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ON MINIMAL REGULAR DIGRAPHS WITH GIRTH 4

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1. INTRODUCTION

A problem which has been studied is to find the minimum number $f(r, g)$ of vertices an r-regular digraph with (directed) girth g can have. Behzad, Chartrand, and Wall [1] showed by construction that $f(r, g) \leqq r(g - 1) + 1$ and conjectured equality always holds. Later Behzad [2] verified this conjecture for $r = 2$, any g. Bermond [3] established its validity for $r = 3$, any g and for a few values of g when $r = 4$ or 5. Recently Hamidoune [5, 6] proved the conjecture for $r = 4$, any g and showed that a vertex-transitive r-regular digraph with girth g has at least $r(g - 1) + 1$ vertices.

The conjecture follows easily for all r when $g = 2$ or 3, and this leads one to consider the case $g = 4$. Except for those values of r included above, there are no other known cases of equality when $g = 4$. However, Cacetta and Haggkvist [4] have obtained the lower bound

$$f(r, 4) > \tfrac{1}{2}(4 + \sqrt{2})\, r + 1 \approx 2{\cdot}71r + 1\,. \tag{1.1}$$

In this note, we study $f(r, 4)$ and prove that r-regular, girth 4 digraphs satisfying an additional restriction (namely, $\hat{\delta} \leqq 6$, see Theorem 3.1) have at least $3r + 1$ vertices. This enables us to conclude that $f(r, 4) = 3r + 1$ for $r \leqq 23$. We also improve the lower bound (1.1) by showing that

$$f(r, 4) > 2{\cdot}85r + 1{\cdot}52\,. \tag{1.2}$$

Note that (1.2) implies $f(r, 4) = 3r + 1$ for $r \leqq 10$.

We use standard terminology and notation throughout. Thus $D = (V, E)$ denotes a *digraph* (without loops and multiple arcs) with *vertex set* $V = V(D)$ and *arc set* $E = E(D)$. For $v \in V$, $\Gamma^+(v) = \{u : (v, u) \in E\}$ *is the out-neighborhood* of v and $\Gamma^-(v) = \{u : (u, v) \in E\}$ is the *in-neighborhood* of v; $d_D^+(v) = |\Gamma^+(v)|$ is the *outdegree* of v, while $d_D^-(v) = |\Gamma^-(v)|$ is the *indegree* of v. The *minimum outdegree* and *minimum indegree* of vertices of D are denoted by

$$\delta_D^+ = \min_{v \in V} d_D^+(v)\,, \quad \delta_D^- = \min_{v \in V} d_D^-(v)\,.$$

For $A \subseteq V$, $\langle A \rangle$ is the *digraph induced by D on the vertex set A*; $\Gamma^+(A) = \{u : \exists v \in A,\ (v, u) \in E\}$ and $\Gamma^-(A) = \{u : \exists v \in A,\ (u, v) \in E\}$. Hence for $v \in A$, $d^+_{\langle A \rangle}(v) = |\Gamma^+(v) \cap A|$, $\delta^+_{\langle A \rangle} = \min_{v \in A} d^+_{\langle A \rangle}(v)$, and so forth.

If D has a (directed) cycle, the minimum length of a cycle of D is called the *girth* of D. If $d^+_D(v) = d^-_D(v) = r$ for every $v \in V$, then D is called *r-regular*.

Finally, we introduce the following key notion for this paper. We define $\hat{\delta}^+_D$ and $\hat{\delta}^-_D$ to be the minimum outdegree (respectively, indegree) of the digraphs induced on the out-neighborhoods (respectively, in-neighborhoods) of the vertices of D:

$$\hat{\delta}^+_D = \min_{v \in V} \delta^+_{\langle \Gamma^+(v) \rangle} = \min_{v \in V, u \in \Gamma^+(v)} d^+_{\langle \Gamma^+(v) \rangle}(u)\,,$$

$$\hat{\delta}^-_D = \min_{v \in V} \delta^-_{\langle \Gamma^-(v) \rangle} = \min_{v \in V, u \in \Gamma^-(v)} d^-_{\langle \Gamma^-(v) \rangle}(u)\,.$$

We then set

$$\hat{\delta}_D = \min \{\hat{\delta}^+_D, \hat{\delta}^-_D\}\,.$$

Thus $\hat{\delta}_D = p$ implies that each vertex in $\langle \Gamma^+(v) \rangle$ has outdegree at least p and each vertex in $\langle \Gamma^-(v) \rangle$ has indegree at least p, for all $v \in V$.

2. LOWER BOUND FOR $f(r, 4)$

The following lemma due to Caccetta and Haggkvist [4] will be useful in obtaining our lower bound for $f(r, 4)$.

Lemma 2.1. *Let D be a digraph with girth at least* 4. *Then*

$$\delta^{\pm}_D < \frac{3 - \sqrt{5}}{2} (|V(D)| - 1)\,.$$

As an immediate corollary we have the following.

Corollary 2.2. *Let D be a digraph with girth at least* 4 *and* $|V(D)| \leqq 6$. *Then there exists vertices* $u, v \in V(D)$ *such that* $d^+_D(u) \leqq 1$ *and* $d^-_D(v) \leqq 1$.

We now state and prove the main result of this section which improves an inequality of Caccetta and Haggkvist [4].

Theorem 2.3. *Let* $\alpha = (3 - \sqrt{5})/2$. *Then*

$$f(r, 4) > (3 - \alpha^2)\, r + 1 + \alpha(\alpha + 1) > 2{\cdot}85r + 1{\cdot}52\,.$$

Proof. Let $D = (V, E)$ be an r-regular digraph with girth g at least 4. By reversing arcs if necessary, we may assume $\hat{\delta}_D = \hat{\delta}^-_D$. Let $v \in V$ and $u \in \Gamma^-(v)$ be vertices such that $d^-_{\langle \Gamma^-(v) \rangle}(u) = \hat{\delta}_D$. Applying Lemma 2.1 to $\langle \Gamma^-(v) \rangle$, we obtain $\hat{\delta}_D < \alpha(r - 1)$.

We use the following notation (see Figure 1):

$$X = \Gamma^-(v), \quad Y = \Gamma^+(v), \quad Z = V - (X \cup Y \cup \{v\}),$$

$$Z' = \Gamma^-(u) \cap Z, \quad Z'' = Z - Z', \quad Y' = \Gamma^+(u) \cap Y.$$

Since $g \geqq 4$, it follows that $Z' = \Gamma^-(u) - X$; since D is r-regular, $|Z'| = r - \hat{\delta}_D$.

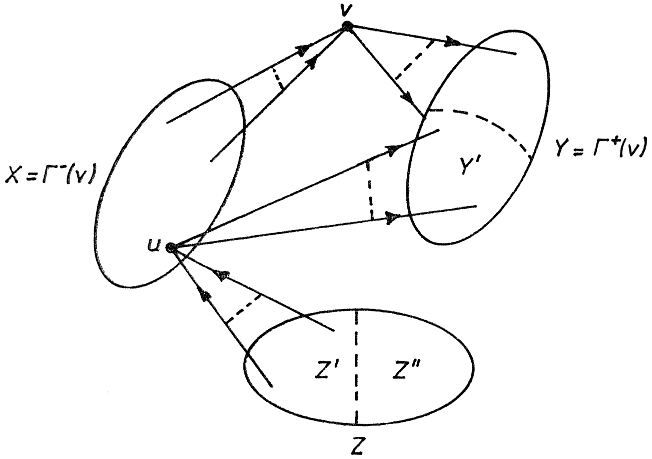

Figure 1

Using definitions, we calculate that

$$|Y'| = d^+_{\langle \Gamma^+(u) \rangle}(v) \geqq \delta^+_{\langle \Gamma^+(u) \rangle} \geqq \hat{\delta}^+_D \geqq \hat{\delta}_D .$$

Since $g \geqq 4$, $\Gamma^+(Y') \subseteq Y \cup Z''$. This along with the r-regularity of D implies that

$$\delta^+_{\langle Y' \rangle} \geqq r - |Z''| - (|Y| - |Y'|) = |Y'| - |Z''| .$$

Applying Lemma 2.1 to $\langle Y' \rangle$, we obtain

$$|Z''| \geqq |Y'| - \delta^+_{\langle Y' \rangle} > |Y'| - \alpha(|Y'| - 1) \geqq (1 - \alpha)\,\hat{\delta}_D + \alpha .$$

Then

$$|Z| = |Z'| + |Z''| = (r - \hat{\delta}_D) + |Z''| > (r - \hat{\delta}_D) + (1 - \alpha)\,\hat{\delta}_D + \alpha =$$
$$= r - \alpha(\hat{\delta}_D - 1) > r - \alpha(\alpha(r - 1) - 1) = (1 - \alpha^2)\,r + \alpha(\alpha + 1) .$$

Hence

$$|V)D)| = 2r + 1 + |Z| > 2r + 1 + (1 - \alpha^2)\,r + \alpha(\alpha + 1) =$$
$$= (3 - \alpha^2)\,r + 1 + \alpha(\alpha + 1) .$$

The theorem now follows.

Corollary 2.4. $f(r, 4) = 3r + 1$ *for* $r \leqq 10$.

Using the above results, we can also verify the stronger conjecture of Caccetta and Haggkvist [4] for small r.

Theorem 2.5. *Let D be a digraph with girth* 4 *and minimum outdegree* $\delta_D^+ \geqq r$. *Then* $|V(D)| \geqq 3r + 1$ *for* $r \leqq 6$.

Proof. By eliminating arcs if necessary, we may assume that $d_D^+(v) = r$ for every $v \in V(D)$. If D is r-regular, the conclusion follows from Corollary 2.4. So we assume D is not r-regular. Then there exists a vertex w such that $d_D^-(w) \geqq r + 1$. Since $|\Gamma^+(w)| = r \leqq 6$, we may apply Corollary 2.2 to $\langle\Gamma^+(w)\rangle$ to obtain a vertex $u \in \Gamma^+(w)$ with $d^+_{\langle\Gamma^+(w)\rangle}(u) \leqq 1$. Then $|\Gamma^+(u) - \Gamma^+(w)| \geqq r - 1$, and since the girth is 4,

$$|V(D)| \geqq |\Gamma^-(v)| + |\{v\}| + |\Gamma^+(v)| + |\Gamma^+(u) - \Gamma^+(w)| \geqq 3r + 1\,.$$

We remark that in the notation of [4], [6], the preceding theorem implies that $h(r, 4) = 3r + 1$ for $r \leqq 6$.

3. RESTRICTED r-REGULAR DIGRAPHS

The main result of this section is the verification of the conjectured inequality $f(r, 4) \geqq 3r + 1$ for a restricted class of r-regular digraphs of girth 4. This restricted class is large enough to include all such digraphs with $r \leqq 23$.

Theorem 3.1. *Let* $D = (V, E)$ *be an r-regular digraph with girth* $g \geqq 4$ *and* $\hat{\delta}_D \leqq 6$. *Then* $|V| \geqq 3r + 1$.

Proof. We use the same notation as in the proof of Theorem 2.3. Thus $\hat{\delta}_D = \hat{\delta}_D^-$, u and v are vertices with $u \in \Gamma^-(v)$ and $d^-_{\langle\Gamma^-(v)\rangle}(u) = \hat{\delta}_D$, and Figure 1 applies. If $\hat{\delta}_D = 0$, then it follows that $\Gamma^-(u) \subseteq Z$ so that $|Z| \geqq r$ and $|V| \geqq 3r + 1$. Now suppose $\hat{\delta}_D = 1$. Then $|Z'| = r - 1$. Since $|Y'| = d^+_{\langle\Gamma^+(v)\rangle}(u) \geqq \hat{\delta}_D \geqq 1$, there exists a vertex $y' \in Y'$. Since $|Y| = r$, $d^+_{\langle Y\rangle}(y') \leqq r - 1$; since $g \geqq 4$, $\Gamma^+(y') \subseteq Y \cup Z''$. The r-regularity of D now implies $Z'' \neq \emptyset$ so that $|Z| = |Z'| + |Z''| \geqq r$. Hence $|V| \geqq 3r + 1$.

As $\hat{\delta}_D$ increases, the analysis becomes more complicated. Here we proceed with the case $\hat{\delta}_D = 6$; similar, but simpler, arguments are available for $\hat{\delta}_D = 2, 3, 4, 5$.

So our assumptions are that $d^-_{\langle\Gamma^-(v)\rangle}(u) = \hat{\delta}_D = 6$. We assume $|V| < 3r + 1$, equivalently $|Z''| \leqq 5$, and obtain a contradiction.

Let $X' = \Gamma^-(u) \cap \Gamma^-(v)$ and $X'' = \Gamma^+(u) \cap \Gamma^-(v)$, so that $X' \cap X'' = \emptyset$ and $|X'| = 6$. We then have the following.

(1) $|Y'| = |\Gamma^+(u) \cap \Gamma^+(v)| = 6$ or 7: Since $g \geqq 4$, $\delta^+_{\langle Y'\rangle} \geqq (r - |Z''|) - (|Y| - |Y'|) = |Y'| - |Z''| \geqq |Y'| - 5$. Now $g \geqq 4$ implies that $|Y'| \geqq 2(|Y'| - 5) + 1$ or $|Y'| \leqq 9$. On the other hand, $|Y'| = d^+_{\langle\Gamma^+(u)\rangle}(v) \geqq \hat{\delta}_D = 6$. Suppose

$|Y'| = 8$ or 9. Then $\delta^+_{\langle Y' \rangle} \geqq |Y'| - 5 \geqq 3$ and applying Theorem 2.5 to $\langle Y' \rangle$, we conclude $|Y'| \geqq 10$, a contradiction. So $|Y'| = 6$ or 7.

(2) If $|Y'| = 6$, then $|X''| = r - 12$: Let $Y' = \{y'_1, \ldots, y'_6\}$. By Corollary 2.2 and $\delta^+_{\langle Y' \rangle} \geqq |Y'| - 5 = 1$ (see (1)), we may suppose $d^+_{\langle Y' \rangle}(y'_1) = 1$. Using $g = 4$, $|Y - Y'| = r - 6$, $|Z''| \leqq 5$, and $d^+_D(y'_1) = r$, we conclude that $Z'' \subseteq \Gamma^+(y'_1)$ and $|Z''| = 5$. Now $y'_1 \in \Gamma^+(u)$ and since $d^+_{\langle Y' \rangle}(y'_1) = 1$, $|\Gamma^+(u) \cap \Gamma^+(y'_1)| \leqq$ $\leqq |Z''| + 1 = 6$. Since $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) \geqq \hat{\delta}_D = 6$, we conclude that $Z'' \subseteq \Gamma^+(u)$. Since $d^+(u) = r$, we now conclude that $|X''| = |\Gamma^+(u) \cap \Gamma^-(v)| = r - 1 -$ $- |Y'| - |Z''| = r - 12$.

(3) If $|Y'| = 7$, then $|X''| = r - 12$ or $r - 13$: Let $Y' = \{y'_1, \ldots, y'_7\}$. Since D is r-regular, it follows that $\delta^+_{\langle Y' \rangle} \geqq 2$. If $\langle Y' \rangle$ were not 2-regular, then an argument like that used in the proof of Theorem 2.5 gives $|Y'| > 7$. Hence $\langle Y' \rangle$ is 2-regular. It now follows that $|Z''| = 5$ and $Z'' \cup (Y - Y') \subseteq \Gamma^+(y'_i)$ for $i = 1, \ldots, 7$. Let $\hat{Z}'' = Z'' \cap \Gamma^+(u)$. Then since $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) \geqq 6$, it follows that $|\hat{Z}''| = 4$ or 5 and $|X''| = |\Gamma^+(u) \cap \Gamma^-(v)| = r - 12$ or $r - 13$. We note for later use that since $|\hat{Z}''| \leqq 5$, it follows from Corollary 2.2 that there exists a $z'' \in \hat{Z}''$ such that $d^+_{\langle \hat{Z}'' \rangle}(z'') \leqq 1$.

(4) If $|Y'| = 7$, then $\Gamma^+(z'') \cap \Gamma^-(y'_1) \neq \emptyset$: We defer the proof to the next section. It follows from (4) that when $|Y'| = 7$, D has a cycle of length 3, contradicting $g = 4$.

(5) If $|Y'| = 6$, then $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 6$ or 7 for vertex y'_1 of Y' with $d^+_{\langle Y' \rangle}(y'_1) = 1$: Refering to (2), we conclude that $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) = 6 = \hat{\delta}_D$. Now reversing the direction of all arcs of D and replacing v by u and u by y'_1 in (1), we conclude that $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 6$ or 7.

If $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 7$, then applying (4) to the digraph obtained from D by reversing all arcs, we obtain a contradiction.

(6) The case $|Y'| = 6$ and $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 6$ cannot occur: We defer the proof to the next section.

Thus the proof of theorem is complete once (4) and (6) are established.

Suppose $\hat{\delta}^+_D = \hat{\delta}_D \geqq 7$ for an r-regular digraph with girth $g \geqq 4$. Then $\delta^+_D \geqq 7$ and it follows from Lemma 2.1 that

$$r > \frac{3 + \sqrt{5}}{2} \delta^+_D + 1 > 19\,.$$

Using Theorem 3.1 we now conclude that $f(r, 4) = 3r + 1$ whenever $r \leqq 19$. This can be improved as follows.

Theorem 3.2. $f(r, 4) = 3r + 1$ for $r \leqq 23$.

Proof. Let D be an r-regular digraph with girth $g \geqq 4$. By Theorem 3.1 it suffices to show that if $\hat{\delta}_D \geqq 7$ and $|V(D)| < 3r + 1$, then $r > 23$. If $\hat{\delta}_D \geqq 9$, then it follows from Lemma 2.1 as above, that $r > 24$. So suppose $\hat{\delta}_D = 7$ or 8 and $|V(D)| < 3r + 1$. We may also assume $r > 19$. We continue with the notation used in the proof of Theorem 3.1.

Since $|V(D)| < 3r + 1$, $|Z''| \leqq \hat{\delta}_D - 1$. For $y' \in Y'$, we have

$$d^+_{\langle Y \rangle}(y') \geqq r - |Z''| \geqq r - (\hat{\delta}_D - 1),$$

while for $y \in Y$, $d^+_{\langle Y \rangle}(y) \geqq \hat{\delta}_D$. Hence

$$\sum_{y \in Y} d^+_{\langle Y \rangle}(y) \geqq |Y'| (r - \hat{\delta}_D + 1) + (r - |Y'|) \hat{\delta}_D =$$

$$= |Y'| (r - 2\hat{\delta}_D + 1) + r\hat{\delta}_D \geqq \hat{\delta}_D(r - 2\hat{\delta}_D + 1) + r\hat{\delta}_D =$$

$$= 2r\hat{\delta}_D - \hat{\delta}_D(2\hat{\delta}_D - 1),$$

since $|Y'| \geqq \hat{\delta}_D$ and $r > 2\hat{\delta}_D + 1 (r > 19, \hat{\delta}_D = 7$ or $8)$. It follows that there exists a vertex $\bar{y}$ of Y such that

$$d^-_{\langle Y \rangle}(\bar{y}) \geqq 2\hat{\delta}_D - \frac{1}{r} \hat{\delta}_D(2\hat{\delta}_D - 1).$$

Suppose, to the contrary, that $r \leqq 23$. Then since $\delta_D = 7$ or 8, it follows from the above inequality that $d^-_{\langle Y \rangle}(\bar{y}) \geqq 11$. It follows from Lemma 2.1 that there exists a vertex $\bar{\bar{y}} \in \Gamma^+(\bar{y}) \cap Y$ such that

$$d^+_{\langle \Gamma^+(\bar{y}) \cap Y \rangle}(\bar{\bar{y}}) < \frac{3 - \sqrt{5}}{2} (|\Gamma^+(\bar{y}) \cap Y| - 1).$$

Then

$$|\Gamma^+(\bar{\bar{y}}) \cap Y - \Gamma^+(y) \cap Y| \geqq |\Gamma^+(\bar{\bar{y}}) \cap Y| - d^+_{\langle \Gamma^+(\bar{y}) \cap Y \rangle}(\bar{\bar{y}}) >$$

$$> \hat{\delta}_D - \frac{3 - \sqrt{5}}{2} (|\Gamma^+(\bar{y}) \cap Y| - 1).$$

Thus

$$r = |Y| \geqq |\Gamma^-(\bar{y}) \cap Y| + |\{\bar{y}\}| + |\Gamma^+(\bar{y}) \cap Y| + |\Gamma^+(\bar{\bar{y}}) \cap Y - \Gamma^+(\bar{y}) \cap Y| >$$

$$> 11 + 1 + |\Gamma^+(\bar{y}) \cap Y| + \hat{\delta}_D - \frac{3 - \sqrt{5}}{2} (|\Gamma^+(\bar{y}) \cap Y| - 1) =$$

$$= 12 + \hat{\delta}_D + \left(1 - \frac{3 - \sqrt{5}}{2}\right) |\Gamma^+(y) \cap Y| + \frac{3 - \sqrt{5}}{2} \geqq$$

$$\geqq 12 + \hat{\delta}_D + \left(1 - \frac{3 - \sqrt{5}}{2}\right) \hat{\delta}_D + \frac{3 - \sqrt{5}}{2} > 23,$$

since $\hat{\delta}_D \geqq 7$. This contradiction completes the proof of the theorem.

In this section we complete the proof of Theorem 3.1 by establishing the claims in (4) and (6). The details are not particularly illuminating, which was the reason for their deferral until this last section.

Proof of (4). Suppose $|Y'| = |\Gamma^+(u) \cap Y| = 7$. We need to show that $\Gamma^+(z'') \cap \cap \Gamma^-(y'_1) \neq \emptyset$ where $d^+_{\langle \hat{Z}'' \rangle}(z'') \leqq 1$ and $y'_1 \in Y'$.

Case 1. $|\hat{Z}''| = |\Gamma^+(u) \cap Z''| = 4$. Then $|X''| = r - 12$ and $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) = 6$ (see (3)). By reversing the directions of all arcs and replacing v by u and u by y'_1 and using $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) = 6$, we conclude from (1), (2), and (3) that $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 6$ or 7 and $|\Gamma^-(y'_1) \cap \Gamma^+(u)| = r - 12$ or $r - 13$. We then have

$$|\Gamma^-(y'_1) \cap X''| \geqq r - 13 - 4 + 1 = |X''| - 4 .$$

But $Z'' \subseteq \Gamma^+(y'_i)$ (see (3)) so that $\Gamma^+(z'') \cap Y' = \emptyset$. Hence $\Gamma^+(z'') \cap \Gamma^+(u) \subseteq \subseteq \hat{Z}'' \cup X''$. Since $\delta_D = 6$ and $d^+_{\langle \hat{Z}'' \rangle}(z'') \leqq 1$, it follows that $|\Gamma^+(z'') \cap X''| \geqq 5$ so that $\Gamma^+(z'') \cap \Gamma^-(y'_1) \neq \emptyset$.

Case 2. $|\hat{Z}''| = 5$, that is $\hat{Z}'' = Z''$. Then $\Gamma^+(u) = Z'' \cup Y' \cup X'' \cup \{v\}$, $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) = 7$, and $|X''| = r - 13$ (see (3)). Let $T = \Gamma^-(y'_1) \cap \Gamma^-(u)$. By reversing arcs and replacing v by u and u by y'_1 in the argument used in (1), we conclude that $|T| \geqq 6$ and $\delta^-_{\langle T \rangle} \geqq |T| - 6$. Since $g \geqq 4$, $|T| \geqq 2(|T| - 6) + 1$ or $|T| \leqq 11$. Applying Theorem 2.5 to $\langle T \rangle$, we now conclude that $|T| \geqq 3(|T| - 6) + 1$ or $|T| \leqq 8$. Then

$$d^-_{\langle \Gamma^+(u) \rangle}(y'_1) = |\Gamma^-(y'_1) \cap \Gamma^+(u)| \geqq r - 8 - 6 - 1 = r - 15,$$ and since $\langle Y' \rangle$ is 2-regular,

$$|\Gamma^-(y'_1) \cap X''| \geqq r - 15 - 2 - 1 = r - 18 = |X''| - 5 .$$

First suppose that $|\Gamma^-(y'_1) \cap X''| > |X''| - 5$. Then $\Gamma^+(z'') \cap \Gamma^+(u) \subseteq X'' \cup Z''$. Since $d^+_{\langle Z'' \rangle}(z'') \leqq 1$, $|\Gamma^+(z'') \cap X''| \geqq \delta_D - 1 \geqq 5$, and we conclude that $\Gamma^+(z'') \cap \cap \Gamma^-(y'_1) \neq \emptyset$.

Now suppose that $|\Gamma^-(y'_1) \cap X''| = |X''| - 5$. Then it follows that $d^-_{\langle \Gamma^+(u) \rangle}(y'_1) = = r - 15$, $|T| = 8$, and $X - X' - X'' \subseteq \Gamma^-(y'_1)$. If $\Gamma^+(z'') \cap (X - X' - X'') \neq \emptyset$, then $\Gamma^+(z'') \cap \Gamma^-(y'_1) \neq \emptyset$. So assume $\Gamma^+(z'') \cap (X - X' - X'') = \emptyset$. Then it follows that $\Gamma^+(z'') \subseteq Z'' \cup X'' \cup Y''$. Since $|Y''| = r - |Y'| = r - 7$ and since $d^+_{\langle Z'' \rangle}(z'') \leqq 1$, it follows that $|\Gamma^+(z'') \cap X''| \geqq r - (r - 7) - 1 = 6$. Thus again we conclude $\Gamma^+(z'') \cap \Gamma^-(y'_1) \neq \emptyset$. This completes the proof of (4).

Proof of (6). Suppose $|Y'| = 6$ and $|\Gamma^-(y'_1) \cap \Gamma^-(u)| = 6$, where $d^+_{\langle Y' \rangle}(y'_1) = 1$. By (2), $|X''| = r - 12$. By (5), $d^+_{\langle \Gamma^+(u) \rangle}(y'_1) = 6 = \delta_D$, so that by reversing the direction of all arcs and replacing u by y'_1 and v by u in (2), we conclude that

$$|\Gamma^-(y'_1) \cap \Gamma^+(u)| = d^-_{\langle \Gamma^+(u) \rangle}(y'_1) = r - 12 .$$

It follows from $|Y'| = 6$, $d^+_{\langle Y' \rangle}(y'_i) \geqq 1$ $(i = 1, \ldots, 6)$, and $g \geqq 4$ that $d^-_{\langle Y' \rangle}(y'_1) \leqq 3$.

We now distinguish three cases.

Case 1. $d^-_{\langle Y'\rangle}(y'_1) = 3$. It then follows that

$$|\Gamma^-(y'_1) \cap X''| = |\Gamma^-(y'_1) \cap \Gamma^+(u)| - 3 - 1 = r - 12 - 4 = |X''| - 4 .$$

Since $d^+_{\langle Y'\rangle}(y'_i) \geqq 1$ for $Y' = \{y'_1, \ldots, y'_6\}$, we may assume that (y'_1, y'_2) and (y'_2, y'_3) are edges (see Figure 2).

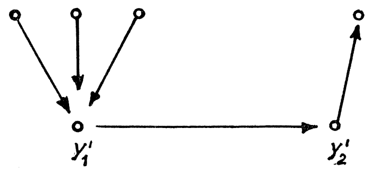

Figure 2. The 6 vertices of $\langle Y'\rangle$ for case 1.

Since $|Z''| = 5$, by Corollary 2.2 there exists a vertex $z'' \in Z''$ with $d^+_{\langle Z''\rangle}(z'') \leqq 1$. Since $d^+_{\langle Y'\rangle}(y'_3) \geqq 1$, it follows from $g \geqq 4$ that $d^-_{\langle Y'\rangle}(y'_2) \leqq 3$. First suppose $d^-_{\langle Y'\rangle}(y'_2) = 3$. Then $d^+_{\langle Y'\rangle}(y'_3) = 1$ and so $Z'' \subseteq \Gamma^+(y'_3)$. Hence $\Gamma^+(z'') \cap Y' = \emptyset$. Therefore $|\Gamma^+(z'') \cap \Gamma^+(u)| \geqq 6$, implies $|\Gamma^+(z'') \cap X''| \geqq 5$. Hence $\Gamma^-(y'_1) \cap \cap \Gamma^+(z'') \neq \emptyset$ contradicting $g \geqq 4$. Now suppose $d^-_{\langle Y'\rangle}(y'_2) \leqq 2$. Since $d^+_{\langle Y'\rangle}(y'_2) = 1$, it follows that $Z'' \subseteq \Gamma^+(y'_2)$ and $|\Gamma^-(y'_2) \cap X''| \geqq (r - 12) - 2 - 1 = |X''| - 3$. Since $g \geqq 4$, $\Gamma^+(z'') \cap Y' \subseteq \{y'_3\}$ and hence $|\Gamma^+(z'') \cap X''| \geqq 5 - 1 = 4$. Hence $\Gamma^+(z'') \cap \Gamma^-(y'_1) \neq \emptyset$, contradicting $g \geqq 4$.

Case 2. $d^-_{\langle Y'\rangle}(y'_1) \leqq 1$. Then as before it follows that $|\Gamma^-(y'_1) \cap X''| \geqq |X''| - 2$. So if $|\Gamma^+(Z'') \cap X''| \geqq 3$, there is a cycle of length 3. Now assume that $|\Gamma^+(Z'') \cap \cap X''| \leqq 2$, so that the number of arcs from Z'' to X'' is at most $5 \cdot 2 = 10$. Counting arcs we obtain

$$|(Z'' \times Z'') \cap E| + |(Z'' \times X'') \cap E| + |(Z'' \times Y') \cap E| =$$
$$= \sum_{z \in Z''} d^+_{\langle \Gamma^+(u)\rangle}(z) \geqq 5\hat{\delta}_D = 5 \cdot 6 = 30 ;$$
$$|(Y' \times Y') \cap E| + |(Y' \times Z'') \cap E| = \sum_{i=1}^{6} d^+_{\langle \Gamma^+(u)\rangle}(y'_i) \geqq 6 \cdot \hat{\delta}_D = 36 .$$

Moreover since $d^+_{\langle Y''\rangle}(y_i) \geqq 1$ for $i = 1, \ldots, 6$, $\langle Y''\rangle$ contains a cycle of length of at least 4 and consequently $|(Y' \times Y') \cap E| \leqq 13$. So

$$|(Y' \times Z'') \cap E| \geqq 36 - 13 = 23$$

and

$$|(Z'' \times Y') \cap E| \leqq 5 \cdot 6 - 23 = 7 .$$

It now follows that

$$|(Z'' \times Z'') \cap E| \geqq 30 - 10 - 7 = 13 ,$$

which contradicts the obvious fact that $\langle Z''\rangle$ has at most $\binom{5}{2} = 10$ arcs.

Case 3. $d^-_{\langle Y' \rangle}(y'_1) = 2$. Then it follows that $|\Gamma^-(y'_1) \cap X''| = |X''| - 3$. If $|\Gamma^+(Z'') \cap X''| \geqq 4$, there is a cycle of length 3, contradicting $g \geqq 4$. Now assume $|\Gamma^+(Z'') \cap X''| \leqq 3$, so that the number of arcs from Z'' to X'' is atmost $5 . 3 = 15$. Let $\Gamma^-(y'_1) \cap Y' = \{y'_5, y'_6\}$ and $Y'_0 = \{y'_2, y'_3, y'_4\}$ (see Figure 3).

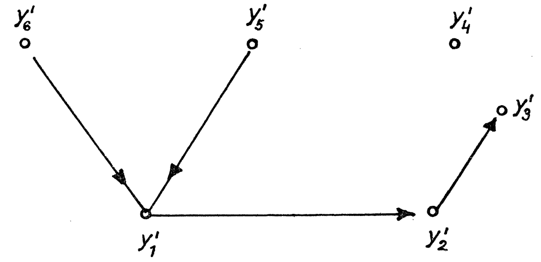

Figure 3. The 6 vertices of $\langle Y' \rangle$ for Case 3.

It is easy to verify that $\sum_{i=2}^{4} d^+_{\langle Y' \rangle}(y'_i) \leqq 7$. Hence $|(Y'_0 \times Z'') \cap E| \geqq 3\hat{\delta}_D - 7 = 11$, so that $|(Z'' \times Y'_0) \cap E| \leqq 5 . 3 - 11 = 4$. Since $g \geqq 4$, it follows that $(Z'' \times Y') \cap \cap E = (Z'' \times Y'_0) \cap E$. But counting arcs again we obtain

$$|(Z'' \times Z'') \cap E| + |(Z'' \times X'') \cap E| + |(Z'' \times Y'_0) \cap E| \geqq 5\hat{\delta}_D = 30 .$$

Hence

$$|(Z'' \times Z'') \cap E| \geqq 30 - 15 - 4 = 11 ,$$

contradicting once again that $\langle Z'' \rangle$ has at most 10 arcs.

This completes the proof.

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