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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS

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INTRODUCTION

We consider solutions $u(x, t)$ of second order parabolic equations

$$(1.1) \qquad -u_t + L_x u = -f(x, t)$$

in a cylindrical space time region $\Omega \times (0, \infty)$, where Ω is a bounded domain in E^n, with the operator L_x given in divergence form

$$(1.2) \qquad L_x u = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(a_{jk}(x, t) \frac{\partial u}{\partial x_k} \right);$$

the coefficients a_{jk} being bounded measurable functions satisfying the ellipticity condition

$$(1.3) \qquad \sum_{j,k=1}^{n} a_{jk}(x, t)\, \xi_j \xi_k \geqq \lambda |\xi|^2 \quad (\lambda > 0)$$

uniformly in $\Omega \times (0, \infty)$ and the symmetry condition $a_{jk} = a_{kj}$. We assume that $u(x, t)$ is given initially:

$$(1.4) \qquad u(x, 0) = \phi(x) \quad \text{on} \quad \Omega ;$$

and that on $\partial\Omega \times (0, \infty)$ u is to satisfy either a homogeneous Dirichlet, Robin or Neumann boundary condition, i.e. either

$$(1.5) \qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty),$$

or

$$(1.6) \qquad \frac{\partial u}{\partial v} + \beta u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty) \quad (\beta \geqq 0,\ \beta \not\equiv 0),$$

or

$$(1.7) \qquad \frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty),$$

respectively, where in the last two conditions $\partial u/\partial v$ denotes the outward conormal derivative on $\partial\Omega \times \{t\}$:

$$\frac{\partial u}{\partial \nu} = \sum_{j,k=1}^{n} a_{jk}(x, t) \frac{\partial u}{\partial x_k}(x, t)\, m_j(x) \quad (x \in \partial\Omega) \tag{1.8}$$

with $m_j(x)$ representing the jth component of the exterior unit normal to $\partial\Omega$. For brevity, in the sequel we shall refer to the problems of solving (1.1), (1.4) under the boundary conditions (1.5), (1.6) and (1.7) as problems I, II and III, respectively. The notion of solution is to be taken in the classical sense, and for the purposes of such an interpretation it will be convenient to regard the coefficients a_{jk} as well as the boundary, $\partial\Omega$, to be sufficiently smooth. These smoothness assumptions will not, however, play an essential role in the derivation of our results, and with the excpetion of the assumption that $\partial\Omega$ be smooth for problems II and III, they can be ultimately dispensed with.

We propose to analyze the asymptotic behavior of solutions to these problems assuming that the right side of (1.1), $f \to 0$ as $t \to \infty$ in an appropriate sense. Suppose, to begin with, that $f \to 0$ uniformly in Ω as $t \to \infty$, then it is well known that for problem I the solutions u will tend to zero uniformly in Ω as $t \to \infty$. The same result also holds for problem II if, for example, we assume β to be uniformly and positively bounded away from zero. There is a corresponding result for problem III in which, under the additional assumption that $\lim_{t\to\infty} \int_0^t \int_\Omega f(\xi, \tau)\, d\xi\, d\tau$ exists, we may conclude that the solutions $u(x, t)$ tend uniformly to suitable constants.

Our goal will be to obtain the same asymptotic results assuming, instead of $f \to 0$ uniformly in Ω, that f tends to zero in the $L_p(\Omega)$ sense:

$$\int_\Omega |f(x, t)|^p\, dx \to 0 \quad \text{as} \quad t \to \infty \tag{1.9}$$

for p sufficiently large in relation to n, the dimension of x space. Specifically, assuming that

$$p > n/2 \text{ when } n \geqq 2 \quad \text{and} \quad p \geqq 1 \text{ when } n = 1 \tag{1.10}$$

we will establish the following theorems.

Theorem 1.1. *Under the stated assumptions* (1.9), (1.10) *regarding f, the solutions $u(x, t)$ of problem* I *tend to zero uniformly in Ω as $t \to \infty$.*

Theorem 1.2. *Under exactly the same assumptions* (1.9), (1.10), *the solutions $u(x, t)$ of problem* II *also tend to zero uniformly in Ω as $t \to \infty$ provided that the coefficient function $\beta(x, t)$ in* (1.6), *which is assumed to be a non-negative bounded measurable function on $\partial\Omega$ for each $t > 0$, satisfies the following condition: For some positive ε, the measures of the sets*

$$E_\varepsilon(t) = [x \in \partial\Omega : \beta(x, t) \geqq \varepsilon] \tag{1.11}$$

should be uniformly bounded away from zero, i.e.

$$\text{(1.12)} \qquad \text{meas}\,[E_\varepsilon(t)] \geqq \delta > 0 \quad \textit{for all} \quad t > 0\,,$$

for a suitable positive δ.

Remarks. A sufficient condition guaranteeing that β satisfy (1.12) is that $\beta(x, t)$ be positively bounded away from zero uniformly in x and t:

$$\text{(1.13)} \qquad \beta(x, t) \geqq \varepsilon > 0 \quad \text{for} \quad (x, t) \in \partial\Omega \times (0, \infty)\,.$$

(This is the assumption that is usually made to assure that $u \to 0$ in problem II; see, for example, [3, p. 166].)

A less stringent condition comparable to (1.13) which still implies that β satisfies (1.12) is to assume that $\beta(x, t)$ is bounded from below by a non-negative function $\gamma(x)$:

$$\beta(x, t) \geqq \gamma(x) \geqq 0 \quad \text{for} \quad (x, t) \in \partial\Omega \times (0, \infty)\,,$$

whose integral over $\partial\Omega$ is positive:

$$\int_{\partial\Omega} \gamma(x)\, \mathrm{d}\sigma(x) > 0\,,$$

$\mathrm{d}\sigma$ denoting the element of area on $\partial\Omega$. Clearly this condition allows β to vanish at some points of $\partial\Omega$, although there must still be a fixed "substantial" subset of $\partial\Omega$ over which $\beta(x, t)$ will be positively bounded away from zero for $t > 0$. The reason for the formulation in terms of the condition (1.12) is to allow the "substantial" subset $E_\varepsilon(t)$ over which $\beta(x, t)$ is required to be positively bounded away from zero to vary with t.

Finally for problem III we have

Theorem 1.3. *Assume, as before, that f tends to zero in the L_p sense (1.9) with p satisfying the condition (1.10). Assume in addition that the limit*

$$\text{(1.14)} \qquad \lim_{t \to \infty} \int_0^t \int_\Omega f(x, \tau)\, \mathrm{d}x\, \mathrm{d}\tau = A$$

exists. Then the solutions $u(x, t)$ of problem III *tend to constants*:

$$u(x, t) \to \frac{1}{|\Omega|}\,[I + A] \quad as \quad t \to \infty$$

uniformly in Ω, where I *denotes the integral of the initial data and $|\Omega|$ the measure of Ω*:

$$I = \int_\Omega \phi(x)\, \mathrm{d}x\,, \quad |\Omega| = \text{meas}\,[\Omega]\,.$$

Theorems of this type are useful in studying the asymptotic behavior of solutions of semilinear parabolic systems, in particular, for systems of reaction diffusion equations. For such systems one can sometimes obtain the L_p asymptotic behavior relatively easily; one then invokes theorems of the above type to obtain the corresponding uniform asymptotic behavior. In fact Theorem 1.1 for the case of the heat equation, $-u_t + c\,\Delta u = -f$, was developed by the writer in [14] to study the asymptotic behavior of the reaction diffusion system

$$-u_t + a\,\Delta u = juv\,, \quad -v_t + b\,\Delta v = kuv \tag{1.15}$$

in precisely the manner indicated. Subsequently, Gardner in [7] used exactly the same argument to analyze a generalization of (1.15). The same technique has also been used by Webb [20] to obtain the asymptotic behavior of solutions to a functional equation which models epidemic phenomena. Recently, a further application of this method was given by Gardner [8] to analyze systems occurring in mathematical ecology.

The proofs of Theorems 1.1, 1.2 and 1.3 will be based on representation formulas which express the solutions u of problems I, II and III in terms of the initial data ϕ and right side f by means of integral operators whose kernels are the appropriate fundamental solutions F of the corresponding problem:

$$u(x, t) = \int_0^t \int_\Omega F(x, t; \xi, \tau) f(\xi, \tau)\, d\xi\, d\tau + \int_\Omega F(x, t; \xi, 0)\, \phi(\xi)\, d\xi\,.$$

These fundamental solutions are usually referred to, more specifically, as the Green's, Robin's and Neumann's functions, G, R and N, for problems I, II and III, respectively. Their definitions and basic properties will be described in Section 2 which is devoted to a discussion of all the needed preliminaries. Then in Section 3, using a technique due to Nash [17], we will derive estimates for the fundamental solutions G, R and N, from which, as a consequence of the representation formulas (1.16), Theorems 1.1, 1.2 and 1.3 will follow easily. The proofs of these theorems together with some further results of a related nature will then be presented in Section 4, the final section of the paper.

2. PRELIMINARIES

This section contains a discussion of various preliminaries which will be needed for our later developments. In the first part of this section we will list for ease of reference, the basic properties of the fundamental solutions for problems I, II and III that will be used in the sequel; while in the second part of this section we will present some inequalities which will play a crucial role in deriving the desired estimates for these fundamental solutions.

(a) Fundamental Solutions. We first turn to a description of the fundamental solutions G, R and N; and in doing so it will be convenient to refer to them generically by $F = F(x, t; \xi, \tau)$, whenever we wish to describe a property which is identical for all of them. To begin with they are defined as follows: Foı fixed $(\xi, \tau) \in \Omega \times (0, \infty)$, as functions of x and t they are to satisfy the equation

$$\left[-\frac{\partial}{\partial t} + L_x\right] F(x, t; \xi, \tau) = 0 \quad \text{for} \quad x \in \Omega\,, \quad t > \tau \tag{2.1}$$

together with the appropriate homogeneous boundary condition

$$G(x, t; \xi, \tau) = 0 \quad \text{fos} \quad x \in \partial\Omega\,, \quad t > \tau\,, \tag{2.2}$$

$$\left[\frac{\partial}{\partial v} + \beta(x, t)\right] R(x, t; \xi, \tau) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \quad t > \tau\,, \tag{2.3}$$

$$\frac{\partial}{\partial v}\,[N(x, t; \xi, \tau)] = 0 \quad \text{for} \quad x \in \partial\Omega\,, \quad t > \tau\,, \tag{2.4}$$

for problems I, II and III, respectively.

As $(t - \tau) \downarrow 0$, they are to behave like δ functions in the sense that for ϕ a uniformly continuous function in Ω

$$\lim_{(t-\tau)\downarrow 0} \int_\Omega F(x, t; \xi, \tau)\, \phi(\xi)\, \mathrm{d}\xi = \phi(x) \tag{2.5}$$

is to hold uniformly in any compact subset of Ω; while in addition, if $\phi \in C_0(\Omega)$ (the set of continuous functions with compact support contained in Ω) (2.5) should hold uniformly in Ω.

The equation and boundary conditions (2.1)–(2.4) are to be fulfilled in the classical Gnse. As far as ıegularity properties are concerned this will be taken to mean that se, R and N are to be twice continuously differentiable with respect to x and once continuously differentiable with respect to t, for $x \in \Omega$ and $t > \tau$; while as functions of x, for fixed t, we will demand that they be $C^1(\bar{\Omega})$ functions. To assure this will require that the coefficients a_{jk} and boundary $\partial\Omega$ be sufficiently smooth, and we will assume this to be the case; although it should be pointed out that with the exception of the smoothness assumption on $\partial\Omega$ in problems II and III, these kind of regularity assumptions will have no "quantitative" effect on the estimates for the fundamental solutions to be derived in the section that follows. For the construction of fundamental solutions with the stated properties as well as those to be mentioned further on see [2], [3], [11], [12], [13], [15] and [19].

In addition to the aforementioned properties of the fundamental solutions $F(x, t; \xi, \tau)$ we also need to take note of the properties which they possess as functions of ξ and τ for fixed $(x, t) \in \Omega \times (0, \infty)$. It turns out that with respect to ξ and τ they satisfy an adjoint equation together with adjoint boundary conditions. In view of the symmetry of the a_{jk}'s the adjoint equation here takes the form

$$\left[\frac{\partial}{\partial \tau} + L_\xi\right] F(x, t; \xi, \tau) = 0 \quad \text{for} \quad \xi \in \Omega\,, \quad \tau < t\,, \tag{2.6}$$

where by L_ξ we mean the operator

$$L_\xi v(\xi, \tau) = \sum_{j,k=1}^{n} \frac{\partial}{\partial \xi_j}\left(a_{jk}(\xi, \tau)\frac{\partial v}{\partial \xi_k}(\xi, \tau)\right);$$

while the adjoint boundary conditions are given by

$$G(x, t; \xi, \tau) = 0 \quad \text{for} \quad \xi \in \partial\Omega\,, \quad \tau < t\,, \tag{2.7}$$

$$\left[\frac{\partial}{\partial v} + \beta(\xi, \tau)\right] R(x, t; \xi, \tau) = 0 \quad \text{for} \quad \xi \in \partial\Omega\,, \quad \tau < t\,, \tag{2.8}$$

$$\frac{\partial}{\partial v}\left[N(x, t; \xi, \tau)\right] = 0 \quad \text{for} \quad \xi \in \partial\Omega\,, \quad \tau < t \tag{2.9}$$

for problems I, II and III, respectively, where $\partial/\partial v$ here denotes the same outward conormal derivative defined in (1.8), but this time taken with respect to ξ rather than x, i.e.

$$\frac{\partial v}{\partial v}(\xi, \tau) = \sum_{j,k=1}^{n} a_{jk}(\xi, \tau)\frac{\partial v}{\partial \xi_k}(\xi, \tau)\, m_j(\xi)\,.$$

Also, corresponding to the property (2.5), we have here the property, that for $\psi(x)$ uniformly continuous in Ω

$$\lim_{(t-\tau)\downarrow 0} \int_\Omega F(x, t; \xi, \tau)\, \psi(x)\, \mathrm{d}x = \psi(\xi) \tag{2.10}$$

holds uniformly on any compact subset of Ω; while if $\psi \in C_0(\Omega)$, (2.10) holds uniformly in Ω.

Again the equation and boundary conditions (2.6)–(2.9) are to be satisfied in the classical sense, so that as functions of ξ and τ for $\xi \in \Omega$ and $\tau < t$, the fundamental solutions F should be twice continuously differentiable with respect to ξ and once continuously differentiable with respect to τ; furthermore, as functions of ξ, for τ fixed, they should belong to $C^1(\bar{\Omega})$. To assure this will require no further smoothness assumptions on the coefficients a_{jk} and boundary $\partial\Omega$ than have already been made to guarantee the corresponding regularity conditions for the F's as functions of x and t.

The importance of these fundamental solutions for our purposes is that, as already noted in the introduction, they allow us to represent any sufficiently regular solution of problem I, II or III in the form

$$u(x, t) = \int_0^t \int_\Omega F(x, t; \xi, \tau) f(\xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau + \int_\Omega F(x, t; \xi, 0)\, \phi(\xi)\, \mathrm{d}\xi\,; \tag{2.11}$$

and it is these representation formulas applied to classical solutions u of problems I, II or III which will be our point of departure in proving the asymptotic results stated in the introduction.

Finally, we need to record one further property of the fundamental solutions F that will play an essential role in deriving the estimates for them; namely, the "semi-group" identity:

$$F(x, t; \xi, \tau) = \int_\Omega F(x, t; y, s)\, F(y, s; \xi, \tau)\, \mathrm{d}y \quad (\tau < s < t) \tag{2.12}$$

which follows from considerations based on the uniqueness of solutions to problems I, II and III.

(b) Inequalities. In deriving estimates for the fundamental solutions F we will avail ourselves of some inequalities for functions in the Sobolev spaces $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$, which we take to be the closures of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, respectively, under the norm $(\int_\Omega [|v|^2 + |Dv|^2]\, \mathrm{d}\xi)^{1/2}$, where $\mathrm{D}v$ denotes the gradient of v. In particular, we will be using the well-known Poincaré inequality

$$\int_\Omega |v|^2\, \mathrm{d}\xi \leqq c \int_\Omega |Dv|^2\, \mathrm{d}\xi \tag{2.13}$$

valid for arbitrary $v \in W_0^{1,2}(\Omega)$ with the constant c depending on Ω; it is also valid [16, Th. 3.6.5, p. 83] for all $v \in W^{1,2}(\Omega)$ which have mean value zero:

$$\int_\Omega v\, \mathrm{d}\xi = 0 \tag{2.14}$$

provided that $\partial\Omega$ is sufficiently smooth.

Next, for the estimation of the Robin function R under the assumption that β satisfies condition (1.12), we will need to apply the following variant of Poincaré's inequality.

Proposition 2.1. *Let $S(\varepsilon, \delta)$, $(\varepsilon > 0,\ \delta > 0)$, denote the set of non-negative $L_\infty(\partial\Omega)$ functions $\beta(\xi)$ with the property that*

$$\operatorname{meas}[\xi : \xi \in \partial\Omega,\ \beta(\xi) \geqq \varepsilon] \geqq \delta\,. \tag{2.15}$$

Then for any given $\mu > 0$, there exists a positive constant $a = a(\varepsilon, \delta, \mu, \Omega)$ so that

$$a \int_\Omega v^2(\xi)\, \mathrm{d}\xi \leqq \mu \int_\Omega |Dv(\xi)|^2\, \mathrm{d}\xi + \int_{\partial\Omega} \beta(\xi)\, v^2(\xi)\, \mathrm{d}\sigma(\xi) \tag{2.16}$$

holds for all $v \in W^{1,2}(\Omega)$ and all $\beta \in S(\varepsilon, \delta)$, provided, again, that $\partial\Omega$ is sufficiently smooth.

Remark. The expression $v(\xi)$ entering into the boundary integral on the right of (2.16) is to be understood as the "trace" of the function $v \in W^{1,2}(\Omega)$ on the boundary $\partial\Omega$ (cf. [16], pp. 72–78); as such it is a welldefined $L_2(\partial\Omega)$ function which coincides with the restriction of v to $\partial\Omega$ when v is a $C^1(\bar{\Omega})$ function. Moreover the mapping from a function in $W^{1,2}(\Omega)$ into its trace in $L_2(\partial\Omega)$ is continuous in the sense that if $v_n \to v$ weakly in $W^{1,2}(\Omega)$, the traces of $v_n \to$ to the trace of v in $L_2(\partial\Omega)$.

Proof of Proposition 2.1. It will be sufficient to show that there exists a positive constant $a = a(\varepsilon, \delta, \mu, \Omega)$ so that

$$(2.17) \qquad a \int_\Omega v^2(\xi)\, d\xi \leqq \mu \int_\Omega |D\, v(\xi)|^2\, d\xi + \varepsilon \int_{\partial\Omega} \chi(\xi)\, v^2(\xi)\, d\sigma(\xi)$$

holds for $v \in W^{1,2}(\Omega)$ and χ the characteristic function of any set whose measure $\geqq \delta$; (2.16) will then follow from (2.17) by taking χ to be the characteristic function of the set $[\xi : \xi \in \partial\Omega,\ \beta(\xi) \geqq \varepsilon]$ for any $\beta \in S(\varepsilon, \delta)$.

To establish (2.17), suppose, on the contrary, that it did not hold. Then there would exist a sequence of functions $\{v_j(\xi)\}$ in $W^{1,2}(\Omega)$ together with a sequence of characteristic functions $\{\chi_j(\xi)\}$ defined on $\partial\Omega$ so that

$$\int_\Omega v_j^2(\xi)\, d\xi = 1\,, \quad \int_{\partial\Omega} \chi_j(\xi)\, d\sigma(\xi) \geqq \delta$$

and

$$(2.18) \qquad \mu \int_\Omega |D\, v_j(\xi)|^2\, d\xi + \varepsilon \int_{\partial\Omega} \chi_j(\xi)\, v_j^2(\xi)\, d\sigma(\xi) \leqq j^{-1}$$

for $j = 1, 2, \ldots$. Now as this implies that the Sobolev norms of the v_j's are uniformly bounded:

$$(2.19) \qquad \int_\Omega v_j^2(\xi)\, d\xi + \int_\Omega |D\, v_j(\xi)|^2\, d\xi \leqq 1 + (\mu j)^{-1} \leqq 1 + \mu^{-1}$$

$j = 1, 2, \ldots$, by a known compactness result [16, Th. 3.4.4, p. 75] we will be able to extract a subsequence $\{v_{j_k}\}$ so that $v_{j_k} \to v$ weakly in $W^{1,2}(\Omega)$ and at the same time $v_{j_k} \to v$ strongly in $L_2(\Omega)$. Since, by (2.18), $\int_\Omega |Dv_{j_k}|^2\, d\xi \leqq (\mu j_k)^{-1}$, $k = 1, 2, \ldots$, it follows that $Dv \equiv 0$ which in turn implies that $v \equiv$ constant $= c$; furthermore, since $v_{j_k} \to v \equiv c$ strongly in $L_2(\Omega)$, and, therefore, $1 = \int_\Omega v_{j_k}^2(\xi)\, d\xi \to \int_\Omega c^2\, d\xi = c^2|\Omega|$, we see that

$$(2.20) \qquad c = |\Omega|^{-1/2} > 0\,.$$

To obtain a contradiction to this, we note that by the continuity properties of the mapping sending a function in $W^{1,2}(\Omega)$ into its trace in $L_2(\partial\Omega)$, the fact that $v_{j_k} \to c$ weakly in $W^{1,2}(\Omega)$ implies that the traces of v_{j_k} on $\partial\Omega$ converge strongly to c in $L_2(\partial\Omega)$:

$$\int_{\partial\Omega} |v_{j_k} - c|^2\, d\sigma(\xi) \to 0 \quad \text{as} \quad k \to \infty\,.$$

If we now make use of the estimate

$$0 \leqq c\delta \leqq \int_{\partial\Omega} \chi_{j_k} c\, d\sigma = \int_{\partial\Omega} \chi_{j_k}(c - v_{j_k})\, d\sigma + \int_{\partial\Omega} \chi_{j_k} v_{j_k}\, d\sigma \leqq$$

$$\leqq \left(\int_{\partial\Omega} |c - v_{j_k}|^2\, d\sigma\right)^{1/2} |\partial\Omega|^{1/2} + \left(\int_{\partial\Omega} \chi_{j_k} v_{j_k}^2\, d\sigma\right)^{1/2} |\partial\Omega|^{1/2}\,,$$

then, sending $k \to \infty$, the preceding in conjunction with

$$\left(\int_{\partial\Omega} \chi_{J_k} v_{j_k}^2 \, d\sigma\right)^{1/2} \leqq (\varepsilon j_k)^{-1/2},$$

which follows from (2.18), allow us to conclude that $c\delta = 0$, in contradiction to (2.20).

In addition to various versions of Poincaré's inequality, we will need to make use of a special case of the Gagliardo-Nirenberg inequality ([5], [6], [18]):

$$\left(\int_\Omega v^2(\xi) \, d\xi\right)^{1/2} \leqq c\left(\int_\Omega |D\, v(\xi)|^2 \, d\xi\right)^{n/2(n+2)} \left(\int_\Omega |v(\xi)| \, d\xi\right)^{2/n+2} \tag{2.21}$$

valid for all $v \in W_0^{1,2}(\Omega)$ with the constant c being independent of Ω. As in the case of the Poincaré inequality (2.13), (2.21) is also valid for $v \in W^{1,2}(\Omega)$ which have mean value zero: $\int_\Omega v \, d\xi = 0$, provided that $\partial\Omega$ is sufficiently smooth; in this case the constant c will depend on Ω.

To establish (2.21) in the latter case, we assume that $\partial\Omega$ is so smooth that every function $v \in W^{1,2}(\Omega)$ has an extension $V \in W_0^{1,2}(G)$, where G is a fixed but arbitrary domain containing $\bar{\Omega}$, with the extension having the properties

$$\int_G [|V|^2 + |DV|^2] \, d\xi \leqq K \int_\Omega [|v|^2 + |Dv|^2] \, d\xi \tag{2.22}$$

and

$$\int_G |V| \, d\xi \leqq K \int_\Omega |v| \, d\xi$$

for a suitable constant K. For the construction of such extensions cf. [16, Thm. 3.4.3, p. 74] or [1, pp. 83–94].

Applying (2.21) to the extension $V \in W_0^{1,2}(G)$ and then using (2.22) we find that

$$\left(\int_\Omega v^2 \, d\xi\right)^{1/2} \leqq \left(\int_G V^2 \, d\xi\right)^{1/2} \leqq c\left(\int_G |DV|^2 \, d\xi\right)^{n/2(n+2)} \left(\int_G |V| \, d\xi\right)^{2/n+2} \leqq$$
$$\leqq c\left(K \int_\Omega [|v|^2 + |Dv|^2] \, d\xi\right)^{n/2(n+2)} \left(K \int_\Omega |v| \, d\xi\right)^{2/n+2}.$$

Finally, since v has means value zero, we may apply Poincaré's inequality (2.13) to estimate the expression $\int_\Omega |v|^2 \, d\xi$ appearing in the first integral on the right of the last member in terms of $\int_\Omega |Dv|^2 \, d\xi$, and doing so we obtain (2.21) under the assumption that v has mean value zero.

3. ESTIMATES FOR FUNDAMENTAL SOLUTIONS

In this section we will derive the needed estimates for the fundamental solutions G, R and N. We begin with the consideration of Green's function. Basing ourselves on a technique due to Nash [17] we will establish

Theorem 3.1. *For Green's function we have the estimate*

$$\left(\int_\Omega G^q(x, t; \xi, \tau)\, d\xi\right)^{1/q} \leqq c(t-\tau)^{-n/2\bar{q}} \quad (t > \tau,\ x \in \Omega), \tag{3.1}$$

for any $q \in [1, \infty)$, as well as $q = \infty$, with $\bar{q}$ denoting the Hölder conjugate of q and c a positive constant.

In the statement of Theorem 3.1 we are using the following notational conventions which will hold throughout this section: The letters b and c with or without subscripts will be reserved for positive constants which generally depend on λ, n and Ω; moreover, they will not necessarily represent the same constant on each appearance. When $q = \infty$, the expression $(\int_\Omega |g(\xi)|^q\, d\xi)^{1/q}$ is to be interpreted as the essential supremum of $|g(\xi)|$ over Ω. By the Hölder conjugate $\bar{q}$ of q we mean the extended real valued function defined by

$$\bar{q} = \begin{cases} q/(q-1) & \text{if } 1 < q < \infty \\ \infty & \text{if } q = 1 \\ 1 & \text{if } q = \infty\,. \end{cases}$$

The exponent $n/2\bar{q}$ occurring on the right of (3.1) is then well-defined except when $q = 1$ in which case it is to be interpreted as zero.

For the proof of Theorem 3.1 we will need

Lemma 3.2. *The Green's function G has the following properties*

$$G(x, t; \xi, \tau) \geqq 0 \quad (x \in \Omega,\ \xi \in \Omega,\ t > \tau), \tag{3.2}$$

$$\int_\Omega G(x, t; \xi, \tau)\, d\xi \leqq 1 \quad (x \in \Omega,\ t > \tau), \tag{3.3}$$

and

$$\lim_{\tau \uparrow t} \int_\Omega G^2(x, t; \xi, \tau)\, d\xi = \infty \quad (x \in \Omega), \tag{3.4}$$

Proof. We use the fact that for $\phi \in C_0(\Omega)$

$$u(x, t) = \int_\Omega G(x, t; \xi, \tau)\, \phi(\xi)\, d\xi \tag{3.5}$$

provides us with a classical solution of the problem

$$\begin{aligned} -u_t + L_x u = 0 \quad & \text{in} \quad \Omega \times (\tau, \infty)\,, \\ u = 0 \quad & \text{on} \quad \partial\Omega \times (\tau, \infty)\,, \\ u(x, \tau) = \phi(x) \quad & \text{for} \quad x \in \Omega \end{aligned}$$

which is continuous in $\bar{\Omega} \times [\tau, \infty)$. Consequently if $\phi \geqq 0$, it follows by the maximum principle that $u(x, t) \geqq 0$ for $x \in \Omega$, $t > \tau$. This shows that the integral operator on the right of (3.5) takes nonnegative functions $\phi \in C_0(\Omega)$ into nonnegative functions; clearly this is only possible if the kernel G of this integral operator is itself nonnegative, which proves (3.2).

Next by another application of the maximum principle, if $\phi \geqq 0$, $u(x, t)$ assumes it maximum on the initial line; consequently

$$\int_\Omega G(x, t; \xi, \tau)\,\phi(\xi)\,d\xi \leqq \sup_\Omega \phi(\xi) \quad (x \in \Omega,\ t > \tau).$$

Replacing $\phi(\xi)$ in the preceding by $\phi_n(\xi)$, where $\{\phi_n(\xi)\}$ is a sequence of non-negative $C_0(\Omega)$ functions with the property that $\phi_n(\xi) \leqq 1$ and $\phi_n(\xi) \to 1$ as $n \to \infty$ pointwise in Ω, we obtain

$$\int_\Omega G(x, t; \xi, \tau)\,\phi_n(\xi)\,d\xi \leqq \sup_\Omega \phi_n(\xi) \leqq 1 \quad (x \in \Omega,\ t > \tau);$$

and passing to the limit as $n \to \infty$, using the Lebesgue dominated convergence theorem, this yields

$$\int_\Omega G(x, t; \xi, \tau)\,d\xi \leqq 1 \quad (x \in \Omega,\ t > \tau)$$

which proves (3.3).

Finally, to prove (3.4), we apply Schwarz's inequality:

$$(3.6)\qquad \left|\int_\Omega G(x, t; \xi, \tau)\,\phi(\xi)\,d\xi\right| \leqq \left(\int_\Omega G^2(x, t; \xi, \tau)\,d\xi\right)^{1/2} \left(\int_\Omega \phi^2(\xi)\,d\xi\right)^{1/2}$$

for $t > \tau$, with ϕ an arbitrary $C_0(\Omega)$ function. Suppose now that (3.4) were not true for some $x \in \Omega$; then for this x we would be able to find an increasing sequence $\{\tau_j\}$ converging to $t : \tau_j \uparrow t$, so that $(\int_\Omega G^2(x, t; \xi, \tau_j)\,d\xi)^{1/2}$ was bounded, say, by M. Inserting τ_j in for τ in (3.6) we would then have

$$\left|\int_\Omega G(x, t; \xi, \tau_j)\,\phi(\xi)\,d\xi\right| \leqq M \left(\int_\Omega \phi^2(\xi)\,d\xi\right)^{1/2}$$

for $j = 1, 2, \ldots$. Passing to the limit as $j \to \infty$, making use of (2.5), this would imply that

$$|\phi(x)| = \left|\lim_{j\to\infty} \int_\Omega G(x, t; \xi, \tau_j)\,\phi(\xi)\,d\xi\right| \leqq M \left(\int_\Omega \phi^2(\xi)\,d\xi\right)^{1/2}$$

for all $\phi \in C_0(\Omega)$; but such a point estimate at x for any $\phi \in C_0(\Omega)$ in terms of its $L_2(\Omega)$ norm is clearly impossible. It follows that (3.4) must hold.

Proof of Theorem 3.1. We begin by showing that

$$(3.7)\qquad \int_\Omega G^2(x, t; \xi, \tau)\,d\xi \leqq c(t - \tau)^{-n/2} \quad (t > \tau,\ x \in \Omega).$$

To prove it we shall study the expression on the left as a function of τ and derive a differential inequality for it. Accordingly, regarding x and t as fixed, we set

$$B(\tau) = \int_\Omega G^2(x, t; \xi, \tau)\,\mathrm{d}\xi\,. \tag{3.8}$$

Differentiating B with respect to τ, and making use of the fact that as a function of ξ and τ, G satisfies the adjoint equation (2.6), we find that

$$B'(\tau) = 2\int_\Omega \frac{\partial G}{\partial \tau}\, G\,\mathrm{d}\xi = -2\int_\Omega (L_\xi G)\, G\,\mathrm{d}\xi\,.$$

Since by (2.7) G vanishes for $\xi \in \partial\Omega$, an integration by parts in the integral on the right then leads to

$$B'(\tau) = 2\int_\Omega \sum_{j,k=1}^{n} a_{jk}(\xi, \tau)\left(\frac{\partial G}{\partial \xi_j}\right)\left(\frac{\partial G}{\partial \xi_k}\right)\mathrm{d}\xi\,;$$

from which, in view of the ellipticity condition (1.3), we obtain

$$B'(\tau) \geqq 2\lambda \int_\Omega |D_\xi G|^2\,\mathrm{d}\xi \quad (t > \tau)\,. \tag{3.9}$$

We now apply the Gagliardo-Nirenberg inequality (2.21) to G. (It is applicable because as a function of ξ, $G \in C^1(\bar{\Omega})$ and vanishes for $\xi \in \partial\Omega$ which implies that $G \in W_0^{1,2}(\Omega)$ [9, p. 147]). This gives

$$B(\tau) = \int_\Omega G^2\,\mathrm{d}\xi \leqq c\left(\int_\Omega |D_\xi G|^2\,\mathrm{d}\xi\right)^{n/n+2}\left(\int_\Omega |G|\,\mathrm{d}\xi\right)^{4/n+2} \leqq \tag{3.10}$$

$$\leqq c\left(\int_\Omega |D_\xi G|^2\,\mathrm{d}\xi\right)^{n/n+2} \quad (t > \tau)\,,$$

since $\int_\Omega |G|\,\mathrm{d}\xi \leqq 1$ because of (3.2) and (3.3).

Combining (3.9) and (3.10) we arrive at

$$B'(\tau) \geqq c[B(\tau)]^{(n+2)/n} \quad \text{for} \quad \tau < t\,.$$

Solving this differential inequality, bearing in mind that by (3.4) $B(\tau) \to \infty$ as $\tau \uparrow t$, we obtain

$$B(\tau) \leqq c(t - \tau)^{-n/2} \quad (\tau < t) \tag{3.11}$$

which is the estimate (3.7).

Precisely the same argument, based on the fact that as a function of x and t, G satisfies the equation (2.1) together with the boundary condition (2.2) enables us to establish the estimate

$$\int_\Omega G^2(x, t; \xi, \tau)\,\mathrm{d}x \leqq c(t - \tau)^{-n/2} \quad (t > \tau,\ \xi \in \Omega)\,. \tag{3.12}$$

Next, to obtain a pointwise bound for G out of the L_2 bounds derived above, we apply the semi-group identity (2.12) for G in conjunction with Schwarz's inequality:

$$(3.13)\qquad G(x, t; \xi, \tau) = \int_\Omega G(x, t; \eta, s)\, G(\eta, s; \xi, \tau)\, d\eta \leqq$$

$$\leqq \left(\int_\Omega G^2(x, t; \eta, s)\, d\eta\right)^{1/2} \left(\int_\Omega G^2(\eta, s; \xi, \tau)\, d\eta\right)^{1/2} \quad (\tau < s < t);$$

and then insert in the estimates (3.7) and (3.12), taking $s = (t + \tau)/2$, which yields

$$(3.14)\qquad G(x, t; \xi, \tau) \leqq c^{1/2}(t - s)^{-n/4}\, c^{1/2}(s - \tau)^{-n/4} \Big|_{s=(t+\tau)/2} = b(t - \tau)^{-n/2}$$

for $x, \xi \in \Omega$, and $t > \tau$.

Finally, to estimate the L_q norm of G we use the interpolation inequality for L_p norms [10, p. 146], with the interpolation being between the L_1 and L_∞ norms of G:

$$\|G\|_q \leqq \|G\|_1^{1/q}\, \|G\|_\infty^{(q-1)/q}.$$

In view of (3.2), (3.3) and (3.14) this gives

$$\left(\int_\Omega G^q(x, t; \xi, \tau)\, d\xi\right)^{1/q} \leqq \left(\int_\Omega G(x, t; \xi, \tau)\, d\xi\right)^{1/q} \left(\sup_{\xi\in\Omega} G(x, t; \xi, \tau)\right)^{(q-1)/q} \leqq$$

$$\leqq 1\left(b(t - \tau)^{-n/2}\right)^{(q-1)/q} \leqq c(t - \tau)^{-n/2\bar{q}}$$

for $t > \tau$, $x \in \Omega$, with c independent of q, the desired estimate (3.1).

The estimate of Theorem 3.1 is precise for $t - \tau$ near zero, however for "large" values of $t - \tau$ it can be improved, and this is the content of the result which follows.

Theorem 3.3. *For $(t - \tau) \geqq 1$ the Green's function G satisfies the estimate*

$$(3.15)\qquad G(x, t; \xi, \tau) \leqq c_1 e^{-c_2(t-\tau)} \quad (x \in \Omega,\ \xi \in \Omega)$$

Proof. Using the notation and results of Theorem 3.1, we combine the inequality (3.9):

$$B'(\tau) \geqq 2\lambda \int_\Omega |D_\xi G|^2\, d\xi \quad (\tau < t)$$

with Poincaré's inequality (2.13)

$$B(\tau) = \int_\Omega G^2\, d\xi \leqq c \int_\Omega |D_\xi G|^2\, d\xi$$

(which is applicable since $G \in W_0^{1,2}$) to conclude that

$$B'(\tau) \geqq c\, B(\tau) \quad (\tau < t).$$

Solving this differential inequality we obtain

$$e^{-c(s-t)} B(s)\big|_{s=\tau} \leqq e^{-c(s-t)} B(s)\big|_{s=t-1/2} \quad (\tau \leqq t - 1/2);$$

which on making use of (3.11) to estimate $B(t - 1/2)$ yields $B(\tau) \leqq c_1 e^{-c(t-\tau)}$ for $(t - \tau) \geqq 1/2$, i.e.

$$\int_\Omega G^2(x, t; \xi, \tau)\, d\xi \leqq c_1 e^{-c(t-\tau)} \quad \text{for} \quad (t - \tau) \geqq 1/2 .$$

A similar argument gives us the estimate

$$\int_\Omega G^2(x, t; \xi, \tau)\, dx \leqq c_1 e^{-c(t-\tau)} \quad \text{for} \quad (t - \tau) \geqq 1/2 .$$

Hence using the semi-group identity for G, in conjunction with Schwarz's inequality as in (3.13) we find that

$$G(x, t; \xi, \tau) \leqq \left(\int_\Omega G^2(x, t; \eta, s)\, d\eta\right)^{1/2} \left(\int_\Omega G^2(\eta, s; \xi, \tau)\, d\eta\right)^{1/2} \leqq$$

$$\leqq c_1^{1/2} e^{-c(t-s)/2} c_1^{1/2} e^{-c(s-\tau)/2}\Big|_{s=(t+\tau)/2} = c_1 e^{-c_2(t-\tau)} \quad \text{for} \quad (t - \tau) \geqq 1 .$$

Next we turn to the estimation of the Neumann's function N.

Theorem 3.4. *For the Neumann's function N we have the estimate*

$$\left(\int_\Omega |N(x, t; \xi, \tau) - \mu|^q\, d\xi\right)^{1/q} \leqq c(t - \tau)^{-n/2\bar{q}} \quad (t > \tau,\ x \in \Omega) \tag{3.16}$$

for $q \in [1, \infty)$, as well as $q = \infty$, where $\mu = |\Omega|^{-1}$.

For the proof we will again need some preliminary results which we state in

Lemma 3.5. *The Neumann's function N has the following properties*

$$N(x, t; \xi, \tau) \geqq 0 \quad (x \in \Omega,\ \xi \in \Omega,\ t > \tau), \tag{3.17}$$

$$\int_\Omega N(x, t; \xi, \tau)\, d\xi = 1 \quad (x \in \Omega,\ t > \tau), \tag{3.18}$$

$$\int_\Omega N(x; t; \xi, \tau)\, dx = 1 \quad (\xi \in \Omega,\ t > \tau) \tag{3.19}$$

and

$$\lim_{\tau \uparrow t} \int_\Omega N^2(x, t; \xi, \tau)\, d\xi = \infty \quad (x \in \Omega) . \tag{3.20}$$

Proof. The proof of the positivity for N is similar to the proof of the positivity for G given in Lemma 3.2. We use the fact that for $\phi \in C_0(\Omega)$

$$u(x, t) = \int_\Omega N(x, t; \xi, \tau)\, \phi(\xi)\, d\xi$$

provides us with a classical solution to the equation $-u_t + L_x u = 0$ in $\Omega \times (\tau, \infty)$ which is continuous in $\bar{\Omega} \times [\tau, \infty)$, takes on the initial values ϕ on Ω at $t = \tau$ and satisfies the boundary condition $\partial u/\partial \nu = 0$ on $\partial\Omega \times (\tau, \infty)$. If now $\phi \geqq 0$, then (assuming that $\partial\Omega$ has the interior sphere property) by the theory of parabolic inequalities [3, pp. 52–53] we will have $u(x, t) \geqq 0$ for $(x, t) \in \Omega \times (\tau, \infty)$; from which the positivity of N follows immediately.

Next, to establish (3.18), fix $x \in \Omega$ and $t > 0$, and set

$$I(\tau) = \int_\Omega N(x, t; \xi, \tau) \, d\xi .$$

Differentiating with respect to τ, we then find, since N satisfies the adjoint equation (2.6) and homogeneous boundary condition (2.9), that

$$I'(\tau) = \int_\Omega \frac{\partial N}{\partial \tau} \, d\xi = -\int_\Omega L_\xi(N) \, d\xi = -\int_{\partial\Omega} \frac{\partial N}{\partial \nu} \, d\sigma(\xi) = 0 .$$

Consequently for $\tau < t$, $I(\tau) \equiv$ constant. To evaluate this constant we apply (2.5) (with $\phi \equiv 1$) which gives us

$$\lim_{\tau \uparrow t} I(\tau) = \lim_{\tau \uparrow t} \int_\Omega N(x, t; \xi, \tau) \, d\xi = 1$$

pointwise in Ω. Hence $I(\tau) \equiv 1$ for $\tau < t$ which proves (3.18). The proof of (3.19) is similar. Finally, the proof of (3.20) is exactly the same as the proof of (3.4).

Proof of Theorem 3.4. The steps in the proof parallel those in the proof of Theorem 3.1. We will first establish the estimate

$$\int_\Omega |N(x, t; \xi, \tau) - \mu|^2 \, d\xi \leqq c(t - \tau)^{-n/2} \quad (t > \tau, \ x \in \Omega) \tag{3.21}$$

by deriving a differential inequality for the quantity

$$B(\tau) = \int_\Omega |N(x, t; \xi, \tau) - \mu|^2 \, d\xi . \tag{3.22}$$

Basing ourselves on the observation that by (2.6) and (2.9), N and hence $N - \mu$ satisfy the adjoint equation

$$\frac{\partial}{\partial \tau} (N - \mu) + L_\xi(N - \mu) = 0 \tag{3.23}$$

and homogeneous boundary condition $(\partial/\partial \nu)(N - \mu) = 0$, it follows, on differentiating B with respect to τ, utilizing the equation (3.23), and then integrating by parts, that

$$B'(\tau) = 2\int_\Omega (N-\mu)\frac{\partial}{\partial\tau}(N-\mu)\,\mathrm{d}\xi = -2\int_\Omega (N-\mu)\,L_\xi(N-\mu)\,\mathrm{d}\xi =$$

$$= 2\int_\Omega \sum_{j,k=1}^{n} a_{jk}(\xi,\tau)\frac{\partial}{\partial\xi_j}(N-\mu)\frac{\partial}{\partial\xi_k}(N-\mu)\,\mathrm{d}\xi .$$

Therefore, on account of the ellipticity condition (1.3)

$$B'(\tau) \geqq 2\lambda \int_\Omega |D_\xi(N-\mu)|^2\,\mathrm{d}\xi \quad (\tau < t). \tag{3.24}$$

Next we note that in view of (3.18) $N-\mu$ has mean value zero: $\int_\Omega (N-\mu)\,\mathrm{d}\xi = 0$; and so we may apply the Gagliardo-Nirenberg inequality (2.21) to it:

$$\begin{aligned} B(\tau) = \int_\Omega (N-\mu)^2\,\mathrm{d}\xi &\leqq c\left(\int_\Omega |D_\xi(N-\mu)|^2\,\mathrm{d}\xi\right)^{n/n+2}\left(\int_\Omega |N-\mu|\,\mathrm{d}\xi\right)^{4/n+2} \leqq \\ &\leqq 2^{4/n+2}c\left(\int_\Omega |D_\xi(N-\mu)|^2\,\mathrm{d}\xi\right)^{n/n+2} \quad (\tau < t), \end{aligned} \tag{3.25}$$

since

$$\int_\Omega |N-\mu|\,\mathrm{d}\xi \leqq \int_\Omega N\,\mathrm{d}\xi + \mu\int_\Omega \mathrm{d}\xi = 2 \tag{3.26}$$

because of (3.17) and (3.18). Combining (3.24) and (3.25) we find that

$$B'(\tau) \geqq c[B(\tau)]^{(n+2)/n} \quad \text{for} \quad \tau < t\,;$$

and solving this differential inequality as in the proof of Theorem 3.1 we obtain the estimate (3.21).

A similar argument based on the fact that as a function of x and t, N satisfies equation (2.1) together with the boundary condition (2.4) results in the estimate

$$\int_\Omega |N(x,t;\xi,\tau) - \mu|^2\,\mathrm{d}x \leqq c(t-\tau)^{-n/2} \quad (t > \tau,\ \xi \in \Omega). \tag{3.27}$$

For the next step in the proof we observe that because of (3.18) and (3.19), the semi-group identity (2.12) for N implies the same kind of identity for $N-\mu$:

$$\begin{aligned} N(x,t;\xi,\tau) - \mu &= \\ = \int_\Omega [N(x,t;\eta,s) - \mu]\,[N(\eta,s;\xi,\tau) - \mu]\,\mathrm{d}\eta &\quad (\tau < s < t). \end{aligned} \tag{3.28}$$

With this identity in hand, the estimates (3.21) and (3.27) then lead to

$$|N(x,t;\xi,\tau) - \mu| \leqq c(t-\tau)^{-n/2} \quad (t > \tau,\ x,\ \xi \in \Omega) \tag{3.29}$$

in exactly the same way as we obtained (3.14) in the proof of Theorem 3.1.

Finally, to estimate the L_q norm of $N - \mu$ we again use the interpolation inequality

(3.30) $$\|N - \mu\|_q \leqq \|N - \mu\|_1^{1/q} \|N - \mu\|_\infty^{(q-1)/q} .$$

The inequality (3.29) gives us an estimate for $\|N - \mu\|_\infty$, the L_∞ norm of $N - \mu$, while from (3.26) we have the estimate $\|N - \mu\|_1 \leqq 2$ for the L_1 norm of $N - \mu$. Inserting these estimates into (3.30) we obtain

$$\left(\int_\Omega |N(x, t; \xi, \tau) - \mu|^q \, \mathrm{d}\xi\right)^{1/q} \leqq 2^{1/q}(c(t - \tau)^{-n/2})^{(q-1)/q}$$

for $t > \tau$, $x \in \Omega$, from which the desired inequality (3.16) follows.

For large $t - \tau$ it is again possible to improve the estimate (3.16) of the preceding theorem, and this improvement is described in

Theorem 3.6. *For* $(t - \tau) \geqq 1$ *the Neumann's function N satisfies the estimate*

(3.31) $$|N(x, t; \xi, \tau) - \mu| \leqq c_1 \mathrm{e}^{-c_2(t-\tau)} \quad (x \in \Omega, \ \xi \in \Omega) .$$

Proof. The proof is virtually identical with the proof of Theorem 3.3 so we will only sketch it. For the expression $B(\tau)$ defined by (3.22) we derive the differential inequality

(3.32) $$B'(\tau) \geqq c\, B(\tau) \quad (\tau < t)$$

by combining (3.24) with Poincaré's inequality (2.13) for $N - \mu$:

$$B(\tau) = \int_\Omega |N - \mu|^2 \, \mathrm{d}\xi \leqq c \int_\Omega |D_\xi(N - \mu)|^2 \, \mathrm{d}\xi$$

(which is applicable since $N - \mu$ has mean value zero). Solving (3.32) as in Theorem 3.3 we arrive at the estimate $B(\tau) \leqq c_1 \mathrm{e}^{-c(t-\tau)}$ for $(t - \tau) \geqq 1/2$, i.e.

(3.33) $$\int_\Omega |N(x, t; \xi, \tau) - \mu|^2 \, \mathrm{d}\xi \leqq c_1 \mathrm{e}^{-c(t-\tau)} \quad \text{for} \quad (t - \tau) \geqq 1/2 .$$

A similar argument yields the estimate

(3.34) $$\int_\Omega |N(x, t; \xi, \tau) - \mu|^2 \, \mathrm{d}x \leqq c_1 \mathrm{e}^{-c(t-\tau)} \quad \text{for} \quad (t - \tau) \geqq 1/2 .$$

The desired result (3.31) then follows from (3.33) and (3.34) by using the semi-group identity (3.28) exactly as in the proof of Theorem 3.3.

Remark. We wish to point out that, unlike the estimates of Theorems 3.1 and 3.3, the constants in the estimates of Theorems 3.4 and 3.6 depend quantitatively on some measure of smoothness for $\partial\Omega$; this is due to the fact that, ultimately, they are expressible in terms of the constants that figure in the Gagliardo-Nirenberg and

Poincaré inequalities, (2.21) and (2.13), for functions with mean value zero, both of which require a smooth boundary for their validity.

We now turn to the estimation of Robin's function R. For $t - \tau$ near zero this will be acomplished on the basis of

Lemma 3.7. *Assume that the coefficient function β is non-negative, then*

$$0 \leqq R \leqq N . \tag{3.35}$$

Proof. Let $\phi \in C_0(\Omega)$ and set

$$u(x, t) = \int_\Omega [N(x, t; \xi, \tau) - R(x, t; \xi, \tau)]\, \phi(\xi)\, d\xi . \tag{3.36}$$

Then from the properties of N and R, u will be a classical solution of $-u_t + L_x u = 0$ in $\Omega \times (\tau, \infty)$ which is continuous in the closure $\bar{\Omega} \times [\tau, \infty)$, assumes the values zero on $\bar{\Omega}$ at $t = \tau$ and satisfies the boundary condition

$$\frac{\partial u}{\partial v} + \beta u = \int_\Omega \left[\left(\frac{\partial N}{\partial v} + \beta N\right) - \left(\frac{\partial R}{\partial v} + \beta R\right)\right] \phi(\xi)\, d\xi =$$

$$= \int_\Omega \beta N\, \phi(\xi)\, d\xi \geqq 0 \quad \text{on} \quad \partial\Omega \times (\tau, \infty)$$

provided that $\phi \geqq 0$. By the theory of parabolic inequalities (again assuming $\partial\Omega$ to have the interior sphere property) it follows that $u \geqq 0$ in $\Omega \times (\tau, \infty)$, and hence that the kernel of the operator (3.36), $N - R \geqq 0$. This proves that $R \leqq N$. The proof that $R \geqq 0$ is conducted along similar lines, and is in fact practically the same as the proof that $N \geqq 0$ given in Lemma 3.5.

Corollary 3.8. *For $0 < t - \tau \leqq 1$ the Robin's function satisfies the estimate*

$$\left(\int_\Omega R^q(x, t; \xi\tau)\, d\xi\right)^{1/q} \leqq b(t - \tau)^{-n/2\bar{q}} \quad (x \in \Omega), \tag{3.37}$$

with q any number in $[1, \infty)$, as well as $q = \infty$.

Proof. Using the notation $\| \ \|_q$ for the L_q norm, it follows from (3.35) of the preceding lemma, that

$$\|R\|_q \leqq \|N\|_q \leqq \|N - \mu\|_q + \|\mu\|_q .$$

Inserting in the estimate (3.16) for $\|N - \mu\|_q = (\int_\Omega |N - \mu|^q\, d\xi)^{1/q}$ and recalling that $\mu = |\Omega|^{-1}$, this results in

$$\left(\int_\Omega R^q(x, t; \xi, \tau)\, d\xi\right)^{1/q} \leqq c(t - \tau)^{-n/2\bar{q}} + |\Omega|^{-1+1/q} \leqq b(t - \tau)^{-n/2\bar{q}}$$

for $0 < t - \tau \leqq 1$, $x \in \Omega$; the required inequality (3.37).

Just as G and $N - \mu$ decayed exponentially for $t - \tau$ large, the same is true for R. More precisely we have

Theorem 3.9. *Assume that the coefficient function β is non-negative and satisfies condition* (1.12), *then for* $(t - \tau) \geqq 1$ *the Robin's function R satisfies the estimate*

$$R(x, t; \xi, \tau) \leqq c_1 e^{-c_2(t-\tau)} \quad (x, \xi \in \Omega). \tag{3.38}$$

Remark. Here the constant c_2 will depend not only on λ, n and Ω but also on the ε and δ appearing in condition (1.12).

Proof. The proof proceeds very much like the proofs of Theorems 3.3 and 3.6. We first establish an estimate for the quantity

$$B(\tau) = \int_\Omega R^2(x, t; \xi, \tau)\, d\xi$$

by deriving a differential inequality for it in a by now familiar way: We differentiate B with respect to τ, make use of the equation (2.6) satisfied by R and then integrate by parts. In this case this leads to

$$B'(\tau) = 2\int_\Omega R \frac{\partial R}{\partial \tau}\, d\xi = -2\int_\Omega R L_\xi R\, d\xi =$$

$$= 2\int_\Omega \sum_{j,k=1}^{n} a_{jk}(\xi, \tau)\left(\frac{\partial R}{\partial \xi_j}\right)\left(\frac{\partial R}{\partial \xi_k}\right) d\xi - 2\int_{\partial\Omega} R \frac{\partial R}{\partial \nu}\, d\sigma(\xi).$$

Applying the ellipticity condition (1.3) and the boundary condition (2.8), it follows that

$$B'(\tau) \geqq 2\lambda \int_\Omega |D_\xi R|^2\, d\xi + 2\int_{\partial\Omega} \beta R^2\, d\sigma(\xi).$$

But from the variant on Poincaré's inequality (2.16), there exists a positive constant $a = a(\varepsilon, \delta, \lambda, \Omega)$ so that

$$a\, B(\tau) = a\int_\Omega R^2\, d\xi \leqq \lambda \int_\Omega |D_\xi R|^2\, d\xi + \int_{\partial\Omega} \beta R^2\, d\sigma(\xi).$$

Combining the two preceding inequalities we obtain

$$B'(\tau) \geqq c\, B(\tau) \quad (\tau < t);$$

and solving this differential inequality exactly as in the proof of Theorem 3.3 we arrive at $B(\tau) \leqq c_1 e^{-c(t-\tau)}$ for $t - \tau \geqq 1/2$, i.e.

$$\int_\Omega R^2(x, t; \xi, \tau)\, d\xi \leqq c_1 e^{-c(t-\tau)} \quad \text{for} \quad t - \tau \geqq 1/2. \tag{3.39}$$

A similar argument yields

$$(3.40) \qquad \int_\Omega R^2(x, t; \xi, \tau)\, dx \leqq c_1 e^{-c(t-\tau)} \quad \text{for} \quad t - \tau \geqq 1/2 .$$

The desired result (3.38) then follows from (3.39) and (3.40) by applying the semi-group identity (2.12) for R:

$$R(x, t; \xi, \tau) = \int_\Omega R(x, t; \eta, s)\, R(\eta, s; \xi, \tau)\, d\eta \quad (\tau < s < t),$$

togethei with Schwarz's inequality in exactly the same way as was done in the proof of Theorem 3.3.

4. PROOFS OF THE MAIN RESULTS

In this section we give the proofs of the asymptotic results stated in Section 1, after which we briefly consider some further results of a similar character. We begin with the

Proof of Theorem 1.1. The proof is based on the representation formula (2.11) for solutions u of problem I which we here write in the form

$$(4.1) \qquad u(x, t) = v(x, t) + w(x, t),$$

wheie

$$(4.2) \qquad v(x, t) = \int_0^t \int_\Omega G(x, t; \xi, \tau) f(\xi, \tau)\, d\xi\, d\tau$$

and

$$(4.3) \qquad w(x, t) = \int_\Omega G(x, t; \xi, 0)\, \phi(\xi)\, d\xi .$$

From the estimate (3.15) for G, the integral represented by w clearly tends to zero uniformly in Ω as $t \to \infty$; thus we need only show that $v \to 0$ uniformly in Ω as $t \to \infty$.

For this purpose we first make use of our hypothesis (1.9): $\int_\Omega |f(\xi, \tau)|^p\, d\xi \to 0$ as $\tau \to \infty$ (with p satisfying condition (1.10)) to assure the existence of a positive number T so that, for given $\varepsilon > 0$

$$(4.4) \qquad \left(\int_\Omega |f(\xi, \tau)|^p\, d\xi\right)^{1/p} < \varepsilon \quad \text{holds for} \quad \tau \geqq T.$$

Using this T, we decompose the τ integration in the integral defining v, and then apply

Hölder's inequality together with (4.4) to obtain

$$(4.5)\qquad |v(x,t)| = \left|\int_0^T\int_\Omega Gf\,d\xi\,d\tau + \int_T^t\int_\Omega Gf\,d\xi\,d\tau\right| \leqq$$

$$\leqq \int_0^T\int_\Omega G(x,t;\xi,\tau)\,|f(\xi,\tau)|\,d\xi\,d\tau + \varepsilon\int_T^t\left(\int_\Omega G^q(x,t;\xi,\tau)\,d\xi\right)^{1/q} d\tau \quad (t > T)$$

where q denotes the Hölder conjugate of $p : q = \bar{p}$. In a moment we are going to show that

$$(4.6)\qquad J = \sup_{\substack{x\in\Omega\\ t>T+1}} \int_T^t\left(\int_\Omega G^q(x,t;\xi,\tau)\,d\xi\right)^{1/q} d\tau < \infty\,.$$

Temporarily granting this, we find from the preceding inequality (4.5), on making use of the estimate (3.15), that for $t > T+1$

$$\sup_{x\in\Omega} |v(x,t)| \leqq c_1 e^{-c_2(t-T)} \int_0^T\int_\Omega |f(\xi,\tau)|\,d\xi\,d\tau + \varepsilon J\,;$$

from which the desired uniform convergence of $v(x, t)$ to zero in Ω follows immediately.

It remains to establish the finiteness of the expression J defined in (4.6). To achieve this we write

$$\int_T^t\left(\int_\Omega G^q\,d\xi\right)^{1/q} d\tau = \int_T^{t-1}\left(\int_\Omega G^q\,d\xi\right)^{1/q} d\tau + \int_{t-1}^t\left(\int_\Omega G^q\,d\xi\right)^{1/q} d\tau\;\; (t > T+1),$$

and note that the estimate (3.15) is applicable to the G appearing in the first integral on the right. Applying it, and making use of (3.1) to estimate the second integral on the right we find that

$$\int_T^t\left(\int_\Omega G^q(x,t;\xi,\tau)\,d\xi\right)^{1/q} d\tau \leqq \int_T^{t-1} c_1 e^{-c_2(t-\tau)}\,|\Omega|^{1/q}\,d\tau + \int_{t-1}^t c(t-\tau)^{-n/2\bar{q}}\,d\tau \leqq$$

$$\leqq c_1|\Omega|^{1/q}\int_1^\infty e^{-c_2 s}\,ds + c\int_0^1 s^{-n/2p}\,ds \quad (x\in\Omega,\; t > T+1)$$

since, by definition, $q = \bar{p}$ so that $\bar{q} = p$. The desired finiteness of J now follows from the observation that both of the last two integrals are finite, the second one because of the condition (1.10) : $p > n/2$ when $n \geqq 2$ and $p \geqq 1$ when $n = 1$. This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is identical with the proof of Theorem 1.1. We merely let R play the role of G in the argument just given, replacing the estimates (3.1) and (3.15) for G by the corresponding estimates (3.37) and (3.38) for R.

The argument proving Theorem 1.1 is also the main ingredient in the proof of

Theorem 1.3. To prove the latter we start from the representation formula (2.11):

$$u(x, t) = \int_{\Omega} N \, \phi(\xi) \, d\xi + \int_0^t \int_{\Omega} N f(\xi, \tau) \, d\xi \, d\tau$$

for solutions of problem III, and setting

$$m(t) = \frac{1}{|\Omega|} \left[\int_{\Omega} \phi(\xi) \, d\xi + \int_0^t \int_{\Omega} f(\xi, \tau) \, d\xi \, d\tau \right],$$

recalling that $\mu = |\Omega|^{-1}$, re-write it in the form

$$u(x, t) - m(t) = \int_{\Omega} (N - \mu) \, \phi(\xi) \, d\xi + \int_0^t \int_{\Omega} (N - \mu) f(\xi, \tau) \, d\xi \, d\tau .$$

Now exactly as in the proof of Theorem 1.1 we can show that the right side $\to 0$ uniformly in Ω as $t \to \infty$. (We let $N - \mu$ play the role of G in that proof, and replace the estimates (3.1) and (3.15) by (3.16) and (3.31), respectively.) Therefore, since, in view of hypothesis (1.14), $m(t)$ converges to the quantity

$$\frac{1}{|\Omega|} \left[\int_{\Omega} \phi(\xi) \, d\xi + \lim_{t \to \infty} \int_0^t \int_{\Omega} f(\xi, \tau) \, d\xi \, d\tau \right] = \frac{1}{|\Omega|} [I + A],$$

it follows that $u(x, t)$ converges to the same quantity uniformly in Ω as $t \to \infty$, which proves Theorem 1.3.

Finally we want to point out that the conclusions of Theorems 1.1, 1.2 and 1.3 remain valid if in place of the assumption (1.9) with p satisfying (1.10) we assume that $f \in L_p[\Omega \times (0, \infty)]$ for some $p > n/2 + 1$, i.e.

$$\int_0^{\infty} \int_{\Omega} |f(\xi, \tau)|^p \, d\xi \, d\tau < \infty \quad \text{for} \quad p > n/2 + 1 . \tag{4.7}$$

The proofs of these assertions are very similar to the proofs of the corresponding assertions in Theorems 1.1, 1.2 and 1.3. As a sample we give the proof of the analogue of Theorem 1.1 under the assumption (4.7).

Our point of departure is again the representation formula (4.1) and as in the proof of Theorem 1.1 we need only show that the term $v(x, t)$ of that formula tends to zero uniformly in Ω as $t \to \infty$. To this end we again use the decomposition for the integral defining v indicated in (4.5) with T this time chosen, in accordance with hypothesis (4.7), so that

$$\left(\int_T^{\infty} \int_{\Omega} |f(\xi, \tau)|^p \, d\xi \, d\tau \right)^{1/p} < \varepsilon .$$

Applying Hölder's inequality we thus obtain

$$|v(x, t)| \leq \int_0^T \int_{\Omega} G(x, t; \xi, \tau) \, |f(\xi, \tau)| \, d\xi \, d\tau + \varepsilon \left(\int_T^t \int_{\Omega} G^q(x, t; \xi, \tau) \, d\xi \, d\tau \right)^{1/q} \quad (t > T)$$

where $q = \bar{p}$. Therefore if we can show that the quantity

$$I = \sup_{\substack{x \in \Omega \\ t > T+1}} \left(\int_T^t \int_\Omega G^q(x, t; \xi, \tau) \, d\xi \, d\tau \right)^{1/q} < \infty$$

it will follow that

$$\sup_{x \in \Omega} |v(x, t)| \leqq e^{-c(t-T)} \left(\int_0^T \int_\Omega |f(\xi, \tau)| \, d\xi \, d\tau \right) + \varepsilon I \quad (t > T + 1)$$

and hence that v converges uniformly to zero in Ω as $t \to \infty$; thereby proving the result.

To prove I finite we proceed similarly to the analysis of J by using the decomposition

$$\int_T^t \int_\Omega G^q \, d\xi \, d\tau = \int_T^{t-1} \int_\Omega G^q \, d\xi \, d\tau + \int_{t-1}^t \int_\Omega G^q \, d\xi \, d\tau \quad (t > T + 1),$$

and then inserting in the estimates (3.15) and (3.1) into the integrals on the right. Here this yields

$$\int_T^t \int_\Omega G^q(x, t; \xi, \tau) \, d\xi \, d\tau \leqq \int_T^{t-1} c_1^q \, e^{-qc_2(t-\tau)} \, |\Omega| \, d\tau + \int_{t-1}^t c^q(t - \tau)^{-nq/2\bar{q}} \, d\tau \leqq$$

$$\leqq c_1^q \, |\Omega| \int_1^\infty e^{-qc_2 s} \, ds + c^q \int_0^1 s^{-n/2(p-1)} \, ds \quad (x \in \Omega, \; t > T + 1),$$

since $q/\bar{q} = \bar{p}/p = 1/p - 1$. The condition $p > n/2 + 1$ in the assumption (4.7) then implies the finiteness of the very last integral on the right, and hence the finiteness of I; which completes the proof.

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