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PSEUDO-SIMILARITY AND PARTIAL UNIT REGULARITY

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I. INTRODUCTION

In a recent paper [6] the notion of pseudo-similarity was introduced. This notion is defined in the following way. Let $R_{m\times n}$ denote the $m \times n$ matrices over a ring R with unity. If $A \in R_{m\times m}$ and $B \in R_{n\times n}$, we say that A is *pseudo-similar to* B, $A \approx B$, if there exist $X \in R_{m\times n}$ and two possibly distinct $X^-, X^= \in R_{n\times m}$ such that

$$(1) \qquad X^- A X = B\,, \quad X B X^= = A\,, \quad X X^- X = X\,, \quad X X^= X = X\,.$$

It was proved in [6] that for $R = F$, a field, and $A, B \in F_{n\times n}$, pseudosimilarity implies similarity, and hence pseudo-similarity is equivalent to similarity in this case. Subsequently, it was proved in [7] that pseudosimilarity implies similarity in a unit regular ring. In another paper [2] on pseudo-similarity, the possible ranks of X, X^-, $X^=$ were determined in the field case, and the class of pairs of matrices A and B, for which $A \approx B$, was characterized for fixed matrices X, X^-, and $X^=$. Other articles which use the idea of pseudo-similarity are [4] and [8].

Let Z denote the ring of integers. For $A, B \in Z_{n\times n}$ it follows from the result proved in [6] that $A \approx B$ over the integers implies A is similar to B over the rationals. Generally, however, similarity of integral matrices over the rationals does not imply similarity over the integers, see [10, p. 55]. But we shall establish here that $A \approx B$ over the integers does imply that A is similar to B over the integers, and thus pseudo-similarity is equivalent to similarity for integral matrices. In fact we will prove this same result for matrices over a much more general type of ring. We do this by introducing a new kind of regularity, partial unit regularity, and show that a ring R with unity is partially unit regular if and only if pseudo-similarity implies similarity in R. As a result, in rings R where regular matrices $A \in R_{n\times n}$ have a certain normal form, pseudo-similarity coincides with similarity in $R_{n\times n}$. We will as well obtain equivalent conditions for the regular matrices $A \in R_{n\times n}$ to have this normal form. Examples and related results are also given.

We first recall some definitions for elements in a ring R. The element $a \in R$ is regular if there is a solution to the equation $axa = a$. These solutions are usually called inner-or 1-inverses of a and will be denoted by a^-. The element $a \in R$ (with unity) is unit regular if there is a unit $u \in R$ such that $aua = a$, where the unit elements are the elements which have a two-sided inverse. The ring R (with unity) is [unit] regular if every $a \in R$ is [unit] regular. We will assume that all rings have a unity 1. If R is a ring with unity, then $R_{n \times n}$ is a ring with unity and the above definitions apply to $R_{n \times n}$ as well.

II. PSEUDO-SIMILARITY AND PARTIAL UNIT REGULARITY

We now introduce a new kind of regularity.

Definition. A ring R is called *partially unit regular* if every regular element is unit regular.

Examples of partially unit regular (p.u.r. for short) rings are numerous. A trivial example of a p.u.r. ring is a unit regular ring. So every division ring is p.u.r. The ring of integers is a p.u.r. ring. Inded, any ring R where the only regular elements are 0 and the units is p.u.r. In particular, integral domains are p.u.r. rings, for in these rings,

$$a(1 - a^-a) = 0\,, \quad a \neq 0 \Rightarrow 1 - a^-a = 0\,.$$

Also, the ring Z/p^m (the integers modulo p^m, p a prime) is p.u.r. since every element in Z/p^m is a unit or is divisible by p.

Of more interest are the rings R, for which $R_{n \times n}$ for all n, and hence R, are p.u.r. What is somewhat surprising, are the many examples of this which we can find. We remark that these matrix rings will furnish us with examples of p.u.r. rings, other than unit regular rings, where not every regular element is 0 or a unit. But first we characterize p.u.r. rings.

Theorem 1. *The ring R is p.u.r. if and only if pseudo-similarity implies similarity in R.*

Proof. To prove this theorem we use the equivalence of (1) and (8) in Theorem 2B in [7].

($\Rightarrow$) Suppose that

(2) $$x^-ax = b\,, \quad xbx^= = a\,, \quad xx^-x = x\,, \quad xx^=x = x\,.$$

Since R is p.u.r., $xux = x$ for some unit u in R. Now, let

$$q = (1 - xx^= + xu)\,u^{-1}\,(1 - x^-x + ux)\,.$$

Then, from [7], q is a unit and $q^{-1}aq = b$, so that a is similar to b.

($\Leftarrow$) We assume that pseudo-similarity implies similarity. Let x be any regular element in R and suppose that (2) holds for some $a, b, x^-, x^= \in R$ (for example, $a = xx^-$, $b = x^-x$, $x^= = x^-$). Then, from our assumption and the theorem in [7], we have that x is unit regular. Thus, R is p.u.r.

As a consequence of Theorem 1, in each of the rings mentioned above, pseudo-similarity coincides with similarity.

We remark that in the proof of Theorem 1 the element q is independent of a and b. Thus, assuming x is unit regular, if $a \approx b$, via $x, x^-, x^=$, for any number of pairs of elements a and b, then a is similar to b for all those pairs.

Now let us turn to matrices.

Corollary 1. *Suppose $R_{n \times n}$ is such that for each regular matrix $A \in R_{n \times n}$,*

$$PAQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

for some units P, $Q \in R_{n \times n}$. Then, $R_{n \times n}$ is p.u.r. and thus pseudo-similarity coincides with similarity in $R_{n \times n}$.

Proof. Suppose A is regular and

$$PAQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where P and Q are units in $R_{n \times n}$. It is then immediate that QP is a unit inner inverse of A. The result then follows from Theorem 1.

Examples. (1) Let R be any elementary divisor ring, so that for each $A \in R_{n \times n}$ there exist units P and Q in $R_{n \times n}$ where $PAQ = D$, a diagonal matrix, [9]. Examples of such rings are division rings, principal ideal domains, and Z/p^m [1], [5]. We further assume that the only regular elements of R are 0 and the units, as is the case with these examples. Then, $R_{n \times n}$ is a p.u.r. ring. To see this note that if A is regular then D is regular. Letting (without loss of generality)

$$D = \begin{bmatrix} s_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & s_r & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ 0 & & & & & & 0 \end{bmatrix}$$

where each $s_i \neq 0$, it then follows that each s_i is regular. Hence each s_i is a unit. Thus "new" units P and Q may be constructed such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

(2) Let $R = \mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials in n variables with complex

coefficients. In [11] it is shown that $A \in R_{n\times n}$ has a 1-inverse if and only if there exist units P, $Q \in R_{n\times n}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} A$. Thus, $R_{n\times n}$ is p.u.r.

(3) We give an example where $R_{n\times n}$ is p.u.r. but where the condition of the above corollary is not necessarily met. Let R be a unit regular ring. It was shown in [9] that R is then an elementary divisor ring, from which it follows that $R_{n\times n}$ also is unit regular and thus p.u.r. Since in general unit regular elements do not have to be units, the condition of the corollary does not hold in general here, that is, every regular matrix A in $R_{n\times n}$ may not be unit equivalent to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

We shall take up this question shortly in Section IV.

If A and B are square matrices over R, but of different sizes, then we can augment zeros to the matrices $X, X^-, X^=, A$ (or B), of (1) to obtain some generalizations. The case where $m > n$ is given below.

Theorem 2. *Let $A \in R_{m\times m}$, $B \in R_{n\times n}$, where $R_{m\times m}$ is p.u.r. and $m > n$. Then, $A \approx B$ over R if and only if A is similar to*

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

over R.

Proof. The proof is straightforward and is omitted.

III. MATRICES OVER PRINCIPAL IDEAL DOMAINS

In this section R will denote a principal ideal domain, examples of which are the ring of integers, any field, and the ring of polynomials in a single variable with coefficients from a field. In the previous section we saw that $R_{n\times n}$ is a p.u.r. ring so that pseudo-similarity is equivalent to similarity in $R_{n\times n}$. Here we look at the situation using the block form of the matrices. To do so we note that the results of [2] and [3] which we use are valid for matrices over any principal ideal domain.

Suppose $A \approx B$ as in (1), where $A, B, X, X^-, X^= \in R_{n\times n}$. Let $\operatorname{rank} X = r$. Then, from Theorem 6 in [3], we have units P and Q and a matrix L such that

$$PXQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad X^- = Q\begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix}P.$$

We next use Theorem 1 in [3] to write

$$X^{=} = Q\begin{bmatrix} I_r & Z_2 \\ Z_3 & Z_4 \end{bmatrix}P$$

for some matrices Z_2, Z_3, Z_4.

Now set $QP = U$. Then, using matrices, the expression for q in the proof of Theorem 1 becomes

$$P^{-1}\begin{bmatrix} I & -Z_2 \\ 0 & I \end{bmatrix}Q^{-1}$$

which we shall denote by T. Now, as in Theorem 4 in [2], we can write

$$A = P^{-1}\begin{bmatrix} W & WZ_2 \\ 0 & 0 \end{bmatrix}P, \qquad B = Q\begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

for some $r \times r$ matrix W. It can then be verified by direct multiplication that $T^{-1}AT = B$, so that A is similar to B over R (as was shown in Theorem 1).

We note in the case that $X^{-} = X^{=}$ we can take $Z_2 = 0$ in the above. In general the matrix T^{-1} which we obtained turns out to be a unit inner inverse of X.

In [2] the possible ranks of $X, X^{-}, X^{=}$ were determined in the field case in terms of the core-nilpotent decomposition of A and B. With $A, B \in F_{n\times n}$, $A \approx B$ as in (1), and t denoting the number of 1×1 Jordan Blocks [0] in the Jordan form of A and B, it was shown that

$$\operatorname{rank} X \geqq n - t\,.$$

Conversely, for each positive integer r, $n - t \leqq r \leqq n$, and for all positive integers p, q, $r \leqq p \leqq n$, $r \leqq q \leqq n$, there exist X of rank r, X^{-} of rank p, $X^{=}$ of rank q, such that (1) holds. For matrices over a principal ideal domain R it is an open question as to whether for each r, $n - t \leqq r < n$, there exist X of rank r such that (1) holds. However, in this case, for a fixed X satisfying (1) there do exist X^{-} and $X^{=}$ of the various possible ranks satisfying (1).

Theorem 3. *Suppose* $A \approx B$ *as in* (1) *where* $A, B, X, X^{-}, X^{=} \in R_{n\times n}$ *and let* $\operatorname{rank} X = r$. *Then, with this fixed* X, *for all positive integers* p, q, $r \leqq p \leqq n$, $r \leqq q \leqq n$, *there exist* X^{-} *of rank* p, *and* $X^{=}$ *of rank* q, *such that* (1) *holds.*

Proof. Let $A \approx B$ via $X, X^{-}, X^{=}$. As in the above we can write

$$X = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}, \quad X^{-} = Q\begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix}P, \quad X^{=} = Q\begin{bmatrix} I_r & Z_2 \\ Z_3 & Z_4 \end{bmatrix}P,$$

$$A = P^{-1}\begin{bmatrix} W & WZ_2 \\ 0 & 0 \end{bmatrix}P, \quad B = Q\begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

for some units P, Q and some matrices L, Z_2, Z_3, Z_4 and W. But then with any choices for L, Z_3, and Z_4, it is easy to check that the equations in (1) still hold. Hence, setting $Z_3 = 0$ and choosing matrices L and Z_4 with appropriate ranks, our result then follows.

IV. REDUCTION OF A TO NORMAL FORM

We shall now investigate when a regular matrix A has the normal form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

under unit equivalence as in Corollary 1. There of couse all matrices were square. However, to arrive at conditions which hold for matrices over arbitrary rings we must initially consider possibly non-square units. We start with some more definitions and notation.

Let R be a ring. By R^m we denote the $m \times 1$ column vectors over R, considered as a right R-module. Similarly, ${}^nR$ denotes the $1 \times n$ row vectors over R, considered as a left R-module. We recall that a basis for any module, if one exists, is a linearly independent subset which spans the module.

Now let $B \in B_{m \times n}$. By $R(B)$ we mean the submodule of R^m of all right linear combinations of the columns of B and by $RS(B)$ the submodule of ${}^nR$ of all left linear combinations of the rows of B. As usual $N(B) = \{x \in R^n \mid Bx = 0\}$ a submodule of R^n. We say that B is invertible if and only if there exists $X \in R_{n \times m}$ such that

$$BX = I_m \quad \text{and} \quad XB = I_n .$$

The inverse of B, if one exists, is unique and is denoted as usual by B^{-1}.

Lemma 1. *The following are equivalent for $B \in R_{m \times n}$.*
(i) *B is invertible.*
(ii) *The columns of B form a basis for R^m.*
(iii) *The rows of B form a basis for ${}^nR$.*

Proof. (ii) $\Rightarrow$ (i) By the spanning property there exists $X \in R_{n \times m}$ such that $BX = I_m$. Then $B(I_n - XB) = 0$ and so by the linear independence $I_n - XB = 0$, or $XB = I_n$. Thus, $X = B^{-1}$.

The rest of the proof should be clear.

Definition. Let W_1 be a submodule of the R-module W. To say that the basis B_1 of W_1 has a *complementary basis* B_2 will mean that $B_1 \cup B_2$ is a basis of W.

Theorem 4. *The following are equivalent for $A \in R_{n \times n}$.*

(i) $$A = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V$$

for some invertible matrices $U \in R_{n \times m}$, $V \in R_{m \times n}$ and some r.

(ii) *$R(A)$ and $RS(A)$ have finite bases and these bases have finite complementary bases.*

(iii) *$R(A)$ and $N(A)$ have finite bases and these bases have finite complementary bases.*

Proof. (i) $\Rightarrow$ (ii) Let

$$A = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V$$

as given in (i) and let

$$U = [U_1\ U_2], \quad V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $U_1 \in R_{n \times r}$, $V_1 \in R_{r \times n}$. Then $A = U_1 V_1$ and since V is invertible there exists $V_1^- \in R_{n \times r}$ such that $V_1 V_1^- = I_r$.

Now clearly $R(A) \subseteq R(U_1)$, and since $AV_1^- = U_1 V_1 V_1^- = U_1$, $R(A) = R(U_1)$. Hence, the columns of U_1 form a basis of $R(A)$, with complementary basis the columns of U_2. Similarly, $RS(A) = RS(V_1)$, and the rows of V_1 form a basis of $RS(A)$, with complementary basis the rows of V_2.

(ii) $\Rightarrow$ (iii) Let a basis for $RS(A)$ be arranged as rows in a matrix V_1, a complementary basis be arranged as rows in a matrix V_2, and let

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where V is say $p \times n$. Then V is invertible and we partition V^{-1} conformably with V as

$$V^{-1} = [Q_1\ Q_2].$$

Hence, $Q_1 V_1 + Q_2 V_2 = I_n$ and $V_1 Q_2 = 0$.

We now claim that the columns of Q_2 form a basis for $N(A)$ (with complementary basis the columns of Q_1). Indeed, since $RS(A) = RS(V_1)$, there exist Y, Z such that $V_1 = YA$ and $A = ZV_1$. Hence, $AQ_2 = ZV_1 Q_2 = 0$, so that $R(Q_2) \subseteq N(A)$.

Conversely, let $Ax = 0$. Then $Q_1 V_1 x = Q_1 YAx = 0$, and so $(I_n - Q_2 V_2)x = 0$, so that $x = Q_2 V_2 x$. Hence, $x \in R(Q_2)$ and $R(Q_2) = N(A)$. The above claim is thus true and our implication holds.

(iii) $\Rightarrow$ (i) Let a basis for $N(A)$ be arranged as columns in a matrix Q_2, a complementary basis be arranged as columns in a matrix Q_1, and let

$$Q = [Q_1\ Q_2]$$

where Q is say $n \times q$. Then Q is invertible and we partition Q^{-1} conformably with Q as

$$Q^{-1} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

We first claim that the columns of AQ_1 form a basis for $R(A)$.

Clearly $R(AQ_1) \subseteq R(A)$. Also, $A = A(Q_1P_1 + Q_2P_2) = AQ_1P_1$, so that $R(AQ_1) = R(A)$. Now let $AQ_1x = 0$. Then $Q_1x \in N(A) \cap R(Q_1) = R(Q_2) \cap R(Q_1) = \{0\}$, or $Q_1x = 0$. Since $P_1Q_1 = I$ we then have $x = 0$. Hence the columns of AQ_1 form a linearly independent set, and the above claim is valid.

Next, we are given some basis for $R(A)$, say the columns of U_1, with a complementary basis, say the columns of U_2. Then there exist T, S such that $AQ_1 = U_1T$ and $U_1 = AQ_1S$. Hence $AQ_1 = AQ_1ST$ and $U_1 = U_1TS$. It then follows that $S = T^{-1}$.

Lastly, we have

$$AQ = [AQ_1 \; 0] = [U_1T \; 0] = [U_1T \; U_2]\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

or

$$A = [U_1T \; U_2]\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

(where the sizes of the three matrices on the right-hand side are $n \times q$, $q \times q$, $q \times n$, respectively). Now,

$$[U_1T \; U_2] = [U_1 \; U_2]\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

a product of two invertible matrices, so that $[U_1T \; U_2]$ is invertible. The proof of the theorem is now complete.

We recall that $R_{n\times n}$ is said to be finite if $AB = I_n \Rightarrow BA = I_n$, for example see [7].

Lemma 2. *If $R_{n\times n}$ is finite, then $R_{k\times k}$ is finite for all k, $k \leqq n$, and all invertible matrices with row or column dimension equal to n are square.*

Proof. That $R_{k\times k}$ is finite for all k, $k \leqq n$, is easy to show. Next, suppose that

$$AX = I_m \quad \text{and} \quad XA = I_n$$

where $A \in R_{m\times n}$, $X \in R_{n\times m}$. Assume that $m < n$ and set

$$A = [G \; H], \quad X = \begin{bmatrix} W \\ Z \end{bmatrix}$$

where $G, W \in R_{m\times m}$. Then $XA = I_n$ yields $WG = I_m$ and $ZH = I_{n-m}$. Since $R_{m\times m}$ is finite we have $GW = I_m$. Then, $AX = I_m$ says that $GW + HZ = I_m$, so that $HZ = 0$ and hence $HZH = 0$. But $ZH = I_{n-m}$ implies $HZH = H$, and so $H = 0$. Hence, $I_{n-m} = 0$, a contradiction.

In a similar manner, if we assume $n < m$ we also arrive at a contradiction. Thus $m = n$ and the proof of the lemma is complete.

Recall that if $E \in R_{n\times n}$ and $E^2 = E$, then E is called an *idempotent matrix*.

Theorem 5. *The following are equivalent:*

(i) *For each regular matrix* $A \in R_{n \times n}$,

$$A = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V$$

for some invertible matrices $U \in R_{n \times m}$, $V \in R_{m \times n}$, *and some* r.

(ii) *If* $E \in R_{n \times n}$ *and* E *is idempotent, then*

$$E = N \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1}$$

for some invertible matrix $N \in R_{n \times m}$ *and some* r.

(iii) *If* $E \in R_{n \times n}$ *and* E *is idempotent, then* $R(E)$ *has a finite basis.*
In which case $R_{n \times n}$ *is finite p.u.r. and all invertible matrices with row or column dimension equal to* n *are square.*

Proof. (i) $\Rightarrow$ (ii) Since $E^2 = E$, E is regular. So,

$$E = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V$$

for some invertible matrices $U \in R_{n \times m}$, $V \in R_{m \times n}$, and some r. Now set

$$F = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

$Y = UFU^{-1}$, and $W = I - E + Y$. Hence, $YE = UFU^{-1}UFV = UFV = E$ and $Y = UFU^{-1} = UF(VV^{-1})U^{-1} = EV^{-1}U^{-1}$, so that $EY = Y$. It can then be verified directly that $W^{-1} = I + E - Y$.

Now, since $EUF = UF$, we have $WUF = (I - E + Y)UF = UF - UF + YUF = UF$. Also,
$EWU = E(I - E + Y)U = (E - E + Y)U = YU = UF$. Hence, $EWU = WUF$, and thus $E = NFN^{-1}$, where $N = WU$.

(ii) $\Rightarrow$ (iii) This follows from Theorem 4.

(iii) $\Rightarrow$ (i) Since A is regular, A has an inner inverse, say A^-. Hence, $E = AA^-$ is idempotent and so $R(E)$ has a finite basis. Now $R(E) = R(A)$. Moreover,

$$R^n = R(E) \dotplus R(I - E)$$

where $\dotplus$ denotes a direct sum. But $I - E$ is also idempotent and so $R(I - E)$ has a finite basis. Hence, $R(A)$ has a finite basis with a finite complementary basis.

Likewise, $N(A) = R(I - A^-A)$, where A^-A and $I - A^-A$ are idempotent. Now

$$R^n = R(A^-A) \dotplus R(I - A^-A).$$

Hence, $N(A)$ has a finite basis with a finite complementary basis. The result in (i) then follows from Theorem 4.

To finish the proof of the theorem, suppose that $AB = I_n$, where $A, B \in R_{n \times n}$.

Hence, $ABA = A$, so that A is regular. Then from (i),

$$A = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V$$

for some invertible matrices $U \in R_{n \times m}$, $V \in R_{m \times n}$, and some r. So,

$$U^{-1}AB = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} VB$$

or

$$U^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} VB$$

or

$$I_m = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} VBU .$$

It then follows that $r = m$, so that $A = UV$. Thus A is invertible and $R_{n \times n}$ is finite. It now follows from Lemma 2 that all invertible matrices with row or column dimension equal to n are square. Hence, U and V are square, and from Corollary 1, $R_{n \times n}$ is p.u.r.

Corollary 2. *Suppose that $R_{n \times n}$ satisfies the conditions of the above theorem. Then, for all k, $k \leqq n$, $R_{k \times k}$ also satisfies those conditions. Thus, for all k, $k \leqq n$, $R_{k \times k}$ is finite p.u.r.*

Proof. Let $F^2 = F \in R_{k \times k}$, $k < n$, and let E be the $n \times n$ idempotent matrix

$$\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $R(E) = R\left(\begin{bmatrix} F \\ 0 \end{bmatrix}\right)$, and since we are assuming $R_{n \times n}$ satisfies (i–iii) of Theorem 5, we have that $R\left(\begin{bmatrix} F \\ 0 \end{bmatrix}\right)$ has a finite basis. To prove our corollary we show that $R(F)$ then has a finite basis.

Let a basis for $R\left(\begin{bmatrix} F \\ 0 \end{bmatrix}\right)$ be arranged as columns in an $n \times q$ matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where B_1 is $k \times q$. Then, by the spanning property, there exists $T \in R_{q \times k}$ such that

$$\begin{bmatrix} F \\ 0 \end{bmatrix} = BT .$$

Also, since the columns of B are in $R\left(\begin{bmatrix} F \\ 0 \end{bmatrix}\right)$, there exists $S \in R_{k \times q}$ such that

$$B = \begin{bmatrix} F \\ 0 \end{bmatrix} S .$$

So, $B = BTS$, and by the linear independence of the columns of B, we obtain $TS = I_q$. But then $0 = B_2 T$ implies $B_2 = 0$. Hence, the columns of B_1 are linearly independent, and since $F = B_1 T$, the columns of B_1 form a basis for $R(F)$.

From Theorem 5 we then have equivalent conditions for each regular matrix $A \in R_{n \times n}$ to be *unit* equivalent to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Under these conditions pseudo-similarity thus coincides with similarity in $R_{n \times n}$. Moreover, with these conditions, the only regular elements of R are 0 and the units. For suppose $a \in R$ is regular and $a \neq 0$. Then

$$A = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}$$

is regular in $R_{n \times n}$. Hence, we have

$$U^{-1} A V^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

for some units $U, V \in R_{n \times n}$. So, equating (1, 1)-entries, we see that $ras = 1$ for some $r, s \in R$. From the above corollary R is finite so that $asr = 1$ and $sra = 1$. Thus, a is a unit.

V. OTHER OPEN QUESTIONS

(1) All of our examples where $R_{n \times n}$ is p.u.r. have been where R is p.u.r. Does R being p.u.r. always imply that $R_{n \times n}$ is p.u.r.?

(2) In [8] the notion of semi-similarity is defined for two matrices A and B over a ring. Conditions are given for two matrices over a division ring to be semi-similar. Are there similar conditions for matrices over more general types of rings, such as $Z_{n \times n}$?

(3) Can one characterize canonical forms under the pseudo- and semi-similarity relations?

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