Czechoslovak Mathematical Journal

Luděk Zajíček

On differentiation of metric projections in finite dimensional Banach spaces

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 3, 325–336

Persistent URL: http://dml.cz/dmlcz/101883

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *33 (108)*, PRAHA 24. 10. 1983, No 3

ON DIFFERENTIATION OF METRIC PROJECTIONS IN FINITE DIMENSIONAL BANACH SPACES

LUDĚK ZAJÍČEK, Praha

(Received November 7, 1978)

1. INTRODUCTION

We will consider a real finite-dimensional Banach space X and a nonempty closed set $M \subset X$. For $x \in X$ denote by $d_M(x)$ the distance from the point x to the set M. The metric projection P_M of the space X on the set M is defined as the (possibly) multivalued operator $P_M(x) = \{y \in M;\ \|x - y\| = d_M(x)\}$. Of course, since X is finite dimensional, we always have $P_M(x) \neq \emptyset$. If P_M is singlevalued at a point x, $P_M(x) = \{P\}$, we will write also $P_M(x) = P$. The symbol E_n denotes the n-dimensional Euclidean space.

Kruskal [8] constructed a closed convex subset M of E_3 for which there exists a point $x \in E_3$ and a vector $v \in E_3$, $v \neq o$, such that the directional derivative

$$D_v P_M(x) := \lim_{h \to 0+} (P_M(x + hv) - P_M(x))\, h^{-1}$$

does not exist. He asked whether for any closed convex set $M \subset E_n$ the set of the pairs (x, v) for which $D_v P_M(x)$ exists is dense in the space $E_n \times \{v;\ \|v\| = 1\}$. Asplund [1] showed that the answer is affirmative. In fact he proved that if $M \subset E_n$ is an arbitrary closed set, then P_M is Frechet differentiable at almost all points $x \in E_n$. (Note that in the present article we use an extended notion of the Frechet differentiability for multivalued operators (see Definition 3).) In the same article Asplund raised the problem of characterization of finite dimensional spaces in which any metric projection on a closed set is almost everywhere Frechet differentiable. (Note that the answer to Kruskal's question (M convex) follows easily from the fact that for convex M, P_M is a contraction and therefore is a.e. Frechet differentiable.)

In Section 2 of the present article we construct an example (see Theorem 1) of a compact convex set $M \subset E_2$ such that the set of all points $x \notin M$ at which all directional derivatives $D_v P_M(x)$ exist is of the first category. This example also shows that the set of all pairs (x, v) for which $D_v P_M(x)$ exists can be of the first category in

$(E_2 - M) \times \{v; \|v\| = 1\}$. The construction of this example is based on the well known [15] example of a Lipschitz function $f\colon E_1 \to E_1$ which is nondifferentiable at all points of a residual set.

Further, we give a partial solution of Asplund's problem. We prove that any metric projection on a closed set is almost everywhere Frechet differentiable in any strictly convex two dimensional Banach space (Theorem 3) and in any finite dimensional Banach space with a norm which is in a sense (see Definition 4) Euclidean-like (Theorem 4). The theorems are proved by the same rather simple method using the auxiliary notion of an α-monotone operator.

I do not know whether there exists a finite dimensional strictly convex Banach space X and a closed nonvoid set $M \subset X$ such that the metric projection P_M is not Frechet differentiable at almost all points. (Note that if X is not strictly convex then there exists a hyperplane $M \subset X$ such that P_M is singlevalued at no point from $X - M$ [14].)

2. A COUNTEREXAMPLE

For a closed set $M \subset E_n$ denote by N_M the set of all points at which P_M is not Frechet differentiable. By Asplund [1], N_M is of Lebesgue measure zero. Naturally the question arises whether N_M is always a set of the first category. We will construct an example which shows that the answer is negative. In fact we will prove the following stronger theorem.

Theorem 1. *There exists a compact convex set $M \subset E_2$ and a set $A \subset E_2 - M$ of the first category such that for any point $x \in E_2 - (M \cup A)$ the directional derivative $D_v P_M(x)$ exists only for v of the form $v = \lambda(P_M(x) - x)$.*

Proof. It is well known that there exists a Lipschitz function g defined on $(0, 1)$ and a set $D \subset (0, 1)$ of the first category such that for $x \in (0, 1) - D$ the derivative $g'(x)$ does not exist (see e.g. [15] or [10]). For a construction of such a function it is sufficient to choose a measurable set $E \subset (0, 1)$ such that for any open interval $I \subset (0, 1)$ the inequality $0 < \mu(E \cap I) < \mu I$ holds, and to put $g(x) = \int_0^x \chi_E(t)\, dt$. Since by [11] the set of points x at which $g'_+(x)$ or $g'_-(x)$ exists and $g'(x)$ does not exist is a set of the first category we conclude that there exists a set $C \subset (0, 1)$ of the first category such that at $x \in (0, 1) - C$ neither $g'_+(x)$ nor $g'_-(x)$ exist. Now define the function f on $\langle 0, 1 \rangle$ by the equation $f(x) = \int_0^x g(t)\, dt$. The function f is obviously increasing and convex on $\langle 0, 1 \rangle$ and $f' = g$ on $(0, 1)$, $f(0) = 0$, $f'_+(0) = 0$, $0 < f(1)$, $0 < f'_-(1) < \infty$. Therefore there exist unique numbers $c > 0$, $d > 0$ such that for the function $h(x) := c f(x) - d$ the identities $h(1) = -1$, $h'_-(1) = 1$ hold. Let B be the smallest set which contains the graph of the function h and which is symmetric with respect to the lines $x = 0$, $y = 0$, $y = -x$, $y = x$. The set B obviously is the boundary of a compact convex set M and B has at any point $x \in B$ a tangent line which depends continuously on x. For $0 < x < 1$ put $P(x) = (x, h(x))$ and denote by $n(x)$

the unit vector of the outer normal to M at $P(x)$. Let $x_0 \in (0, 1) - C$ and let $p_0 \notin B$ be a point which lies on the outer normal to M at $P(x_0)$. Let $v \in E_n$ be a unit vector, $v \neq n(x_0)$, $v \neq -n(x_0)$. We will show that the directional derivative $D_v P_M(p_0)$ does not exist. Obviously, for sufficiently small t there exists one and only one point $x(t)$ such that $P_M(p_0 + tv) = P(x(t))$. We will further suppose that v is such a vector that $A(v, n(x_0)) = \beta < \pi/2$ and that $t > 0$ implies $x(t) > x_0$. The other cases are quite similar. Now suppose on the contrary that $D_v P_M(p_0)$ exists. Then the function $x(t)$ has a finite right derivative at the point $t = 0$ and therefore the inverse function $t(x)$ has a finite or infinite right derivative at the point x_0. Let $u(x)$ be the common point of the rays $\{P(x) + n(x_0)\lambda, \lambda \geqq 0\}$, $\{p_0 + v\lambda, \lambda \geqq 0\}$ and let $v(x)$ be the common point of the rays $\{P(x) + n(x)\lambda, \lambda \geqq 0\}$, $\{p_0 + v\lambda, \lambda \geqq 0\}$. Thus the functions $u(x), v(x)$ are defined for $x \in \langle x_0, x_0 + \delta\rangle$, where $\delta > 0$ is a sufficiently small number. Put $a(x) = \varrho(p_0, u(x))$, $b(x) = \varrho(u(x), v(x))$, $z(x) = \varrho(P(x), u(x))$ and $\alpha(x) = \operatorname{arctg} h'(x)$ for $x \in \langle x_0, x_0 + \delta\rangle$. Elementary geometric observations give the following relations for $x \in (x_0, x_0 + \delta\rangle$:

$$a(x) = \cos\left(\pi/2 - \alpha(x_0) - \operatorname{arctg}\frac{x - x_0}{h(x) - h(x_0)}\right)\sqrt{[(x - x_0)^2 + (h(x) - h(x_0))^2]}\sin^{-1}\beta\,,$$

(1) $$z(x) = \varrho(p_0, P(x_0)) + \sin\left(\pi/2 - \alpha(x_0) - \operatorname{arctg}\frac{x - x_0}{h(x) - h(x_0)}\right)\cdot \sqrt{[(x - x_0)^2 + (h(x) - h(x_0))^2]} + \cos\left(\pi/2 - \alpha(x_0) - \operatorname{arctg}\frac{x - x_0}{h(x) - h(x_0)}\right)\cdot \sqrt{[(x - x_0)^2 + (h(x) - h(x_0))^2]}\ \operatorname{cotg}\beta\,,$$

(2) $$b(x) = z(x)\sin(\alpha(x) - \alpha(x_0))\ \sin^{-1}(\beta + \alpha(x_0) - \alpha(x))\,.$$

A simple calculation gives $a'_+(x_0) = \sqrt{[1 + (h'(x_0))^2]}\sin^{-1}\beta$. Since $b(x) = t(x) - a(x)$, we obtain that $b'_+(x_0)$ exists (finite or infinite). From (1) it follows that $z'_+(x_0) = \sqrt{[1 + (h'(x_0))^2]}\operatorname{cotg}\beta$. Put $\varphi(w) = \sin(w - \alpha(x_0))\sin^{-1}(\beta + \alpha(x_0) - w)$. From (2) it follows that $\varphi(\alpha(x))$ has a finite or infinite right derivative at x_0. Since $\varphi'(\alpha(x_0)) = \sin^{-1}\beta$ we easily infer that $\alpha'_+(x_0)$ exists (finite or infinite). Therefore $(h')'_+(x_0) = g'_+(x_0)$ exists, a contradiction.

Denote by Q the set of all points $q \notin M$ for which $P_M(q) \in \operatorname{Graph} h$. For $q \in Q$ denote by $x(q)$ the point from $\langle 0, 1\rangle$ such that $P_M(q) = (x(q), h(x(q)))$. We have shown that if $q \in Q^0$ and $x(q) \notin C$, then the directional derivative $D_v P_M(q)$ exists only for v the form $v = \lambda(P_M(q) - q)$. It is an obvious fact that $x(q): Q \to \langle 0, 1\rangle$ is a continuous open surjective mapping. Therefore $x^{-1}(C \cup \{0\} \cup \{1\})$ is a set of the first category in Q (and consequently in E_2). Let A be the smallest set which contains the set $x^{-1}(C \cup \{0\} \cup \{1\})$ and is symmetric with respect to the lines $x = 0$,

$y = 0$, $y = -x$, $y = x$. The symmetry of M implies that A has all the properties from the statement of the theorem.

Corollary. *There exists a compact convex set $M \subset E_2$ such that the set of all pairs (x, v) for which $D_v P_M(x)$ exists is a set of the first category in the space $(E_2 - M) \times \{v \in E_2;\ \|v\| = 1\}$.*

Proof. The mapping

$$f\colon E_2 - M \mapsto \{v \in E_2;\ \|v\| = 1\}\,,\quad f(x) = \frac{P_M(x) - x}{\|P_M(x) - x\|}$$

is continuous and therefore Graph f is a set of the first category in $(E_2 - M) \times \{v \in E_2;\ \|v\| = 1\}$. The set $A \times \{v \in E_2;\ \|v\| = 1\}$ is also a set of the first category in that space. Thus Corollary immediately follows from Theorem 1.

3. THE FRECHET DIFFERENTIATION OF α-MONOTONE OPERATORS

Asplund [1] showed that for an arbitrary closed set $M \subset E_n$ there exists a convex function g defined on E_n such that for any point $x \in E_n$ the subdifferential $\partial g(x)$ includes the set $P_M(x)$. Therefore the Buseman-Feller-Alexandrov theorem on the second differentiation of convex functions implies that P_M is almost everywhere Frechet differentiable. In Mignot's paper [10] it is proved that any (maximal) monotone operator in E_n is almost everywhere Frechet differentiable. Since a subdifferential of a convex function is a monotone operator, Mignot's theorem yields a new simple proof of the Buseman-Feller-Alexandrov theorem. Kenderov [6] showed that there is an easy direct proof of the fact that any metric projection in E_n is (cyclically) monotone. This observation together with Mignot's theorem give an alternative proof of Asplund's theorem mentioned above. The main idea of the present article is the observation that this alternative proof works also in some non-Euclidean spaces. Instead of monotone operators we use α-monotone operators defined below.

Notation. Let H be a ıeal Hilbert space. The for $o \neq u \in H$ and $o \neq v \in H$ we denote by $A(u, v)$ the angle between u and v. We put $A(u, v) = 0$ if $u = 0$ or $v = 0$.

Definition 1. Let H be a Hilbert space and $T\colon H \mapsto H$ a (possibly) multivalued operator defined on a set $D(T)$. Let $0 < \alpha \leqq \pi/2$. We say that T is an *α-monotone operator* if for any $x_1 \in D(T)$, $x_2 \in D(T)$, $y_1 \in T(x_1)$, $y_2 \in T(x_2)$ we have $A(x_2 - x_1, y_2 - y_1) \leqq$ if for any $x_1 \in D(T)$, $x_2 \in D(T)$, $y_1 \in T(x_1)$, $y_2 \in T(x_2)$ we have $A(x_2 - x_1, y_2 - y_1) \leqq \pi - \alpha$.

Notes. (i) T is $\pi/2$-monotone iff it is monotone.
(ii) It is doubtful whether α-monotonicity is an important notion. However, it is a very useful auxiliary notion for our forthcoming investigations.

Lemma 1. *Let H be a real Hilbert space. Let T: $H \mapsto H$ be an α-monotone operator. Then $S = (I + T)^{-1}$ is a Lipschitz (and therefore univalent) oprator.*

Proof. Let $x_1, x_2 \in D(S)$ and $y_1 \in S(x_1)$, $y_2 \in S(x_2)$. Then $x_1 \in (I + T)(y_1)$, $x_2 \in (I + T)(y_2)$. Thus there exist $z_1 \in T(y_1)$ and $z_2 \in T(y_2)$ such that $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$. The α-monotonicity of T implies $A(z_2 - z_1, y_2 - y_1) \leqq \pi - \alpha$. Therefore from the equation $x_2 - x_1 = (y_2 - y_1) + (z_2 - z_1)$ we easily deduce that $\|x_2 - x_1\| \geqq \|y_2 - y_1\| \sin \alpha$. Thus S is a Lipschitz operator.

Definition 2. We say that a (possibly) multivalued operator T: $E_n \to E_n$ is *continuous at a point* $a \in E_n$ if it is singlevalued at a and for any $\varepsilon > 0$ we have $\|T(a) - y\| < \varepsilon$ whenever $y \in T(x)$ and $\|x - a\|$ is sufficiently small.

Definition 3. (see [10]). Let T: $E_n \to E_n$ be a (possibly) multivalued operator. We say that T is (*Frechet*) *differentiable at a point* $a \in E_n$ if it is singlevalued at a and there exists a linear mapping L: $E_n \to E_n$ such that for any $\varepsilon > 0$ we have $\|y - T(a) - L(x - a)\| \leqq \varepsilon\|x - a\|$ whenever $y \in T(x)$ and $\|x - a\|$ is sufficiently small. The mapping L is called the *differential* (*or derivative*) *of f at a* and is denoted by $T'(a)$.

Notes. (i) If T is defined and singlevalued on a neighbourhood of a, then Definition 3 coincides with the usual definition of the Frechet differentiability.

(ii) If T is defined on no neighbourhood of a point a, then $L = T'(a)$ is not necessarily unambiguously determined.

(iii) If T is differentiable at a point a, then T is continuous at a.

(iv) If T and S are differentiable at a point a, then the operator $T + S$ (defined on $D(T) \cap D(S)$) is also differentiable at a.

Lemma 2. *Let T: $E_n \to E_n$ be a multivalued operator which is continuous at a point $a \in E_n$. Let the inverse operator $S := T^{-1}$ be differentiable at $b := T(a)$ with the differential $M = S'(b)$ which is an isomorphism of E_n onto E_n. Then T is differentiable at a with the differential $L := M^{-1}$.*

Proof. Let $x \in E_n$ and $y \in T(x)$. Put $x - a = \Delta x$ and $y - b = \Delta y$. If we define R by the equation

$$\Delta x = M(\Delta y) + R\,, \tag{3}$$

then we deduce from the differentiability of S that for any $\eta > 0$ we have

$$\|R\| < \eta\|\Delta y\| \quad \text{whenever} \quad \|\Delta y\| \quad \text{si sufficiently small}\,. \tag{4}$$

From (4) and from the continuity of T at a we obtain that $\|R\| < \|\Delta y\|/2\|L\|$ whenever $\|\Delta x\|$ is sufficiently small. Consequently, we have by (3)

$$\|\Delta x\| \geqq \|M(\Delta y)\| - \|R\| > \|\Delta y\|/\|L\| - \|\Delta y\|/2\|L\| = \|\Delta y\|/2\|L\|\,. \tag{5}$$

By (3) we obtain that

$$\Delta y = L(\Delta x) - L(R)\,. \tag{6}$$

From (4) and (5) it easily follows that for any $\varepsilon > 0$ we have $\|L(R)\| < \varepsilon\|\Delta x\|$ whenever $\|\Delta x\|$ is sufficiently small. Therefore by (6) T is differentiable at a.

Proposition 1. *Let T: $E_n \to E_n$ be an α-monotone operator. Then T is differentiable at almost all points at which T is continuous.*

Proof. By Lemma 1, $S := (I + T)^{-1}$ is a Lipschitz operator. Therefore there exists [9] a Lipschitz extension $\tilde{S}$ of S defined on the whole E_n. Let N_1 be the set of all points at which $\tilde{S}$ is not differentiable and N_2 the set of all points x at which $\tilde{S}$ is differentiable but the differential $\tilde{S}'(x)$ is not an isomorphism of E_n onto E_n. It is well known (see Theorem 3.2.3 from [4] or Proposition 1.2 from [10]) that $\tilde{S}(N_1 \cup \cup N_2)$ is a set of Lebesgue measure zero. Let $x \notin \tilde{S}(N_1 \cup N_2)$ be a point of continuity of T. Then $I + T$ is continuous at x and S has at the point: $(I + T)(x) \notin N_1 \cup N_2$ a differential which is an isomorphism of E_n onto E_n. Therefore by Lemma 2, $I + T$ is differentiable at x and consequently T is differentiable at x as well. The proof is complete.

4. THE FRECHET DIFFERENTIATION OF METRIC PROJECTIONS

In the following we will investigate a finite dimensional real Banach space X which we will consider as E_n with a nonEuclidean norm $q = \|\ldots\|_q$. The Euclidean norm is denoted by e or $\|\ldots\|_e$. The metric induced by q or by e is denoted by ϱ_q or by ϱ_e, respectively. The angles and the α-monotonicity are taken with respect to the Euclidean scalar product.

It is a simple fact [14] that if X is not strictly convex, then there exists a hyperplane $M \subset X$ such that P_M is multivalued at any point $x \in X - M$ and therefore P_M is not differentiable almost everywhere.

If X is strictly convex, then for any closed $M \subset X$ (see [7] or [16]), P_M is single-valued at all points except a set A_M of σ-finite $(n - 1)$-dimensional Hausdorff measure (and consequently of Lebesgue measure zero) which is also of the first category. It is easy to see that at any point $x \notin A_M$ the operator P_M is continuous. In fact, if $x_n \to x$ and $y_n \in P_M(x_n)$, $y_n \nrightarrow P_M(x)$, then there exists a subsequence of k_n which converges to a point $y \in M$, $y \neq P_M(x)$. Now the continuity of ϱ_q implies $\varrho_q(x, y) = \varrho_q(x, P_M(x))$, which is a contradiction.

The problem arises whether a metric projection on a closed subset of a strictly convex space X is always almost everywhere Frechet differentiable. The differentiability of P_M at almost all points of M (in an arbitrary X) is an easy consequence of the well known Lebesgue density theorem [13]. In fact, almost all points $x \in M$ are points of (ordinary) density of M and at each such a point x the metric projection P_M has the differential $P'_M(x) = I$ (cf. the proof of Theorem 2 from [3]). For points $x \notin M$ the problem seems to be very difficult and we give a partial solution only. We start our investigation with the following simple theorem.

Theorem 2. *Let $M \subset X$ be a closed set and let P_M be Frechet differentiable at a point $x \notin M$. Then $P'_M(x)$ is not an isomorphism of X onto X.*

This theorem is an easy consequence of the following simple lemma. Note that Lemma 3 is a consequence of the more complicated Lemma 4 but we will also give a short proof of Lemma 3.

Lemma 3. *Let $M \subset X$ be a closed set and let P_M be singlevalued at a point $x \notin M$. Let $x' \neq x$ and $P' \in P_M(x')$, $P' \neq P_M(x)$. Put $\Delta x = x' - x$, $P = P_M(x)$, $\Delta P = P' - P$ and $z = P - x$. Then there exists $\gamma > 0$ such that $A(\Delta p, z) < \pi - \gamma$ whenever Δx is sufficiently small.*

Proof. The set $C := \{y; \varrho_q(x, y) < \varrho_q(x, P)\}$ is an open convex set. Therefore there exists $\gamma > 0$ such that $y \in C$ whenever $A(y - P, z) \geqq \pi - \gamma$ and $\|y - p\|_e$ is sufficiently small. Since $P' \notin C$ and P_M is continuous in x we obtain that $A(\Delta P, z) < \pi - \gamma$ whenever Δx is sufficiently small.

Lemma 4. *Let $M \subset X$ be a closed set and let P_M be singlevalued at a point $x_0 \notin M$. Then there exists $\beta > 0$ and a neighbourhood U of x_0 such that whenever $x, x' \in U$, $P \in P_M(x)$, $P' \in P_M(x')$ and $P \neq P'$, then*

$$\pi - \beta \geqq A(\Delta P, z) \geqq \beta\,,$$

where we put $\Delta P = P' - P$ and $z = P - x$.

Proof. Put $z_0 = P_M(x_0) - x_0$. Since (see 24.5.1 of [12]) $D_v\, q(t)$ is an upper semicontinuous function on $E_n \times E_n$ and $D_{-z_0}\, q(z_0) < 0$, there exists $0 < \delta < \|z_0\|_e$ such that

$$(7)\qquad D_v\, q(t) < 0 \quad \text{whenever} \quad \varrho_e(z_0, t) < \delta \quad \text{and} \quad \varrho_e(-z_0, v) < \delta\,.$$

Put $\sigma = \arcsin\,(\delta/\|z_0\|_e)$ and $\beta = \sigma/2$. Choose a neighbourhood U of x_0 and an open ball B with the center in $P_M(x_0)$ such that for any $x \in U$, $P \in P_M(x)$ and $y \in B$ we have $P \in B$, $\varrho_e(y - x, z_0) < \delta$ and $A(y - x, z_0) < \beta$. Let now $x, x' \in U$, $P \in P_M(x)$, $P' \in P_M(x')$ and put $\Delta P = P' - P$, $\Delta x = x' - x$, $z = P - x$. We will prove that

(i) $A(\Delta P, z) \geqq \beta$ and

(ii) $A(\Delta P, z) \leqq \pi - \beta$.

(i) Suppose on the contrary that $A(\Delta P, z) < \beta$. Since $A(z, z_0) < \beta$ we have $A(\Delta P, z_0) < \sigma$ and therefore by the definition of σ there exists $\lambda > 0$ such that $\varrho_e(-\lambda\, \Delta P, -z_0) < \delta$. Since $\varrho_e(z_0, z - \Delta x) = \varrho_e(z_0, P - x') < \delta$, we have by (7) $D_{-\lambda\Delta P} q(z - \Delta x) < 0$ and consequently $D_{\Delta P} q(z - \Delta x) > 0$. The convexity of q further yields

$$q(z - \Delta x + \Delta P) > q(z - \Delta x)\,.$$

and consequently

$$\varrho_q(P + \Delta P, x + \Delta x) > \varrho_q(P, x + \Delta x),$$

which is a contradiction since $P + \Delta P \in P_M(x + \Delta x)$ and $P \in M$.

(ii) Suppose on the contrary that $A(\Delta P, z) > \pi - \beta$. Then $A(\Delta P, -z) < \beta$ and since $A(z, z_0) < \beta$, we have $A(\Delta P, -z_0) < \sigma$. Consequently, there exists $\lambda > 0$ such that $\varrho_e(\lambda \, \Delta P, -z_0) < \delta$. Let now y be a point from the segment joining P and P'. Then $y \in B$ and therefore $\varrho_e(y - x, z_0) < \delta$. Using (7) we obtain $D_{\lambda \Delta P} q(y - x) < 0$ and consequently $q(P - x) > q(P - x + \Delta P)$. In other words, $\varrho_q(P, x) > \varrho_q(P', x)$ and this is a contradiction. The proof of the lemma is complete.

Proposition 2. *Let X be a two-dimensional strictly convex Banach space and let $M \neq \emptyset$ be a closed subset of X. Let P_M be singlevalued at a point $x_0 \notin M$. Then there exists an open neighbourhood V of x_0 and $\alpha > 0$ such that P_M is α-monotone on V.*

Proof. Choose $\beta > 0$ and a neighbourhood U of x_0 according to Lemma 4. Put $\alpha = \beta/2$ and choose an open neighbourhood V of x_0 such that $V \subset U$ and

$$\|\Delta P\|_e < \sin \alpha \|z\|_e \quad \text{whenever} \quad x, x' \in V \tag{8}$$

and ΔP, z are defined as in Lemma 4. Let now $x, x' \in V$, $P \in P_M(x)$, $P' \in P_M(x')$, $\Delta P = P' - P$, $z = P - x$. It is sufficient to prove that $A(\Delta x, \Delta P) \leqq \pi - \alpha$. Suppose on the contrary that this inequality does not hold. Then $A(-\Delta x, \Delta P) < \alpha$ and $\Delta x \neq 0$, $\Delta P \neq 0$. By Lemma 4 we have

$$\beta \leqq A(\Delta P, z) \leqq \pi - \beta. \tag{9}$$

Denote by s the line joining the points x, P. Since $A(-\Delta x, \Delta P) < \alpha$ we obtain by (9) that the points $P + \Delta P$, $P - \Delta x$, $P + \Delta P - \Delta x$ lie in the same halfplane determined by the line s. Denote by p the line joining the points P, $P + \Delta P$. Let A_1 and A_2 be the common points of the line p and the line joining the point x with the point $P - \Delta x$ and $P + \Delta P - \Delta x$, respectively. Obviously

$$\text{the point } A_1 \text{ lies between } A_2 \text{ and } P. \tag{10}$$

By (8), $A(z + \Delta P, z) < \alpha$ and therefore (9) implies that

$$\text{the point } P + \Delta P \text{ lies between } A_2 \text{ and } P. \tag{11}$$

Since the line going through the points $P - \Delta x$, $P - \Delta x + \Delta P$ is parallel to the line p, there exists $\omega > 0$ such that

$$q(P - \Delta x - x) = \omega \, q(A_1 - x) \quad \text{and} \tag{12}$$

$$q(P + \Delta P - \Delta x - x) = \omega \, q(A_2 - x).$$

Since $P \in P_M(x)$ and $P + \Delta P \in M$, we obtain that $q(P + \Delta P - x) \geqq q(P - x)$. Therefore, using (10), (11) and the strict convexity of q, we obtain that

$$q(A_2 - x) > q(A_1 - x)$$

and therefore by (12)

$$\varrho_q(P', x') = q(P + \Delta P - \Delta x - x) > q(P - \Delta x - x) = \varrho_q(P, x')),$$

a contradiction.

Theorem 3. *Let X be a two-dimensional strictly convex Banach space and let $\emptyset \neq M \subset X$ be a closed set. Then P_M is Frechet differentiable at almost all points $x \in X$.*

Proof. Since X is strictly convex we obtain by [7] or [16] that P_M is singlevalued at any point $x \in X - S$, where S is a set of Lebesgue measure zero. At the beginning of the present section we have observed that P_M is Frechet differentiable at almost all points $x \in M$. By Proposition 2, for any point $x \in X - (M \cup S)$ there exists a neighbourhood U_x such that P_M is α-monotone on U_x. Therefore from Proposition 1 it easily follows that P_M is Frechet differentiable at almost all points of $\bigcup\{U_x;\ x \in X - - (M \cup S)\}$. Thus P_M is Frechet differentiable at almost all points $x \in X$.

Notation. *We use the symbols* $D^2 f(a)\,(u, v)$ and $\Delta^2 f(a; h_1, h_2)$ for the second derivative of a function f on E_n at a point $a \in E_n$ and the second difference of f at a, respectively, in the sense of [2].

We will need the following simple lemma. It is a special case of Theorem 204 from [5].

Lemma 5. *Let X be a finite dimensional Banach space and let $G \subset X$ be an open convex set. Let f be a real function on G which has at any point $x \in G$ the second derivative $D^2 f(x)$. Let $a \in G$, $h_1, h_2 \in X$ and $a + h_1$, $a + h_2$, $a + h_1 + h_2 \in G$. Then there exist* $0 < u < 1$, $0 < v < 1$ *such that*

$$\Delta^2 f(a; h_1, h_2) = D^2 f(a + uh_1 + vh_2)(h_1, h_2).$$

Definition 4. Let X be a finite dimensional Banach space with a norm q which belongs to the class $C^2(X - \{o\})$. We say that q is a *Euclidean-like norm* if $D^2 q(x) \,.\, (h, h) > 0$ for any linearly independent $x \neq o$, $h \neq o$.

Notes. (i) Any Euclidean-like norm is obviously strictly convex.

(ii) The Euclidean norm is Euclidean-like.

Proposition 3. *Let X be a finite dimensional Banach space with a Euclidean-like norm q. Let $0 \neq M \subset X$ be a closed set and let P_M be singlevalued at a point $x_0 \notin M$. Then there exist $\alpha > 0$ and a neighbourhood U of x_0 such that P_M is α-monotone on U.*

Proof. Recall that we consider X as E_n with a norm $q = \|\ldots\|_q$ while $e = \|\ldots\|_e$ is the Euclidean norm on X. Put $z_0 = P_M(x_0) - x_0$ and choose, according to Lemma 4, the corresponding $\beta > 0$.

Choose a compact convex set Y such that $z_0 \in Y^0$, $o \notin Y$ and

$$(13) \qquad A(y, z_0) < \beta/4 \quad \text{for any} \quad y \in Y.$$

Using Lemma 4 and the continuity of P_M at x_0 we can choose a neighbourhood U of x_0 such that for any $x, x' \in U$, $P \in P_M(x)$, $P' \in P_M(x')$ we have (we put $\Delta P = P' - P$, $z = P - x$)

$$(14) \qquad \pi - \beta \geqq A(\Delta P, z) \geqq \beta \quad \text{if} \quad \Delta P \neq 0$$

and

$$(15) \qquad P' - X \in Y^0 .$$

Put $M = Y \times \{\xi \in X;\ \|\xi\|_e = 1,\ \pi - \frac{3}{4}\beta \geqq A(z_0, \xi) \geqq \frac{3}{4}\beta\}$. If $(y, \xi) \in M$ then we have $y \neq o$, $\xi \neq o$ and by (13), $A(y, z_0) < \beta/4$. Therefore $\pi - \beta/4 > A(y, \xi) > \beta/4$ and thus y, ξ are linearly independent. Since q is Euclidean-like we obtain that the function $F(y, \xi) := D^2\, q(y)\,(\xi, \xi)$ is a positive continuous function on the compact set M and therefore there exists $m > 0$ such that

$$(16) \qquad D^2\, q(y)\,(\xi, \xi) > m \quad \text{for} \quad (y, \xi) \in M .$$

Similarly, there exists $K > 0$ such that

$$(17) \qquad |D^2\, q(y)\,(\xi, \eta)| < K \quad \text{whenever} \quad y \in Y \quad \text{and} \quad \|\xi\|_e = \|\eta\|_e = 1 .$$

Let $0 < \alpha < \operatorname{arctg}(m/K)$. We will show that P_M is α-monotone on U. Suppose on the contrary that there exist $x, x' \in U$ and $P \in P_M(x)$, $P' \in P_M(x')$ such that $A(\Delta x, \Delta P) > \pi - \alpha$, where $\Delta x = x' - x$, $\Delta P = P' - P$ and $z = P - x$. Consequently, we have

$$(18) \qquad A(-\Delta x, \Delta P) < \alpha , \quad \Delta x \neq 0 , \quad \Delta P \neq o .$$

Consider the second difference

$$\Delta^2 q(z; \Delta P, -\Delta x) = q(P - x + \Delta P - \Delta x) - q(P - x + \Delta P) -$$
$$- q(P - x - \Delta x) + q(P - x) .$$

We have

$$q(P - x + \Delta P - \Delta x) - q(P - x - \Delta x) = \varrho_q(P', x') - \varrho_q(P, x') \leqq 0 ,$$

since $P' \in P_M(x')$ and $P \in M$. Similarly,

$$q(P - x) - q(P - x + \Delta P) = \varrho_q(P, x) - \varrho_q(P', x) \leqq 0 .$$

Thus we obtain

$$(19) \qquad \Delta^2 q(z; \Delta P, -\Delta x) \leqq 0 .$$

On the other hand, by (15), all the points

$$P - x\,, \quad P - x + \Delta P = P' - x\,, \quad P - x - \Delta x = P - x'\,,$$
$$P - x + \Delta P - \Delta x = P' - x'$$

lie in Y^0. Since q is a Euclidean-like norm and $o \notin Y$ we deduce from Lemma 5 that there exist $0 < u < 1$, $0 < v < 1$ such that

$$\Delta^2 q(z; \Delta P, -\Delta x) = D^2 q(P - x + u\,\Delta P - v\,\Delta x)\,(\Delta P, -\Delta x)\,.$$

The convexity of Y implies that the point $y = P - x + u\,\Delta P - v\,\Delta x$ belongs to Y. By (13) and (15) we obtain $A(z, z_0) < \beta/4$ and by (14) and (18), $\pi - \beta \geqq A(\Delta P, z) \geqq \geqq \beta$. Therefore $\pi - \frac{3}{4}\beta \geqq A(\Delta P, z_0) \geqq \frac{3}{4}\beta$ and consequently, if we put $\xi = = \Delta P/\|\Delta P\|_e$, we have $(y, \xi) \in M$. Let s be the orthogonal projection of $-\Delta x$ on the one-dimensional subspace of X generated by ΔP and put $r = -\Delta x - s$. Then

(20) $$\|r\|_e/\|s\|_e < m/K\,.$$

Putting $\eta = r/\|r\|_e$ we obtain

$$\begin{aligned} D^2\,q(y)\,(\Delta P, -\Delta x) &= \|\Delta P\|_e\,D^2\,q(y)\,(\xi, r + s) = \\ &= \|\Delta P\|_e\,D^2\,q(y)\,(\xi, \|r\|_e\,\eta + \|s\|_e\,\xi) = \\ &= \|\Delta P\|_e\,(\|s\|_e\,D^2\,q(y)\,(\xi, \xi) + \|r\|_e\,D^2\,q(y)\,(\xi, \eta)) > \\ &> \|P\|_e\,(\|s\|_e\,m - \|r\|_e\,K)\,, \end{aligned}$$

where we have used (16) and (17). Using (20) we conclude that

$$\Delta^2 q(z; \Delta P, -\Delta x) = D^2\,q(y)\,(\Delta P, -\Delta x) > 0\,,$$

and this is a contradiction with (19).

Theorem 4. *Let X be a finite dimensional Banach space with a Euclidean-like norm and let $\emptyset \neq M \subset X$ be a closed set. Then P_M is Frechet differentiable at almost all points $x \in X$.*

Proof. If we write "Proposition 3" instead of "Proposition 2" in the proof of Theorem 3 we obtain the proof of Theorem 4.

Acknowledgments. I learned about the problem of the category of N_M which is solved in Section 2 from Prof. P. Kenderov.

I wish to thank Prof. J. Kolomý for bringing my attention to Mignot's paper [10].

References

[1] *E. Asplund:* Differentiability of the metric projection in finite-dimensional Euclidean space, Proc. Amer. Math. Soc. *38* (1973), 218–219.

[2] *J. Dieudonné:* Éléments d'analyse, Tome I: Fondements de l'analyse moderne, Paris 1972.

[3] *P. Erdös:* On the Hausdorff dimension of some sets in Euclidean space, Bull. Amer. Math. Soc. *52* (1946), 107—109.
[4] *H. Federer:* Geometric Measure Theory, Springer-Verlag, New York 1969.
[5] *V. Jarník:* Differential Calculus II, Prague 1956 (Czech).
[6] *P. S. Kenderov:* Points of single-valuedness of multivalued monotone mappings in finite dimensional spaces, Serdica *2* (1976), 160—164.
[7] *S. V. Konjagin:* Approximation properties of arbitrary sets in Banach spaces, Dokl. Akad. Nauk SSSR, *239* (1978), No. 2, 261—264 (Russian).
[8] *J. B. Kruskal:* Two convex counterexamples: A discontinuous envelope function and a non-differentiable nearest-point mapping, Proc. Amer. Math. Soc. *23* (1969), 697—703.
[9] *E. J. Mc Shane:* Extension of range of functions, Bull. Amer. Math. Soc. *40* (1934), 837—842.
[10] *F. Mignot:* Contrôle dans les inéquations variatonelles Elliptiques, J. Functional Analysis *22* (1976), 130—185.
[11] *C. J. Neugebauer:* A theorem on derivatives, Acta Sci. Math. (Szeged). *23* (1962), 79—81.
[12] *R. T. Rockafellar:* Convex Analysis, Princeton 1970.
[13] *S. Saks:* Theory of the Integral, New York 1937.
[14] *S. Stečkin:* Approximation properties of sets in normed linear spaces, Rev. Math. Pures Appl. *8* (1963), 5—18 (Russian).
[15] *Z. Zahorski:* Sur l'ensemble des points de nondérivabilité d'une fonction continue, Bull. Soc. Math. France, *74* (1946), 147—178.
[16] *L. Zajíček:* On the points of multivaluedness of metric projections in separable Banach spaces, Comment. Math. Univ. Carolinae *19* (1978), 513—523.

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).