Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 2, 309–313

Persistent URL: http://dml.cz/dmlcz/101879

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

ON k-DOMATIC NUMBERS OF GRAPHS

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(Received March 4, 1982)

In [1] M. Borowiecki and M. Kuzak have generalized the concept of a dominating set in a graph. Let G be an undirected graph without loops and multiple edges, let k be a positive integer. A k-dominating set in the graph G is a subset D of the vertex set $V(G)$ of G with the property that for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ such that $d(x, y) \leqq k$. (The symbol $d(x, y)$ denotes the distance of the vertices x, y in the graph G.) For $k = 1$ the k-dominating sets are dominating sets in the usual sense.

This leads to a generalization of the concept of the domatic number of a graph which was introduced by E. J. Cockayne and S. T. Hedetniemi in [2]. A k-domatic partition of G is a partition of $V(G)$, all of whose classes are k-dominating sets in G. The maximum number of classes of a k-domatic partition of G is called the k-domatic number of G and denoted by $d_k(G)$.

For $k = 1$ we have $d_k(G) = d(G)$, where $d(G)$ is the domatic number of G.

Proposition 1. *Let k, l be positive integers, $k < l$. Let G be an undirected graph. Then $d_k(G) \leqq d_l(G)$.*

Proof. From the definition of a k-dominating set it is clear that each k-dominating set in G is also l-dominating in G and hence each k-domatic partition of G is an l-domatic partition of G. This implies the assertion. $\square$

Proposition 2. *Let G be an undirected graph with n vertices, let $D(G)$ be its diameter. Then $d_k(G) = n$ for each $k \geqq D(G)$.*

Proof. Let $k \geqq D(G)$, let $x \in V(G)$. For each $y \in V(G)$ we have $d(x, y) \leqq D(G) \leqq k$, therefore $\{x\}$ is a k-dominating set in G. The partition of $V(G)$ into one-element sets is a k-domatic partition of G; it has n classes and no partition of $V(G)$ can have more than n classes. This implies $d_k(G) = n$. $\square$

Proposition 3. *Let G be an undirected graph, let G' be its spanning subgraph. Then $d_k(G) \geqq d_k(G')$.*

Proof. The assertion follows from the fact that $V(G') = V(G)$ and the distance of arbitrary two vertices in G' is greater than or equal to that in G. $\square$

Proposition 4. *Let G be an undirected graph, let k be a positive integer. Then $d_k(G)$ is equal to the minimum of k-domatic numbers of all connected components of G.*

The proof is left to the reader.

Theorem 1. *Let G be a connected undirected graph with n vertices, let k be a positive integer. Then*

$$d_k(G) \geqq \min(n, k + 1).$$

Proof. If $n \leqq k + 1$, then the diameter of G is at most k, therefore $d_k(G) = n$. Suppose that $n > k + 1$. Choose a spanning tree T of G. If the diameter of T is less than or equal to k, then so is the diameter of G and $d_k(G) = n$. If the diameter of T is greater than k, let c be a centre of T. Let P be a diametral path in T; the vertex c lies on P. Let P_1, P_2 be two subpaths of P whose union is the whole P and which have exactly one vertex in common, namely c. If T has two centres, then we suppose (without loss of generality) that the centre different from c lies on P_1. Let B_1 be the subtree of T whose vertex set consists of all vertices x with the property that c does not lie between x and any vertex of P_1. We shall colour the vertices of T by the colours $0, 1, \ldots, k$ in the following way. The vertex c is coloured by 0. Any vertex of B_1 is coloured by the colour i such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv -d(c, x) \pmod{(k + 1)}$. Any vertex x of T not lying in B_1 is coloured by the colour i such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv d(c, x) \pmod{(k + 1)}$. In both these cases $d(c, x)$ denotes the distance of c and x in T. As the diameter of T is greater than k, the path P_1 has a length at least $]k/2[$ and contains the vertices of all the colours $[k/2] + 1, \ldots, k$; the path P_2 has a length at least $[k/2]$ and contains the vertices of all the colours $1, \ldots, [k/2]$. (Here and in the sequel for an arbitrary real number a the symbol $[a]$ denotes the greatest integer which is less than or equal to a and the symbol $]a[$ denotes the least integer which is greater than or equal to a.) Let D_i be the set of all vertices of T which are coloured by the colour i (for $i = 0, 1, \ldots, k$). Let i be an arbitrary one from the numbers $0, 1, \ldots, k$; we shall prove that D_i is a k-dominating set in T. Let $x \in V(T) - D_i$; then $x \in D_j$ for some j distinct from i. Suppose $i < j$. If x does not lie in B_1, then on the path connecting x with c there is a vertex y such that $d(c, y) = d(c, x) - j + i$; we have $y \in D_i$ and $d(x, y) = j - i \leqq k$. If x lies in B_1 and $d(c, x) \geqq k + 1$, then there exists a vertex y in B_1 such that $d(c, y) = d(c, x) - k - 1 - i + j$; we have $y \in D_i$ and $d(x, y) = k + 1 + i - j \leqq k$. If x lies in B_1 and $d(c, x) \leqq k$, then $d(c, x) = k + 1 - j$ and there exists a vertex y on P_2 such that $d(c, y) = i$; we have $y \in D_i$ and $d(x, y) = k + 1 - j + i \leqq k$. Now suppose $i > j$. If x lies in B_1, then on the path connecting x with c there is a vertex y such that $d(c, y) = d(c, x) - i + j$; we have $y \in D_i$ and $d(x, y) = i - j \leqq k$. If x does not lie in B_1 and $d(c, x) \geqq k + 1$, then on the path connecting x and c there exists a vertex y such that $d(c, y) = d(c, x) - k - 1 + i - j$; we have $y \in D_i$ and $d(x, y) = k + 1 - i + j \leqq k$. If x does not lie in B_1 and $d(c, x) \leqq k$, then $d(c, x) = j$

and on P_1 there exists a vertex y such that $d(c, y) = k + 1 - i$; then $y \in D_i$ and $d(x, y) = k + 1 - i + j \leqq k$. We have proved that D_i is a k-dominating set in T. As i was chosen arbitrarily, $\{D_0, D_1, \ldots, D_k\}$ is a k-domatic partition of T with $k + 1$ classes and $d_k(T) \geqq k + 1$. According to Proposition 3 we have $d_k(G) \geqq \geqq d_k(T) \geqq k + 1$. □

A graph consisting of one path will be called a snake.

Theorem 2. *Let G be a snake with n vertices, let k be a positive integer. Then*

$$d_k(G) = \min(n, k + 1).$$

Proof. According to Theorem 1 the k-domatic number of G is at least $\min(n, k + 1)$. If $n \leqq k + 1$, it evidently cannot be greater. Thus suppose that $n > k + 1$. Let u be a terminal vertex of G. There are exactly $k + 1$ vertices of G whose distances from u are at most k. If $\mathscr{P}$ is a partition of $V(G)$ into at least $k + 2$ classes, then at least one class of $\mathscr{P}$ contains none of these vertices. This class is not a k-dominating set in G, thus $\mathscr{P}$ is not a k-domatic partition of G. Hence $d_k(G) = = k + 1 = \min(n, k + 1)$. □

Theorem 3. *Let k, n be two positive integers, let $2 \leqq k < n$. Then for each integer m such that $k + 1 \leqq m \leqq n$ there exists a tree T_m with n vertices such that $d_k(T_m) = = m$.*

Proof. According to Theorem 2 a snake with n vertices may be taken as T_{k+1}. Now let $k + 2 \leqq m \leqq n$. Let $a = [n/m]$. Take a snake S with $a(k + 1)$ vertices. Let u be a terminal vertex of S. Let v be the vertex of S adjacent to u. To each vertex of S distinct from v whose distance from u is congruent with 1 modulo $k + 1$ (there are exactly $a - 1$ such vertices) we add $m - k - 1$ new vertices and join them with it by edges. To v we add $n - am + m - k - 1$ new vertices and join them with by edges. We obtain a tree T_m which has evidently n vertices. Now we colour the vertices of T_m by the colours $0, 1, \ldots, m - 1$. If x is a vertex of S, then we colour it by the colour i such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv d(u, x) \pmod{(k + 1)}$. If y is a vertex of S such that $y \neq v$ and $d(u, y) \equiv 1 \pmod{(k + 1)}$, then to y we have added $m - k - 1$ new vertices; we colour them by the colours $k + 1, \ldots, m - 1$. The vertices adjacent to v and not belonging to S will be coloured also by the colours $k + 1, \ldots, m - 1$; some of these colours may be repeated. (We have $n - am + + m - k - 1 \geqq m - k - 1$, because $a \leqq n/m$.) Let D_i be the set of all vertices of T_m coloured by the colour i (for $i = 0, 1, \ldots, m - 1$). We shall prove that each D_i is a k-dominating set in T_m. First suppose $i \leqq k$. Let $x \in V(T_m) - D_i$; then $x \in D_j$ for some $j \neq i$. If $j < i$, then x belongs to S. If $d(u, x) \leqq k$, then $d(u, x) = j$. There exists a vertex y of S such that $d(u, y) = i$; we have $y \in D_i$ and $d(x, y) = i - j \leqq k$. If $d(u, x) \geqq k + 1$, then there exists a vertex y of S such that $d(u, y) = d(u, x) - - k + i - j - 1$; we have $y \in D_i$ and $d(x, y) = k - i + j + 1 \leqq k$. If $i < j \leqq k$, then x belongs to S. There exists a vertex y of S such that $d(u, y) = d(u, x) + i - j$;

we have $y \in D_i$ and $d(x, y) = j - i \leqq k$. If $j > k$, then x does not belong to S and is adjacent to a vertex $z \in D_1$. If $z = v$, then there exists a vertex y of S such that $d(u, y) = i$; we have $y \in D_i$ and $d(x, y) = i \leqq k$. If $z \neq v$, $i \neq 0$, $i \neq 1$, $i \neq 2$, then there exists a vertex y of S such that $d(u, y) = d(u, z) - k + i - 2$; we have $d(x, y) = k - i + 3 \leqq k$. If $i = 1$, then we have $z \in D_i$ and $d(x, z) = 1 \leqq k$. If $i = 0$ or $i = 2$, then the vertex y of S adjacent to z has the property that $y \in D_i$ and $d(x, y) = 2 \leqq k$.

Now suppose $i > k$. Let again $x \in V(T_m) - D_i$; then $x \in D_j$ for some $j \neq i$. If $j \leqq k$, then there exists a vertex z of S such that $d(u, z) = d(u, x) - j + 1$; we have $z \in D_1$ and $d(x, z) = j - 1$. There exists a vertex $y \in D_i$ adjacent to z and $d(x, y) = j \leqq k$. If $j > k$, then x is adjacent to a vertex $z \in D_1$ and there exists another vertex y adjacent to z such that $y \in D_i$, while $d(x, y) = 2 \leqq k$.

Thus we have proved that each D_i is a k-dominating set in T_m and $\{D_0, D_1, \ldots, D_{m-1}\}$ is a k-domatic partition of T_m, which implies $d_k(T_m) \geqq m$. Now let w be the terminal vertex of S distinct from u. There are exactly m vertices (including w itself) whose distance from w in T_m is less than or equal to m. By the same consideration as in the proof of Theorem 2 we prove that $d_k(T_m)$ cannot be greater than m and thus $d_k(T_m) = m$. □

In Fig. 1 there is a tree T_m for $k = 4$, $m = 7$, $n = 23$.

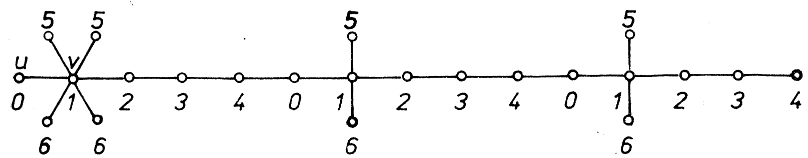

Fig. 1.

Theorem 4. *Let C_n be a circuit with n vertices, let k be a positive integer. Then*

$$d_k(C_n) = \left[\frac{n}{\left] \frac{n}{2k+1} \right[} \right].$$

Proof. If $n < 2k + 1$, then $d_k(C_n) = n$ according to Proposition 2 and

$$n = \left[\frac{n}{\left] \frac{n}{2k+1} \right[} \right].$$

If $n \geqq 2k + 1$, then to each vertex of C_n there exist exactly $2k + 1$ vertices (including this vertex itself) whose distances from this vertex are at most k. Therefore each k-dominating set in C_n has at least $]n/(2k + 1)[$ vertices and each domatic partition of C_n has at most

$$\left[\frac{n}{\left]\frac{n}{2k+1}\right[}\right]$$

classes.

Now denote

$$q = \left[\frac{n}{\left]\frac{n}{2k+1}\right[}\right], \quad r = (2k+1)\,q - n\,, \quad s = \left]r/q\right[\,.$$

The circuit C_n can be divided into q edge-disjoint paths such that $qs - r$ of them have the length $2k + 2 - s$ and the remaining $q + r - qs$ of them have the length $2k + 1 - s$. (The reader may verify that $qs - r < q$ and that the sum of the lengths of the described paths is equal to n.) Let P be the set of the described paths. We colour the vertices of C_n by the colours $0, 1, \ldots, 2k - s$ in the following way. The terminal vertices of the paths of P (each of them common for two of these paths) are coloured by 0. Now we choose a sense of running around C_n. If a path from P has the length $2k + 1 - s$ (or $2k + 2 - s$), we run along it in the chosen sense and colour its inner vertices consecutively by the colours $1, \ldots, 2k - s$ (or $0, 1, \ldots, 2k - s$, respectively). Let D_i be the set of vertices of C_n which are coloured by the colour i for $i = 0, 1, \ldots, 2k - s$. We see that for any fixed i the distance between two vertices of D_i is at most $2k + 2 - s$ for $s \geqq 1$ and at most $2k + 1 - s$ for $s = 0$; thus in both the cases at most $2k + 1$. This implies that any vertex not belonging to D_i has the distance at most k from some vertex of D_i. Hence D_i is a k-dominating set in C_n, $\{D_0, D_1, \ldots, D_{2k-s}\}$ is a k-domatic partition of C_n and $d_k(C_n) \geqq 2k - s + 1$. We shall compute $2k - s + 1$. We have

$$2k - s + 1 = 2k - \left]r/q\right[+ 1 = 2k - \left]((2k+1)\,q - n)/q\right[+ 1 =$$

$$= 2k - (2k+1) + [n/q] + 1 = [n/q] = \left[\frac{n}{\left]\frac{n}{2k+1}\right[}\right].$$

Therefore $d_k(C_n)$ is greater than or equal to this number; as the converse inequality was proved above, it is equal to it. □

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