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ON THE SEMIGROUP OF FULLY INDECOMPOSABLE RELATIONS

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The purpose of this note is to give a sufficient condition for the conjecture in [4] concerning the semigroup of fully indecomposable relations to hold.

A binary relation on a finite set $\Omega_n = \{a_1, a_2, \ldots, a_n\}$ of n elements, $n > 1$,is a subset of $\Omega_n \times \Omega_n = \{(a_i, a_j); a_i, a_j \in \Omega_n\}$. Let $B = B(\Omega_n)$ be the set of all (binary) relations on Ω_n. Then B is a semigroup with the multiplication defined as follows: for ϱ and τ in B, $(a_i, a_j) \in \varrho\tau$ if there is a $a_k \in \Omega_n$ such that $(a_i, a_k) \in \varrho$ and $(a_k, a_j) \in \tau$. Let ω be the universal relation on Ω_n, i.e., $\omega = \Omega_n \times \Omega_n$. Let M_n denote the set of all $n \times n$ matiices over the Boolean algebra of $\{0, 1\}$. Then M_n is a semigroup under the ordinary matrix multiplication, and the map

$$\varrho \to M(\varrho) = (M_{ij})$$

where

$$M_{i,j} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \varrho, \quad \text{and} \\ 0 & \text{otherwise}, \end{cases}$$

is an isomorphism of B onto M_n. Also, let X_n be the set of all directed graphs on n vertices with allowable loops and simple directed edges. Each matrix in M_n can be considered as the adjacency matrix of a directed graph Y in X_n, and it determines Y uniquely up to isomorphism. Also, each graph in X_n with labelled vertices determines a unique matrix in M_n as its adjacency matrix. Hence, there is a one-to-one correspondence among B, M_n and X_n:

$$\varrho \to M(\varrho) \to Y(\varrho).$$

Let $B_0 = B_0(\Omega_n)$ consist of all binary relations on Ω_n with $\mathrm{pr}_1(\varrho) = \mathrm{pr}_2(\varrho) = \Omega_n$ where

$$a_i\varrho = \{x \in \Omega_n; (a_i, x) \in \varrho\}, \quad \varrho a_i = \{y \in \Omega_n; (y, a_i) \in \varrho\},$$

$$\mathrm{pr}_1(\varrho) = \bigcup_{j=1}^{n} \varrho a_j \quad \text{and} \quad \mathrm{pr}_2(\varrho) = \bigcup_{j=1}^{n} a_j\varrho.$$

*) This work was done, while the author was a visiting scholar at the University of Pittsburgh.

Clearly, B_0 is a subsemigroup of B. This means that, if $\varrho \in B_0$, then none of the columns and none of the rows in $M(\varrho)$ consist of all zeros, and every vertex in the graph $Y(\varrho) \in X_n$ is incident with at least one incoming edge and at least one outgoing edge (a loop is considered both as an incoming edge and as an outgoing edge). A relation $\varrho \in B_0$ is said to be decomposable, if there is a π belonging to the group Π of all permutation relations on Ω_n such that $M(\pi \varrho \pi^{-1})$ is of the form

$$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix} \tag{1}$$

where B and D are square matrices of sizes $s \times s$ and $(n - s) \times (n - s)$ respectively, and $1 \leqq s \leqq n - 1$. Otherwise, it is called indecomposable. A relation $\varrho \in B_0$ is said to be partly decomposable, if there exist π_1 and π_2 in Π such that $M(\pi_1 \varrho \pi_2)$ is of the form (1). Otherwise, it is called fully indecomposable. A relation $\varrho \in B_0$ is said to be primitive, if there is a positive integer $k = k(\varrho)$ such that $\varrho^k = \omega$. If k is the least integer such that $\varrho^k = \omega$, then k is said to be the index of ϱ. Let $P = P(\Omega_n)$ and $F = F(\Omega_n)$ be, respectively, the set of all primitive relations in B_0 and the set of all fully indecomposable relations in B_0. Since a fully indecomposable relation is primitive, we have $F \subset P$. A graph Y in X_n is said to be strongly connected if, for any two vertices in Y, there is a directed path in Y from one vertex to the other. If ϱ is decomposable, then the corresponding graph $Y(\varrho)$ is not strongly connected. If ϱ is primitive, then the corresponding graph $Y(\varrho)$ is strongly connected. However, if the graph $Y(\varrho)$ is strongly connected, ϱ may not be primitive.

To any $\varrho \in P$, there is a least integer $l_2 = l_2(\varrho)$ such that $\varrho^{l_2} \in F$. The conjecture on pp. 162–163 in [4] states:

For any $\varrho \in P$, we have $l_2 = l_2(\varrho) \leqq n$ where n is the cardinality of Ω_n, i.e., $|\Omega_n| = n$. It was shown in [1] that the conjecture does not hold in general. To find a necessary and sufficient condition(s) for the conjecture to hold seems to be very difficult. Here we shall prove the following

Theorem. *Let $\varrho \in P = P(\Omega_n)$ with $(a_i, a_i) \in \varrho$ for at least one $a_i \in \Omega_n$. Then $\varrho^{l_2} \in F$ with $l_2 = l_2(\varrho) \leqq n$.*

We note that $(a_i, a_i) \in \varrho$ for at least one $a_i \in \Omega_n$ implies the corresponding graph $Y(\varrho)$ having at least one loop. Thus, for convenience, a relation ϱ is said to be a loop-relation if $(a_i, a_i) \in \varrho$ for at least one $a_i \in \Omega_n$. Consequently, the theorem above can be stated as: If ϱ is a primitive loop-relation, then the conjecture holds, i.e., $\varrho^{l_2} \in F$ with $l_2 = l_2(\varrho) \leqq n$.

In order to prove our theorem, we need the following lemmas:

Lemma 1. *Let $M = M(\varrho)$ be the adjacency matrix of the graph $Y = Y(\varrho)$ with n vertices. Then, in $M^r = (M^r_{i,j})$, $M^r_{g,h}$ is 1 (is 0) if and only if there is at least one directed path (no directed path) of length r in Y from the vertex g to the vertex h.*

Proof. It follows from the definition of adjacency matrix and the definition of matrix multiplication over the Boolean algebra of $\{0, 1\}$.

Lemma 2. *Let Y be a strongly connected graph with n vertices. Then for any two different vertices u and v in Y, there exists a directed path of length at most $n - 1$ in Y from u to v.*

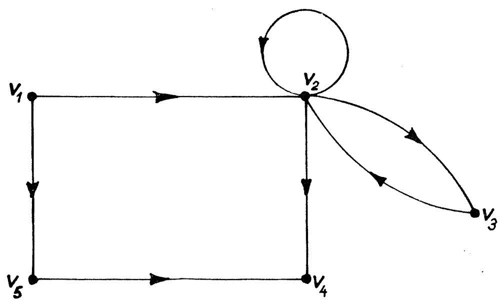
Proof. Since the graph Y is strongly connected, there exists a path from u to v, and the path goes through each of the vertices in Y at most once. Consequently, the path is of length at most $n - 1$.

The following corollary is well known. (For instance see [3] and [2]).

Corollary 2.1. *If ϱ is a primitive loop-relation, then the index of $\varrho \leqq 2n - 2$.*

Proof. Since ϱ is primitive, the corresponding graph $Y = Y(\varrho)$ is strongly connected. By using Lemma 2 and by using the loop there is a directed path of length $2(n - 1)$ in Y from any vertex in Y to all n vertices in Y, i.e., M^{2n-2} consists of all 1's where $M = M(\varrho)$ is the adjacency matrix of Y, and the index of $\varrho \leqq 2n - 2$ follows.

Let U be a subset of the vertex set $V(Z)$ of a graph Z in X_n. We define $N_t(U) = \{v \in V(Z)$; there exists a directed path of length t in Z from a vertex v_i in U to $v\}$, and $|N_t(U)|$ is the cardinality of $N_t(U)$. For example, let Z be the following graph



Then $N_1(\{v_1\}) = \{v_2, v_5\}$, $N_2(\{v_1\}) = \{v_2, v_3, v_4\}$ and $N_2(\{v_1, v_4, v_5\}) = \{v_2, v_3, v_4\}$. Also, $|N_1(\{v_1\})| = 2$ and $|N_2(\{v_1\})| = |N_2(\{v_1, v_4, v_5\})| = 3$. (Note that Z is not strongly connected.)

Lemma 3. *Let $\varrho \in P = P(\Omega_n)$, $Y = Y(\varrho)$ be the corresponding graph and $M = M(\varrho)$ be the adjacency matrix of Y. Then ϱ^r is partly decomposable if and only if there exists a set U_k of k vertices in Y, where $1 \leqq k \leqq n - 1$, such that $|N_r(U_k)| \leqq k$. In other words, ϱ^r is fully indecomposable if and only if for every set U_k of k different vertices in Y and for every $k = 1, 2, \ldots, n - 1$, $|N_r(U_k)| > k$.*

Proof. If ϱ^r is partly decomposable, then there exist π_1 and π_2 in Π such that $M(\pi_1 \varrho^r \pi_2)$ is of the form

(2) $$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

where B and D are square matrices of size $k \times k$ and $(n - k) \times (n - k)$ respectively.

Since B is a $k \times k$ matrix with $1 \leqq k \leqq n-1$, by Lemma 1, we have $|N_r(U_k)| \leqq k$ where U_k consists of the k vertices in Y.

If there exists a set U_k of k vertices in Y, where $1 \leqq k \leqq n-1$, such that $|N_r(U_k)| \leqq$ $\leqq k$, then there exist permutation matrices Q_1 and Q_2 such that $Q_1 M^r Q_2$ is of the form (2), i.e., ϱ^r is partly decomposable.

Let ϱ be a loop-relation in $P = P(\Omega_n)$ and $Y = Y(\varrho)$ be its corresponding graph in X_n with a loop at a fixed vertex w. Let $d_u = d(u, w)$ be the shortest length, of the directed path from the vertex u in Y to w. Let u_1 and u_2 be two different vertices in Y. We define $u_2 \leqq u_1$, if $d_{u_2} \leqq d_{u_1}$. (We note that since u_1 and u_2 are different vertices, $u_2 = u_1$ means $d_{u_2} = d_{u_1}$.)

Lemma 4. *Let ϱ be a loop-relation in $P = P(\Omega_n)$ and $Y = Y(\varrho)$ be its corresponding graph with a loop at a fixed vertex w. If $\{v_1, v_2, \dots, v_k\}$ is a set of k different vertices in Y where $1 \leqq k \leqq n-1$ such that $v_k \leqq v_{k-1} \leqq \dots \leqq v_1$, then $d_{v_i} \leqq n-i$ for $i = 1, 2, \dots, k$.*

Proof. By induction on k. For $k = 1$, by Lemma 2, $d_{v_1} \leqq n-1$. Assume that the lemma holds for any set of $k-1$ vertices in Y. Consider any set U_k of k different vertices in Y. We may assume $U_k = \{v_1, v_2, \dots, v_{k-1}, v_k\}$ with $v_k \leqq v_{k-1} \leqq v_{k-2} \leqq \dots$ $\dots \leqq v_1$. By our inductive hypothesis, $d_{v_i} \leqq n-i$ for $i = 1, 2, \dots, k-1$. There are two cases to be considered:

Case 1. If $d_{v_k} < d_{v_{k-1}}$, i.e., $v_k < v_{k-1} \leqq v_{k-2} \leqq \dots \leqq v_1$, then $d_{v_{k-1}} \leqq n -$ $-(k-1)$ implies $d_{v_k} \leqq n-k$.

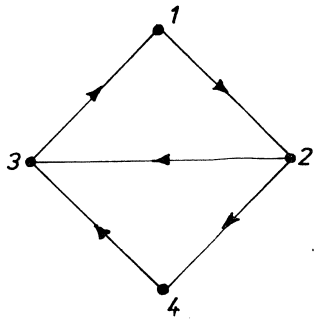
Case 2. If $d_{v_k} = d_{v_{k-1}}$, i.e., $v_k = v_{k-1} \leqq v_{k-2} \leqq \dots \leqq v_1$, then the path of shortest length from v_k to w does not pass through the vertex v_{k-1}, nor does it pass through any of the vertices $v_{k-2}, v_{k-3}, \dots, v_1$. Consequently, d_{v_k} is at most $n-1-(k-1) =$ $= n-k$, i.e., $d_{v_k} \leqq n-k$.

Now the proof of our theorem goes as follows: Since ϱ is a loop-relation in $P =$ $= P(\Omega_n)$, the corresponding strongly connected graph $Y(\varrho)$ in X_n has at least one loop, say, the loop is at the vertex w. Let $M = M(\varrho)$ be the adjacency matrix of Y.

Let $U_k = \{v_1, v_2, \dots, v_k\}$ be a set of any k different vertices in Y where $1 \leqq k \leqq$ $\leqq n-1$. We may assume that $v_k \leqq v_{k-1} \leqq v_{k-2} \leqq \dots \leqq v_1$. Then, by Lemma 4, $d_{v_i} \leqq n-i$ for $i = 1, 2, \dots, k$. Since Y is strongly connected, the directed paths of length i from w, $1 \leqq i \leqq n-1$, pass through at least $i+1$ vertices in Y. Say, these vertices are $w, w_1, w_2, \dots, w_i$ in Y. Again, since Y is strongly connected and since $v_k \leqq v_{k-1} \leqq v_{k-2} \leqq \dots \leqq v_1$ where $1 \leqq k \leqq n-1$, by using the loop at w, (if necessary, use the loop many times) there is at least one directed path of length n from v_i to w, at least one directed path of length n from v_i to $w_1, \dots,$ at least one directed path of length n from v_i to w_i. Hence $|N_n(\{v_i\})| \geqq i+1$ for $i = 1, 2, \dots, k$, i.e., for any $v_i \in U_k$, $|N_n(\{v_i\})| \geqq 2$. For any two different $v_{i_1}, v_{i_2} \in U_k$, we suppose $v_{i_2} \leqq v_{i_1}$, then $|N_n(\{v_{i_1}\})| \geqq 2$ and $|N_n(\{v_{i_2}\})| \geqq 3$. Since $|N_n(\{v_{i_1}, v_{i_2}\})| \geqq$ $\geqq \max\{|N_n(\{v_{i_1}\})|, |N_n(\{v_{i_2}\})|\}$, $|N_n(\{v_{i_1}, v_{i_2}\})| \geqq 3$. Similarly, for any t different

$v_{i_1}, v_{i_2}, \ldots, v_{i_t} \in U_k$ where $3 \leqq t \leqq k$, $|N_n(\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\})| \geqq t+1$. By Lemma 3, ϱ^n is fully indecomposable, i.e., $\varrho^n \in F$, and it follows that $\varrho^{l_2} \in F$ where $l_2 = l_2(\varrho) \leqq n$.

The following example shows that the loop relation in our Theorem is not a neces sary condition for the conjecture to hold: Let $\Omega_4 = \{1, 2, 3, 4\}$ and $\varrho = \{(1, 2), (2, 3), (2, 4), (3, 1), (4, 3)\}$. Then $Y = Y(\varrho)$ and $M = M(\varrho)$ are, respectively,

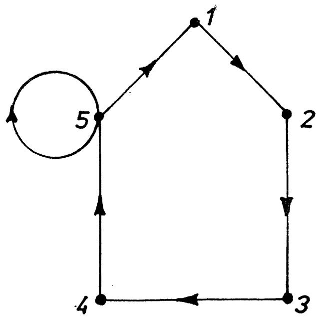


and $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$

Then

$$M^4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in F, \quad \text{i.e.,} \quad \varrho^4 \in F.$$

The following example demonstrates our theorem: Let $\Omega_5 = \{1, 2, 3, 4, 5\}$ and $\varrho = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (5, 5)\}$. Then $Y = Y(\varrho)$ and $M = M(\varrho)$ are, respectively,



and $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$

Then M^2, M^3, M^4 and M^5 are, respectively,

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We note that $M^2 \notin F$, because $|N_2(\{1\})| = 1$. $M^3 \notin F$, because $|N_3(\{2\})| = 1$. $M^4 \notin F$, because $|N_4(\{1, 2\})| = 2$. But $M^5 \in F$, i.e., $\varrho^5 \in F$.

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References

[1] *Chao, C. Y.:* On a conjecture of the semigroup of fully indecomposable relations, Czechoslovak Math. J., *27* (1977), 591—597.

[2] *Dulmage, A. L.* and *Mendelsohn, N. S.:* Gaps in the exponent set of primitive matrices, Illinois J. of Math. *8* (1964), 642—656.

[3] *Heap, B. R.* and *Lynn, M. S.:* The index of primitivity of a non-negative matrices, Numerische Math *6* (1964), 120—141.

[4] *Schwarz, Š.:* The semigroup of fully indecomposable relations and Hall relations, Czechoslovak Math. J., *23* (1973), 151—163.

[5] *Schwarz, Š.:* A new approach to some problems in the theory of nonnegative matrices, Czechoslovak Math. J., *16* (1966), 274—283.

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