha_i}) \quad \text{for each} \quad i \in \{1, \dots, k\},$$

i.e.,

$$\{x\} = \bigcap_{i=1}^{k} T(i; j_{i1}, \dots, j_{i\alpha_i}).$$

Further, $g(x) = g^{j+1}(a)$, and for each $i \in \{1, \dots, k\}$ there are $s_{i1}, \dots, s_{i\alpha_i} \in \{0, \dots \dots, p_i - 1\}$ such that

$$j + 1 \equiv s_{i1} + s_{i2}p_i + \dots + s_{i\alpha_i}p_i^{\alpha_i - 1} \pmod{p_i^{\alpha_i}},$$

hence

$$\{g(x)\} = \bigcap_{i=1}^{k} T(i; s_{i1}, \dots, s_{i\alpha_i}).$$

Therefore g is defined by means of the construction 5.1.

6. AUXILIARY RESULTS

In this paragraph we assume that there exists a subset $C \subseteq A$ such that C is a large cycle. Let $n \in N$ and let A_n^f be the union of all $x \in A$ such that $K_f(x)$ contains a cycle with the cardinality n. (Recall that $K_f(x)$ is the connected component containing x.) We shall prove that for each $x \in A$, the set $K_f(x)$ can be determined by means of $\mathrm{Con}(A, f)$. Further it will be shown that if $x \in A_1^f \cup A_2^f$, then $f(x)$ is uniquely determined by means of $\mathrm{Con}(A, f)$. Some auxiliary results for components with large cycles will be established.

6.1. Lemma. *Let* $y \in A - C$. *The following conditions are equivalent*:

(1) $f(y) = y$.
(2) $\Theta(x, y) = [\{y\} \cup C]$ *for each* $x \in C$.

Proof. It is obvious that (1) implies (2). Let us suppose that (2) is valid. We have $f(x)\,\Theta(x, y) f(y)$ for each $x \in C$, and since $f(x) \in C$, we get $f(y) \in \{y\} \cup C$. If $f(y) \neq y$, then there exists $x' \in C$ such that $f(x') = f(y)$. But in this case we obtain $\Theta(x', y) = [\{x', y\}] \neq [\{y\} \cup C]$, which is a contradiction. Hence (1) holds.

6.2. Lemma. *Suppose that* $y, z \in A - C$, $y \neq z$ *and let* (2) *from* 6.1 *hold for* y, *but do not hold for the element* z. *The following conditions are equivalent*:

(1) $f(z) = y$.
(2) $\Theta(y, z) = [\{y, z\}]$.

Proof. According to the assumption and to 6.1 we have $f(y) = y$, $f(z) \neq z$. The implication (1) $\Rightarrow$ (2) is obvious; the implication (2) $\Rightarrow$ (1) follows immediately from Lemma 1.1.

6.3. Lemma. *Let* $x \in C$, $y_1, y_2 \in A - C$, $y_1 \neq y_2$. *The following conditions are equivalent*:

(1) $\{y_1, y_2\}$ *is a cycle of* $A(f)$.
(2) $\{y_1\} \notin \Theta(y_2, x)$, $\{y_2\} \notin \Theta(y_1, x)$ *and* $\Theta(y_1, y_2) = [\{y_1, y_2\}]$.

Proof. Obviously (1) $\Rightarrow$ (2), and the relation (2) $\Rightarrow$ (1) follows from 1.1 for elements y_1 and y_2 (we get a contradiction in each case except when $\{y_1, y_2\}$ is a cycle).

6.4. Corollary. *The sets* A_1^f *and* A_2^f *can be determined by means of the system* Con (A, f). *If* $x \in A_1^f \cup A_2^f$, *then* $f(x)$ *can be described by means of* Con (A, f).

Proof. The assertion follows from Lemmas 6.1, 6.2, 6.3, 1.2 (c) and 1.3 (d).

6.5. Lemma. *Let* $x \in C$, $n \in N_0 = N \cup \{0\}$. *The set* $C_n^f[x]$ *can be determined by means of* Con (A, f).

Proof. We shall prove the assertion of the lemma by induction. In 3.1 it was proved that $C_0^f[x] = C$ can be determined by means of Con (A, f). Let $n \in N$. Assume that for each $m \in N_0$, $m < n$ the assertion is valid. We shall show that for $y \in A$ the following conditions are equivalent:

(1) $y \in C_n^f[x]$.
(2) The condition (2) from 6.1 does not hold, $C_{n-1}^f[x] \neq \emptyset$, $y \notin \bigcup_{m<n} C_m^f[x]$ and if $z \in C_{n-1}^f[x]$ and $u \in A$ with $\{u\} \notin \Theta(z, y)$, then $u \in \{y\} \cup \bigcup_{m<n} C_m^f[x]$.

Obviously (1) $\Rightarrow$ (2). Let (2) hold and assume that $y \notin C_n^f[x]$. Suppose that $z \in C_{n-1}^f[x]$. From (2) and 6.1 it follows that $f(y) \neq y$. Further we have $y \notin \bigcup_{m<n} C_m^f[x]$

hence $f(y) \notin \{y\} \cup \bigcup_{m<n} C^f_m[x]$. From this, from the fact that $f(y)\,\Theta(z, y)\,f(z)$ and in view of (2) we obtain that $f(y) = f(z)$, which is a contradiction.

6.6. Corollary. *Each connected component K of (A, f) can be determined by means of the system* Con (A, f).

Proof. Let $x \in A$. If x is an element belonging to a large cycle C', then C' can be found by means of Con (A, f) (in view of 3.1). From 6.5 it follows that $C^f_n[x]$ can be determined by means of Con (A, f) for each $n \in N_0$, hence $K_f(x) = \bigcup_{n \in N_0} C^f_n[x]$ can be determined by means of Con (A, f). In the case when $x \in A^f_1 \cup A^f_2$ the assertion follows from 6.4.

6.7. Lemma. *Let $x \in C$, $n \in N$, $y \in C^f_{n+1}[x]$, $z \in C^f_n[x]$. The following conditions are equivalent*:

(1) $f(y) = z$.
(2) $\Theta(y, z) = [y\,\Theta(y, z)]$.

Proof. It is obvious that (1) implies (2). Suppose that $f(y) \neq z$. Then the partition of A corresponding to the congruence relation $\Theta(y, z)$ has at least two nontrivial classes, namely $y\,\Theta(y, z) \neq \{y\}$, $f(y)\,\Theta(y, z) \neq \{f(y)\}$, hence (2) does not hold.

6.7.1. Corollary. *Let $x \in C$, $n \in N$, $y \in C^f_{n+1}[x]$. If $g \in R(f)$, then $g(y) = f(y)$.*

6.8. Lemma. *Let $x, v \in C$, $u \in C^f_1[x]$. The following conditions are equivalent*:

(1) $f(u) = f(v)$.
(2) $\Theta(u, v) = [\{u, v\}]$.

Proof. The assertion follows from Lemma 1.1.

6.9. Lemma. *Let $x, v, z \in C$,* card $C = k$, *$n \in N$, $y \in C^f_{n+1}[x]$. If $n' \in N_0$, $n' \equiv n + 1 \pmod{k}$, $0 \leqq n' < k$, then the following conditions are equivalent*:

(1) $z = f^{n+1+k-n'}(y)$, $v = f(z)$.
(2) *If $y\,\Theta(y, z)\,t$ for $t \in A$, then $t \in \bigcup_{i \in N} C^f_i[x] \cup \{z\}$, and if $f(y)\,\Theta(y, z)\,s$ for $s \in A$, then $s \in \bigcup_{i \in N} C^f_i[x] \cup \{v\}$.*

Proof. Let m be an integer such that $n + 1 = mk + n'$. First assume that (1) is valid. We obtain $\Theta(y, z) = [\{y, f^k(y), f^{2k}(y), \ldots, f^{mk}(y), f^{(m+1)k}(y) = z\}, \{f(y), f^{k+1}(y), \ldots, f(z)\}, \ldots, \{f^{k-1}(y), f^{2k-1}(y), \ldots, f^{n+k-n'}(y)\}]$, hence the condition (2) is satisfied. Now suppose that (2) holds. Since $z \in C$, we obtain $y\,\Theta(y, z) \supseteq \{y, f^k(y), f^{2k}(y), \ldots, f^{mk}(y), f^{n+1+k-n'}(y)\}$. From the fact that $f^{n+1+k-n'}(y) \in C$ and from (2) it follows that $f^{n+1+k-n'}(y) = z$. Further we get $f(y)\,\Theta(y, z) \supseteq \{f(y), f^{k+1}(y), f^{2k+1}(y), \ldots, f(z)\}$ and (2) yields that $f(z) = v$.

Let $x, z, v \in C$. We shall say that the ordered pair $[z, v]$ is determined by the surroundings of the cycle C, if there are $n \in N$ and $y \in C^f_{n+1}[x]$ such that the condition (2) from 6.9 is valid. By $M_f(x)$ we denote the system of all pairs which are determined by the surroundings of the cycle $C = C^f_0[x]$. (From 6.9 it follows that the set $M_f(x)$ is determined by means of Con (A, f).)

Let $x \in C$. The system of all ordered pairs $[u, v]$ such that $v \in C$, $u \in C^f_1[x]$ and that the condition (2) from 6.8 holds will be denoted by the symbol $P_f(x)$. (According to 6.5 and 6.8, $P_f(x)$ is determined by Con (A, f).)

Let us remark that in the following theorem 6.10 the condition (2) can be expressed merely by means of congruence relations of (A, f) ((a) in view of 3.1 and 5.4, (b), (c) and (d) in view of 6.9, 6.7 and 6.8 respectively). Further, also the condition that (A, f) is connected and possesses a large cycle C can be expressed by means of the system Con (A, f) (cf. Corollary 6.6 and Lemma 3.1).

6.10. Theorem. *Suppose that (A, f) is connected and possesses a large cycle $C = C^f_0[x]$, $x \in A$. Let $g \in F(A)$. The following conditions are equivalent:*

(1) $g \in R(f)$.
(2) (a) *C is a cycle of (A, g) and $g \sim f$ on C;*
    (b) $M_g(x) = M_f(x)$;
    (c) *$g(u) = f(u)$ for each $u \in \bigcup_{m \geq 1} C^f_m[x]$;*
    (d) $P_g(x) = P_f(x)$.

Proof. First assume that the condition (1) is satisfied. Then the assertion (a) follows from 3.1, (b) follows from 6.9, (c) from 6.7 and (d) from 6.8. Now suppose that (2) holds. In this proof we shall write M_f, P_f, C^f_n instead of $M_f(x)$, $P_f(x)$ and $C^f_n[x]$. Put k = card C. Assume that $a, b \in A$. Let us prove that $\Theta^f(a, b) = \Theta^g(a, b)$. We shall proceed in six steps: 1) $a, b \in C$; 2) $b \in C$, $a \in C^f_1$; 3) $b \in C$, $a \in C^f_n$, $n > 1$; 4) $a, b \in C^f_1$; 5) $b \in C^f_1$, $a \in C^f_n$, $n > 1$; 6) $b \in C^f_m$, $a \in C^f_n$, $1 < m \leqq n$. (In the steps 3), 5) and 6) we shall proceed by induction.)

1) If $a, b \in C$, then from (a) it follows that $\Theta^f(a, b) = \Theta^g(a, b)$.

2) Suppose that $b \in C$, $a \in C^f_1$. There is a uniquely determined $c \in C$ with $f(a) = f(c)$, i.e., with $[a, c] \in P_f$. From (d) it follows that $[a, c] \in P_g$, i.e., $g(a) = g(c)$. The nontrivial classes corresponding to $\Theta^f(a, b)$ are: $\{a\} \cup b\, \Theta^f(f(a), f(b))$ and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ except $b\, \Theta^f(f(a), f(b))$, and the situation for the operation g is analogous. Since $b, c \in C$, we have $\Theta^f(f(a), f(b)) = \Theta^f(f(c), f(b)) = \Theta^f(c, b)$; similarly, $\Theta^g(g(a), g(b)) = \Theta^g(c, b)$, and from (1) we have $\Theta^f(c, b) = \Theta^g(c, b)$, hence $\Theta^f(f(a), f(b)) = \Theta^g(g(a), g(b))$. From this and according to the classes corresponding to $\Theta^f(a, b)$ and to $\Theta^g(a, b)$ we obtain that the relation $\Theta^f(a, b) = \Theta^g(a, b)$ is valid.

3) Now suppose that $b \in C$, $a \in C^f_n$, $n > 1$ and assume that $\Theta^f(x, y) = \Theta^g(x, y)$ for each $x \in C^f_m$, $y \in C$, $m < n$. Let $n' \equiv n \pmod{k}$, $0 \leqq n' < k$. From (b) and from 6.9 it follows that if we denote $b' = f^{n+k-n'}(a)$, then

(3) $b' = f^{n+k-n'}(a) = g^{n+k-n'}(a), f(b') = g(b')$.

Consider the congruence relation $\Theta^f(a, b)$ reduced to the cycle C (it coincides with $\Theta^f(f(a), f(b))$ reduced to C); we obtain the congruence relation $\Theta^f(f^n(a), f^n(b))$ and further

(4) $\Theta^f(f^n(a), f^n(b)) = \Theta^f(f^n(a), f^{n'}(b)) = \Theta^f(f^{n+(k-n')}(a),$
$f^{n'+(k-n')}(b)) = \Theta^f(f^{n+k-n'}(a), f^k(b)) = \Theta^f(b', b)$.

For the operation g we obtain

(4') $\Theta^g(g^n(a), g^n(b)) = \Theta^g(b', b)$,

and since $b, b' \in C$, we have $\Theta^f(b', b) = \Theta^g(b', b)$, hence

(5) $\Theta^f(f^n(a), f^n(b)) = \Theta^g(g^n(a), g^n(b)) = \Theta^f(b, b') = \Theta^g(b, b')$

and these are the congruence relations $\Theta^f(a, b)$ and $\Theta^g(a, b)$ considered on C. The nontrivial classes corresponding to $\Theta^f(a, b)$ are: $\{a\} \cup b\, \Theta^f(f(a), f(b))$ and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ except $b\, \Theta^f(f(a), f(b))$ (analogously for g). Using the induction hypothesis we obtain that

(6) $\Theta^f(f(a), f(b)) = \Theta^g(f(a). f(b)), \Theta^f(g(a), g(b)) = \Theta^g(g(a), g(b))$,

and (c) implies that $f(a) = g(a)$, thus

(7) $\Theta^f(g(a), g(b)) = \Theta^g(f(a), g(b))$.

From (3) and from the relations

(8) $g(b)\, \Theta^g(b, b')\, g(b'), f(b)\, \Theta^f(b, b') f(b')$

it follows that $g(b)\, \Theta^f(b, b') f(b)$, therefore in view of (5) we have $g(b)\, \Theta^f(f(a), f(b)) f(b)$ and hence

(9) $\Theta^f(f(a), f(b)) \geqq \Theta^f(f(a), g(b))$.

Analogously we obtain

(9') $\Theta^g(g(a), g(b)) \geqq \Theta^g(g(a), f(b))$.

Using (6), (7), (9) and (9') the following relation is obtained:

(10) $\Theta^f(f(a), f(b)) \geqq \Theta^f(f(a), g(b)) = \Theta^f(g(a), g(b)) = \Theta^g(g(a), g(b)) \geqq$
$\Theta^g(g(a), f(b)) = \Theta^g(f(a), f(b)) = \Theta^f(f(a), f(b))$.

We have

(*) $\Theta^f(f(a), f(b)) = \Theta^f(f(a), g(b))$.

Further $\Theta^f(f(a), f(b)) = \Theta^g(g(a), g(b))$ and according to the classes of $\Theta^f(a, b)$ and of $\Theta^g(a, b)$ we obtain that $\Theta^f(a, b) = \Theta^g(a, b)$.

4) Now let $b \in C_1^f$, $a \in C_1^f$. The nontrivial classes corresponding to $\Theta^f(a, b)$ are $\{a, b\}$ and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ (analogously for g). There are uniquely determined $c, d \in C$ with $f(a) = f(c)$ and $f(b) = f(d)$, i.e., $[a, c] \in P_f$, $[b, d] \in P_f$. From (d) it follows that $g(a) = g(c)$ and $g(b) = g(d)$. If $c = d$, then obviously $\Theta^f(a, b) = \Theta^g(a, b)$ (the only nontrivial class is $\{a, b\}$). Let $c \neq d$. Then we have

$$\Theta^f(f(a), f(b)) = \Theta^f(f(c), f(d)) = \Theta^f(c, d) = \Theta^g(c, d) = \\ = \Theta^g(g(c), g(d)) = \Theta^g(g(a), g(b)),$$

and therefore $\Theta^f(a, b) = \Theta^g(a, b)$.

5) Suppose that $b \in C_1^f$, $a \in C_n^f$, $n > 1$. Then with respect to (d) there is $c \in C$ with $[b, c] \in P_f$, i.e., $f(b) = f(c)$, $g(b) = g(c)$. If we suppose that $f^{n-1}(a) \neq b$, then the nontrivial classes corresponding to $\Theta^f(a, b)$ are $\{a, b\}$ and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ (analogously for g).

If we assume that $f^{n-1}(a) = b$, then the nontrivial classes corresponding to $\Theta^f(a, b)$ are $\{a\} \cup b\,\Theta^f(f(a), f(b))$, and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ except of $b\,\Theta^f(f(a), f(b))$ (analogously for g). From the induction hypothesis we have

(11) $\Theta^f(f(a), f(b)) = \Theta^g(f(a), f(b))$, $\Theta^f(g(a), g(b)) = \Theta^g(g(a), g(b))$.

From (c) it follows that $g(a) = f(a)$. We have to prove that

(12) $\Theta^f(f(a), f(b)) = \Theta^f(f(a), g(b))$,

i.e., that

(12′) $\Theta^f(f(a), f(c)) = \Theta^f(f(a), g(c))$.

To prove the relation (12′) it suffices to proceed analogously as in 3) (with c instead of b), where (*) shows us that (12′) is valid.

6) Now suppose that $b \in C_n^f$, $a \in C_m^f$, $1 < n \leqq m$. Then (c) implies that $f(a) = g(a)$, $f(b) = g(b)$ and by the induction hypothesis we obtain

(13) $\Theta^f(f(a), f(b)) = \Theta^g(f(a), f(b)) = \Theta^g(g(a), g(b))$.

If $f^{m-n}(a) \neq b$. then the nontrivial classes corresponding to $\Theta^f(a, b)$ are $\{a, b\}$ and all nontrivial classes corresponding to $\Theta^f(f(a), f(b))$; if $f^{m-n}(a) = b$, then the nontrivial classes corresponding to $\Theta^f(a, b)$ are $\{a\} \cup b\,\Theta^f(f(a), f(b))$ and nontrivial classes corresponding to $\Theta^f(f(a), f(b))$ except $b\,\Theta^f(f(a), f(b))$ (analogously for the operation g). Therefore (13) implies that $\Theta^f(a, b) = \Theta^g(a, b)$ and the proof is complete.

From 6.6 and 3.1 it follows that each set A_k^f, where $k \in N$ (the set of all elements $x \in A$ such that $K_f(x)$ contains a cycle with the cardinality k) can be determined by means of Con (A, f). Therefore we shall write A_k instead of A_k^f. In this paragraph there are studied the algebras (A_k, g), (A_m, g) (k, m being distinct positive integers, $g \in R(f)$) and relations between them. In particular, it will be proved that if $A = \bigcup_{i \in I} A_i$ and g.c.d. $(i, j) = 1$ for each $i, j \in I$, $i \neq j$, then for $g \in F(A)$ we have $g \sim f$ on A if and only if $g \sim f$ on A_i for each $i \in I$.

For $k \in N$ denote by B_k the set of all $x \in A_k$ such that x belongs to a cycle.

7.1.1. Lemma. *Let $k \in N$, $k > 2$ and let $x, x' \in B_k$, $C_0^f[x] \neq C_0^f[x']$. If $C_2^f[x] \neq \emptyset$ and $u' \in C_0^f[x']$, then $f(u')$ can be described by means of* Con (A, f).

Proof. Let $y \in C_2^f[x]$ and $u', v' \in C_0^f[x']$. It is obvious that $f(u') = v'$ implies $v'\ \Theta^f(u', y) f(y)$. (Let us remark that according to 6.7, $f(y)$ can be described by means of Con (A, f).) Now suppose that $v'\ \Theta^f(u', y) f(y)$. Since

$$\Theta^f(u', y) = [\{u', y, f^k(y)\}, \{f(u'), f(y), f^{k+1}(y)\}, \{f^2(u), f^2(y)\}, \dots \\ \dots, \{f^{k-1}(u), f^{k-1}(y)\}]$$

and since $f(y) \notin C_0^f[x']$, $f^{k+1}(y) \notin C_0^f[x']$, we obtain that $v' = f(u')$. Hence $f(u')$ is described by means of Con (A, f).

7.1.2. Lemma. *Let $k \in N$, $k > 2$ and let $x, x' \in B_k$, $C_0^f[x] \neq C_0^f[x']$. Further suppose that $C_2^f[x] \neq \emptyset$. If $u \in C_0^f[x]$, then $f(u)$ can be described by means of* Con (A, f).

Proof. Let $y \in C_2^f[x]$, $u, v \in C_0^f[x]$. From 7.1.1 it follows that $f(x')$ can be described by means of Con (A, f). If $f(u) = v$, then obviously $\{v, f(x')\} \in \Theta^f(u, x')$. Assume that $\{v, f(x')\} \in \Theta^f(u, x')$. We have $\Theta(u, x') = [\{u, x'\}, \{f(u), f(x')\}, \dots \dots, \{f^{k-1}(u), f^{k-1}(x')\}]$, hence $f(u) = v$.

7.1.3. Lemma. *Let $k \in N$, $k > 2$ and let $x, x' \in B_k$, $C_0^f[x] \neq C_0^f[x']$. Further suppose that $C_2^f[x] \neq \emptyset$. If $z \in C_1^f[x]$, then $f(z)$ can be described by means of* Con (A, f).

Proof. Let the assumption of the lemma hold. According to 6.8, by means of Con (A, f) we can find an element $t \in C_0^f[x]$ such that $f(t) = f(z)$. Since $f(t)$ can be described by means of Con (A, f) in view of 7.1.2, the element $f(z)$ can be described by means of Con (A, f) as well.

7.2. Corollary. *Let $k \in N$, $k > 2$ and let $x, x' \in B_k$, $C_0^f[x] \neq C_0^f[x']$. Further suppose that $C_2^f[x] \neq \emptyset$. If $a \in A_k$, then $f(a)$ can be described by means of* Con (A, f).

Proof. Let $a \in A_k$. If $a \in C_n^f[y]$, $n \in N$, $n > 1$, $y \in B_k$, then 6.7 implies that $f(a)$

can be described by means of Con (A, f). In the case when $a \in C_0^f[x] \cup C_1^f[x]$, the assertion follows from 7.1.1 and 7.1.3. If $a \in C_0^f[x'']$, $x \neq x'' \in B_k$, we obtain that $f(a)$ can be described by means of Con (A, f) in view of 7.1.2 (putting x'' instead of x'). Now let $a \in C_1^f[x'']$, $x \neq x'' \in B_k$. According to 6.8, by means of Con (A, f) we can find an element $v \in C_0^f[x]$ such that $f(a) = f(v)$. Since $f(v)$ can be described by means of Con (A, f) in view of 7.1.2, we obtain that $f(a)$ can be described by means of Con (A, f) as well.

7.3. Lemma. *Let $k \in N$, $k > 2$ and let $x_0, x_0' \in B_k$, $C_0^f[x_0] \neq C_0^f[x_0']$. Further suppose that $C_0^f[x_0] = \{x_0, \ldots, x_{k-1}\}$, $C_0^f[x_0'] = \{x_0', \ldots, x_{k-1}'\}$ and that $f(x_0) = x_1, \ldots, f(x_{k-2}) = x_{k-1}$, $f(x_{k-1}) = x_0$. Then the following conditions are equivalent:*

(1) $f(x_0') = x_1', \ldots, f(x_{k-1}') = x_0'$.
(2) $\Theta^f(x_0, x_0') = [\{x_0, x_0'\}, \{x_1, x_1'\}, \ldots, \{x_{k-1}, x_{k-1}'\}]$.

Proof. The assertion is obvious.

7.3.1. Corollary. *Let $k \in N$, $k > 2$ and let $x_0, x_0' \in B_k$, $C_0^f[x_0] \neq C_0^f[x_0']$. Let $g \in R(f)$. Then the operation g reduced to the set $C_0^f[x_0']$ is uniquely determined by the operation g reduced to $C_0^f[x]$.*

If we assume that $k \in N$, $k > 2$, $A = A_k$, then we have three possibilities: (α) A is connected, (β) there are $x, x' \in B_k$ such that $C_0^f[x] \neq C_0^f[x']$ and $C_2^f[x] \neq \emptyset$, (γ) there are $x, x' \in B_k$ such that $C_0^f[x] \neq C_0^f[x']$ and $C_2^f[y] = \emptyset$ for each $y \in B_k$. The case (α) was investigated in Theorem 6.10, the case (β) in Corollary 7.2; the remaining case will be studied in Lemma 7.4.

7.4. Lemma. *Let $k \in N$, $k > 2$, $A = A_k$ and suppose that the above condition (γ) is satisfied. Let $g \in F(A)$. Then the following conditions are equivalent:*

(1) $g \in R(f)$.
(2) (a) $g \sim f$ on $K_f(x)$;
 (b) *if $x' \in C_0^f[x]$, $y \in B_k - C_0^f[x]$ and $z \in C_0^f[y]$ then $g(y) = z$ if and only if $\{g(x'), z\} \in \Theta^f(x', y)$;*
 (c) *$P_g(y) = P_f(y)$ for each $y \in B_k$.*

Proof. First assume that the condition (1) is satisfied. Then (a) obviously holds, (b) follows from 7.3, and (c) from 6.8. Conversely, suppose that (2) is valid. Let $a, b \in A$. We shall prove that $\Theta^f(a, b) = \Theta^g(a, b)$.

Let $y \in B_k - C_0^f[x]$, $x' \in C_0^f[x]$. From (b) it follows that $\{g(x'), g(y)\} \in \Theta^f(x', y)$, hence $\Theta^f(x', y) \geqq \Theta^f(g(x'), g(y))$. Since both the partitions corresponding to $\Theta^f(x', y)$ and to $\Theta^f(g(x'), g(y))$ have k nontrivial classes (having two elements), we obtain that $\Theta^f(x', y) = \Theta^f(g(x'), g(y))$. Similarly, $\{g(g(x')), g(g(y))\} \in \Theta^f(g(x'), g(y))$, i.e., $\{g^2(x'), g^2(y)\} \in \Theta^f(x', y), \ldots, \{g^{k-1}(x'), g^{k-1}(y)\} \in \Theta^f(x', y)$. These relations together with

(3) $\Theta^f(x', y) = [\{x', y\}, \{f(x'), f(y)\}, \ldots, \{f^{k-1}(x'), f^{k-1}(y)\}],$
$\Theta^g(x', y) = [\{x', y\}, \{g(x'), g(y)\}, \ldots, \{g^{k-1}(x'), g^{k-1}(y)\}]$

imply

(4) $\Theta^f(x', y) = \Theta^g(x', y)$

since $x', g(x'), \ldots, g^{k-1}(x')$ are distinct elements and therefore $\{x', y\}$, $\{g(x'), g(y)\}, \ldots, \{g^{k-1}(x'), g^{k-1}(y)\}$ are distinct sets.

Now consider $\Theta^f(a, b)$ and $\Theta^g(a, b)$. At first assume that $a, b \in B_k$. If $a, b \in C_0^f[x]$, then from (a) it follows that $\Theta^f(a, b) = \Theta^g(a, b)$. The cases when $a \in C_0^f[x]$ or $b \in C_0^f[x]$ have been investigated in (4). (In what follows in the proof we shall use this fact without mentioning it.) Hence assume that $a \notin C_0^f[x]$, $b \notin C_0^f[x]$. At first let $b \notin C_0^f[a]$. Then the following relations are valid:

(5) $[\{a, x\}, \{f(a), f(x)\}, \ldots, \{f^{k-1}(a), f^{k-1}(x)\}] = \Theta^f(a, x) = \Theta^g(a, x) =$
$= [\{a, x\}, \{g(a), g(x)\}, \ldots, \{g^{k-1}(a), g^{k-1}(x)\}],$

(6) $[\{b, x\}, \{f(b), f(x)\}, \ldots, \{f^{k-1}(b), f^{k-1}(x)\}] = \Theta^f(b, x) = \Theta^g(b, x) =$
$= [\{b, x\}, \{g(b), g(x)\}, \ldots, \{g^{k-1}(b), g^{k-1}(x)\}].$

(7) $\Theta^f(a, b) = [\{a, b\}, \{f(a), f(b)\}, \ldots, \{f^{k-1}(a), f^{k-1}(b)\}],$

(8) $\Theta^g(a, b) = [\{a, b\}, \{g(a), g(b)\}, \ldots, \{g^{k-1}(a), g^{k-1}(b)\}].$

Let $i \in N_0$, $0 \leqq i < k$. According to (5) and (6) there are $j, j' \in N_0$, $0 \leqq j < k, 0 \leqq j' < k$ such that $\{g^i(a), g^i(x)\} = \{f^j(a), f^j(x)\}$, $\{g^i(b), g^i(x)\} = \{f^{j'}(b), f^{j'}(x)\}$. From the fact that a and x or b and x are from distinct components it follows that $f^j(x) = g^i(x) = f^{j'}(x)$, hence $j = j'$. We obtain

(9) $g^i(a) = f^j(a),\ g^i(b) = f^j(b)$

and the correspondence $i \to j$ is one-to-one, therefore (7)–(9) imply

(10) $\Theta^f(a, b) = \Theta^g(a, b).$

If $a \notin C_0^f[x]$, $b = f^n(a) = g^m(a)$ for some $n, m \in N_0$, $0 \leqq n < k$, $0 \leqq m < k$, then (5) and (6) are valid as well and, analogously as above, we obtain that $f^n(x) = g^m(x)$. Further let Θ be the join of the congruence relations $\Theta^f(x, a)$ and $\Theta^f(a, b)$, i.e., $\Theta = \Theta^f(x, a) \vee \Theta^f(a, f^n(a))$. From the definition of Θ it follows that $\Theta = \Theta^f(x, a) \vee \Theta^f(x, f^n(x))$. Similarly, $\Theta^g(x, a) \vee \Theta^g(a, g^m(a)) = \Theta^g(x, a) \vee \Theta^g(x, g^m(x))$. Then

(11) $\Theta^f(x, a) \vee \Theta^f(a, b) = \Theta^f(x, a) \vee \Theta^f(x, f^n(x)) = \Theta^g(x, a) \vee$
$\vee \Theta^g(x, f^n(x)) = \Theta^g(x, a) \vee \Theta^g(x, g^m(x)) = \Theta^g(x, a) \vee \Theta^g(a, b).$

(We have used (4) and (a).) The classes of the partition corresponding to the left hand side of (11) reduced to the set $C_0^f[a]$ must coincide with classes corresponding to the partition of the right hand side of (11) reduced to the set $C_0^f[a]$. The congruence relation $\Theta^f(x, a)$ joins into the same classes only elements from distinct components (and similarly does $\Theta^g(x, a)$), hence we get

(11.1) $\Theta^f(a, b) = \Theta^g(a, b)$.

Now suppose that $a \notin B_k$. Denote $v = f^k(a)$; then $f(a) = f(v)$ and from (c) we have $g(a) = g(v)$. If $b \in B_k$, then

(12) $\Theta^f(a, b) = \Theta^f(a, v) \vee \Theta^f(v, b)$, $\Theta^g(a, b) = \Theta^g(a, v) \vee \Theta^g(v, b)$.

Since $\Theta^f(a, v) = [\{a, v\}] = \Theta^g(a, v)$ and according to (a) or (10) or (11.1) we have $\Theta^f(v, b) = \Theta^g(v, b)$, thus we get $\Theta^f(a, b) = \Theta^g(a, b)$. If $b \notin B_k$, then denote $v' = f^k(b)$. Since $f(b) = f(v')$, (c) implies that $g(b) = g(v')$, hence $\Theta^f(b, v') = [\{b, v'\}] = \Theta^g(b, v')$. We obtain

(13) $\Theta^f(a, b) = \Theta^f(a, v) \vee \Theta^f(b, v') \vee \Theta^f(v, v') =$
$= \Theta^g(a, v) \vee \Theta^g(b, v') \vee \Theta^g(v, v') = \Theta^g(a, b)$.

The proof is complete.

7.4.1. *Let us remark that the conditions* (a), (b) *and* (c) *can be expressed by means of congruence relations* ((a) in view of 6.10, (b) in view of (a) and 7.3, (c) in view of 6.8).

7.5. Theorem. *Let* $A = \bigcup_{i \in I} A_i$ *and assume that* g.c.d. $(i, j) = 1$ *for each* $i, j \in I$, $i \neq j$. *Let* $g \in F(A)$. *Then* $g \in R(f)$ *if and only if* (A_i, g) *is a subalgebra of* (A, g) *and* $g \sim f$ *on* A_i *for each* $i \in I$.

Proof. If $g \in R(f)$, then obviously $g \sim f$ on A_i for each $i \in I$. Conversely, suppose that $g \sim f$ on each A_i, $i \in I$. Let $a, b \in A$; we shall prove that $\Theta^f(a, b) = \Theta^g(a, b)$. There are $i, j \in I$ such that $a \in A_i$ and $b \in A_j$. If $i = j$, then $\Theta^f(a, b) = \Theta^g(a, b)$, since $g \sim f$ on A_i. Suppose that $i \neq j$. Let $x \in K_f(a) \cap B_i$, $x' \in K_f(b) \cap B_j$. There exist $n, m \in N_0$ such that $a \in C_n^f[x] = C_n^g[x]$, $b \in C_m^f[x'] = C_n^g[x']$ (cf. 6.5), and we can assume that $n \geqq m$. From 6.7.1 it follows that

(1) $f(a) = g(a), \ldots, f^{n-1}(a) = g^{n-1}(a)$,

(2) $f(b) = g(b), \ldots, f^{m-1}(b) = g^{m-1}(b)$.

Since card $C_0^f[x] = i$, card $C_0^f[x'] = j$ and g.c.d. $(i, j) = 1$, we obtain that if $m = 0$ then

(3) $\Theta^f(a, b) = [\{a, f(a), \ldots, f^{n-1}(a)\} \cup C_0^f[x] \cup C_0^f[x']]$,

and if $m > 0$, then

(4) $\Theta^f(a, b) = [\{a, b\}, \{f(a), f(b)\}, \ldots, \{f^{m-1}(a), f^{m-1}(b)\}$,
$\{f^m(a), \ldots, f^{n-1}(a)\} \cup C_0^f[x] \cup C_0^f[x']]$

(analogously for g). According to (1) and (2) we get that the relation $\Theta^f(a, b) = \Theta^g(a, b)$ holds.

The following Proposition 7.6 will not be applied for proving the estimate of the cardinality of the set $R(f)$ (cf. 7.8 and 7.9 below). Nevertheless, it could be used in individual cases to verify whether a mapping $g \in F(A)$ belongs to $R(f)$.

7.6. Proposition. *Let $A = \bigcup_{i \in I} A_i$, where $A_i \neq \emptyset$ for each $i \in I$. For each $i \in I$ let a_i be a fixed element belonging to B_i. If $g \in F(A)$, then the following conditions are equivalent:*

(1) $g \in R(f)$.

(2) (a) (A_i, g) *is a subalgebra of* (A, g) *and* $g \sim f$ *on* A_i *for each* $i \in I$;

(b) $\Theta^f(b_i, b_j) = \Theta^g(b_i, b_j)$ *for each* $b_i \in C_0^f[a_i]$, $b_j \in C_0^f[a_j]$, $i, j \in I$, $i \neq j$.

Proof. If $g \in R(f)$, then (2) is obviously valid. Suppose that the condition (2) is satisfied. Let $x, y \in A$. We shall prove that $\Theta^f(x, y) = \Theta^g(x, y)$. There are $i, j \in I$ such that $x \in A_i$ and $y \in A_j$. In view of (a) it suffices to investigate the case when $i \neq j$ only. Further, there are $n, m \in N_0$ such that $f^n(x) \in B_i$ and $x \in C_n^f[f^n(x)]$, $f^m(y) \in B_j$ and $y \in C_m^f[f^m(y)]$. We can assume that $n \geqq m$.

First let $n = m = 0$. If $x \in K_f(a_i)$ and $y \in K_f(a_j)$, then (b) implies that $\Theta^f(x, y) = \Theta^g(x, y)$. Assume that $x \notin K_f(a_i)$, $y \in K_f(a_j)$. According to (a) and (b) we have

$$\Theta^f(x, y) \vee \Theta^f(a_i, x) = \Theta^f(a_i, y) \vee \Theta^f(a_i, x) =$$
$$= \Theta^g(a_i, y) \vee \Theta^g(a_i, x) = \Theta^g(x, y) \vee \Theta^g(a_i, x) .$$

Since each nontrivial class corresponding to the partition $\Theta^f(a_i, x)$ or $\Theta^g(a_i, x)$ has two elements and no such class is a subset of $K_f(x)$, we obtain that $\Theta^f(x, y) = \Theta^g(x, y)$. Now assume that $x \notin K_f(a_i)$, $y \notin K_f(a_j)$. Analogously as above, we have

$$\Theta^f(x, y) \vee \Theta^f(a_i, x) \vee \Theta^f(a_j, y) = \Theta^f(a_i, x) \vee \Theta^f(a_j, y) \vee \Theta^f(a_i, a_j) =$$
$$= \Theta^g(a_i, x) \vee \Theta^g(a_j, y) \vee \Theta^g(a_i, a_j) = \Theta^g(x, y) \vee \Theta^g(a_i, x) \vee \Theta^g(a_j, y) .$$

All nontrivial classes corresponding to the partitions $\Theta^f(a_i, x)$, $\Theta^g(a_i, x)$, $\Theta^f(a_j, y)$ or $\Theta^g(a_j, y)$ have two elements and no such class is a subset of $K_f(x)$ or of $K_f(y)$, therefore $\Theta^f(x, y) = \Theta^g(x, y)$.

Now let $n = 1$, $m = 0$. Since $g \sim f$ on A_i, there is $z \in K_f(x) \cap B_i$ with $f(x) = f(z)$, $g(x) = g(z)$. Using what we have proved above we obtain

$$\Theta^f(x, y) = [\{x, z\}] \vee \Theta^f(z, y) = [\{x, z\}] \vee \Theta^g(z, y) = \Theta^g(x, y) .$$

Suppose that $n > 1$, $m = 0$. From (a) and 6.7.1 it follows that

(3) $f(x) = g(x), \ldots, f^{n-1}(x) = g^{n-1}(x)$.

Let $n' \in N_0$, $n' \equiv n \pmod{i}$, $0 \leqq n' < i$. Denote $u = f^{n+i-n'}(x)$. According to (a) and 6.9 we have

(4) $u = f^{n+i-n'}(x) = g^{n+i-n'}(x),\ f(u) = g(u)$.

Further,

(5) $\Theta^f(x, y) = \Theta^f(x, f^{n+i-n'}(x)) \vee \Theta^f(f^{n+i-n'}(x), y)$,

(6) $\Theta^g(x, y) = \Theta^g(x, g^{n+i-n'}(x)) \vee \Theta^g(g^{n+i-n'}(x), y)$.

Applying the facts proved above and (4) – (6), we obtain

$$\Theta^f(x, y) = \Theta^f(x, u) \vee \Theta^f(u, y) = \Theta^g(x, u) \vee \Theta^g(u, y) = \Theta^g(x, y) .$$

Now let $n = 1$, $m = 1$. Then (a) and 6.8 imply that there are $t \in B_i$, $u \in B_j$ with $f(x) = f(t)$, $g(x) = g(t)$, $f(u) = f(y)$, $g(u) = g(y)$. Hence

$$\Theta^f(x, y) = [\{x, y\}] \vee \Theta^f(f(x), f(y)) = [\{x, y\}] \vee \Theta^f(f(t), f(u)) =$$
$$= [\{x, y\} \vee \Theta^f(t, u) = [\{x, y\}] \vee \Theta^g(t, u) = [\{x, y\}] \vee \Theta^g(g(t), g(u)) =$$
$$= [\{x, y\}] \vee \Theta^g(g(x), g(y)) = \Theta^g(x, y) .$$

Suppose that $n \geqq m > 1$. We proceed by induction; we assume that $\Theta^f(f(x), f(y)) = \Theta^g(f(x), f(y))$. According to (a) and 6.7.1 we have (3) and

(7) $f(y) = g(y), \ldots, f^{m-1}(y) = g^{m-1}(y)$.

Therefore

$$\Theta^f(x, y) = [\{x, y\}] \vee \Theta^f(f(x), f(y)) = [\{x, y\}] \vee \Theta^g(g(x), f(y)) =$$
$$= [\{x, y\}] \vee \Theta^g(g(x), g(y)) = \Theta^g(x, y) ,$$

completing the proof.

Now we shall establish an estimate of card $R(f)$ for the case of an algebra (A, f) with a large cycle.

Let us recall that in this Part II we assume that there is a subset $C \subseteq A$ such that C is a large cycle.

Assume that $A = \bigcup_{i \in I} A_i$, $A_i \neq \emptyset$ for each $i \in I$. If $i \in I$ and $g \in R(f)$, then $g \sim f$ on A_i. Further, according to 7.2, 7.4 and 6.10 for each $i \in I$, $i > 2$ we have

$$\text{card}\, \{g \in F(A_i)\colon\ g \sim f \text{ on } A_i\} < \aleph_0 .$$

If $i = I \cap \{1, 2\}$, then in view of 6.4

$$\text{card}\, \{g \in F(A_i)\colon\ g \sim f \text{ on } A_i\} = 1 .$$

Therefore we obtain the estimate

(A) card $R(f) \leqq c$

(c being the cardinality of the continuum).

7.7. Example. Let I be the set of all prime numbers greater than 2. Further let (A, f) be a monounary algebra such that $A = \bigcup_{i \in I} A_i$, where A_i is a cycle with i elements for each $i \in I$. According to 5.4 we have

(1) $\operatorname{card} \{g \in F(A_i): g \sim f \text{ on } A_i\} =$
$= \operatorname{card} \{g \in F(A_i): A_i \text{ is a cycle with respect to } g\} \geqq 2,$

and in view of 7.5 the following is valid:

(2) $\operatorname{card} R(f) = \prod_{i \in I} \operatorname{card} \{g \in F(A_i): g \sim f \text{ on } A_i\}.$

Hence (1) and (2) imply that

(3) $\operatorname{card} R(f) = c.$

Thus we have proved the following assertion:

7.8. Theorem. *Let (A, f) be a monounary algebra with a large cycle. Then*

$\operatorname{card} R(f) \leqq c$

and this estimate is the best possible.

Let us recall that $E(A)$ denotes the system of all equivalence relations on A. From 7.8 and 4.12 we obtain

7.9. Corollary. *Let (A, f) be a monounary algebra such that* $\operatorname{Con}(A, f) \neq E(A)$. *Then*

(7.9) $\operatorname{card} R(f) \leqq c$

and this estimate is the best possible.

8. ENDOMORPHISMS AND CONGRUENCE RELATIONS

For a monounary algebra we denote by End (A, f) the set of all endomorphisms of (A, f). Further we put

$$\operatorname{Eq}(f) = \{g \in F(A): \operatorname{End}(A, f) = \operatorname{End}(A, g)\}.$$

In [5] it was proved that for each monounary algebra (A, f) we have

(8.1) $\operatorname{card} \operatorname{Eq}(f) \leqq c.$

In view of the estimates (7.9) of 7.9 and (8.1) the natural question arises whether one of the following assertions is valid:

(a_1) For each monounary algebra (A, f) we have $\operatorname{card} R(f) \leqq \operatorname{card} \operatorname{Eq}(f)$.
(a_2) For each monounary algebra (A, f) we have $\operatorname{card} \operatorname{Eq}(f) \leqq \operatorname{card} R(f)$.

Now we shall give two examples showing that neither (a_1) nor (a_2) holds.

8.1. Example. Let A be the set of all integers and let $f(i) = i + 1$, $g(i) = i - 1$ for each $i \in A$. Then

(a) $g \in \mathrm{Eq}(f)$,
(b) $g \notin R(f)$,
(a′) $\mathrm{Eq}(f) = \{f, g\}$,
(b′) $R(f) = \{f\}$.

The assertions follow from Theorem 1 in [5] and from Theorem 2.6.

8.2. Example. Let A be the set of all integers modulo 8 and let $f(i) \equiv i + 1 \pmod 8$, $g(0) = 1$, $g(1) = 2$, $g(2) = 7$, $g(7) = 4$, $g(4) = 5$, $g(5) = 6$, $g(6) = 3$, $g(3) = 0$. Then

(a) $g \notin \mathrm{Eq}(f)$,
(b) $g \in R(f)$,
(a′) $\mathrm{card}\, \mathrm{Eq}(f) = \mathrm{card}\, \{f, f^3, f^5, f^7\} = 4$,
(b′) $\mathrm{card}\, R(f) = 16$.

The assertions follow from Theorem 2 in [5] and from Theorem 5.4 (there are 16 possibilities to get an operation belonging to $R(f)$ by means of the construction 5.1).

9. PARTIAL AND COMPLETE MONOUNARY ALGEBRAS

Let $A \neq \emptyset$ be a set. We denote by $F_p(A)$ and $F(A)$ the system of all partial and of all complete unary operations on A respectively. An equivalence Θ on a partial monounary algebra (A, f) is said to be a *congruence*, if the following is valid (cf. also [8], p. 177):

$$(\forall x, y \in A - D_f)\,(x \Theta y \Rightarrow f(x)\, \Theta\, f(y))\,,$$

where D_f is the set of all $z \in A$ such that $f(z)$ does not exist.

Let us recall some notations and results from [6]. For a partial monounary algebra (A, f) we put

$$R_p(f) = \{g \in F_p(A) \colon \mathrm{Con}\,(A, r) = \mathrm{Con}\,(A, g)\}\,.$$

Let (A, f) be a partial monounary algebra. Consider the following condition for (A, f):

(α) $\mathrm{Con}\,(A, f) \neq E(A)$ and $f^{-1}(D_f) \neq \emptyset$.

In [6] it was proved that the condition (α) can be expressed by means of $\mathrm{Con}\,(A, f)$. Also, D_f can be found by means of $\mathrm{Con}\,(A, f)$. Put $B = A - D_f$.

9.1. Theorem. *If (A, f) is a partial monounary algebra satisfying the condition* (α), *then* $\mathrm{card}\, R_p(f) \leqq 4$.

A partial monounary algebra (A_1, f_1) is said to be a *d-extension of a monounary algebra* (B_1, g_1), if $B_1 \subseteq A_1$, $D_{f_1} = A_1 - B_1$ and $g_1(x) = f_1(x)$ for each $x \in B_1$.

Denote $f_1 = g'_1$. In [6] it was proved that

$$R_p(f) = \{g \in F_p(A)\colon \operatorname{Con}(A, f) = \operatorname{Con}(A, g)\} =$$
$$= \{g \in F_p(A)\colon \operatorname{Con}(B, f/B) = \operatorname{Con}(B, g/B) \text{ and } D_f = D_g\} =$$
$$= \{g'_1 \in F(B)\colon \operatorname{Con}(B, g_1) = \operatorname{Con}(B, f/B)\}\,.$$

Hence in the case $f^{-1}(D) = \emptyset$ the investigation of $R(f)$ reduces to the investigation of $R(f/B)$, i.e. to the case of complete unary algebra. Therefore 7.9 and 9.1 imply the following result:

9.2. Corollary. *Let (A, f) be a partial monounary algebra such that* $\operatorname{Con}(A, f) \neq E(A)$. *Then*

$$\operatorname{card} R_p(f) \leqq c$$

and this estimate is the best possible.

References

[1] *J. Berman:* On the congruence lattice of unary algebras, Proc. Amer. Math. Soc. *36*, 1972, 34—80.

[2] *Д. П. Егорова, Л. А. Скорняков:* О структуре конгруэнций унарной алгебры, Упор. множества и решетки *4*, 1977, 28—40.

[3] *Д. П. Егорова:* Унары со структурой конгруэнций специального вида, Исслед. по алгебре *5*, 1977, 3—19.

[4] *Д. П. Егорова:* Структура конгруэнций унарной алгебры, Упор. множества и решетки *5*, 1978, 11—44.

[5] *D. Jakubíková:* Systems of unary algebras with common endomorphisms I, II, Czech. Math. J. *29* (1979), 406—420, 421—429.

[6] *D. Jakubíková-Studenovská:* Partial monounary algebras with common congruence relations, Czech. Math. J. *32* (*107*) (1982), 307—326.

[7] *D. Jakubíková-Studenovská:* On congruence relations of monounary algebras I, Czech. Math. J. *32* (*107*) (1982), 437—459.

[8] *B. Jónsson:* Topics in universal algebra, Berlin 1972.

[9] *Г. Ч. Куринной:* Унары с общими конгуэнциями I, II, Харьков. университет, 1980; Всесоюз. инст. научной информации, № 957—80 Деп., № 958—80 Деп.

[10] *Г. Ч. Куринной:* Об определимости унара конгруэнциями, Изв. выс. уч. завед., Математика, *48*, 1981, № 3, 76—78.

[11] *M. Novotný:* O jednom problému z teorie zobrazení, Publ. Fac. Sci. Univ. Masaryk, No 344, 1952/53, 53—64.

[12] *M. Novotný:* Über Abbildungen von Mengen, Pacif. J. Math. *13* (1963), 1359—1369.

[13] *L. A. Skornjakov:* Unars, Colloquium on universal algebra, Esztergom, 1977, 735—743.

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