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ON TWO-PARAMETRIC SYSTEMS OF MATRICES AND DIFFEOMORPHISMS

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The topological structure of trajectories of a diffeomorphism near its fixed or periodic point (see e.g. [12]) essentially depends on the fact whether the matrix of its linear part, computed at this fixed or periodic point, respectively, has eigenvalues on the unit circle S_1, whether they are real or complex, and on the number of such eigenvalues.

If $D^r = D^r(R^n)$ is the set of all C^r diffeomorphisms on R^n endowed with the C^r Whitney topology, then there is a residual set D_1^r (a set which contains a countable intersection of open dense subsets) in D^r such that if $f \in D_1^r$, x_0 is a fixed or periodic point of f and $L_{x_0}(f)$ is the matrix of the linear part of f at x_0, then $L_{x_0}(f)$ has no eigenvalues on S_1 (Kupka-Smale theorem for diffeomorphisms; see [12, Theorem 2.4]).

Denote by $G^r = G^r(R^1, R^n)$ the set of all one-parametric systems of diffeomorphisms on R^n of class C^r endowed with the C^r Whitney topology. The papers of Brunovský [7, 8] contain the following result: *There is a residual subset G_1 in G^r such that if $g \in G_1$, then there exist one-dimensional submanifolds $K_k(g)$ of $R^1 \times R^n$ ($k = 1, 2, 3, \ldots$), for which $(\mu_0, x_0) \in K_k(g)$ implies that x_0 is a k-periodic point of g_{μ_0} ($g_\mu \in D^r$, $g_\mu(x) = g(\mu, x)$) and the following holds:*

(1) *The matrix $A = L_{x_0}(g_{\mu_0})$ has no double eigenvalue on S_1.*
(2) *For every $k \geqq 1$ the matrix A^k has no non-real root of 1 as its eigenvalue.*
(3) *If an eigenvalue of A^k lies on S_1, then there is no other eigenvalue on S_1 except its complex conjugate.*
(4) *The set of all $(\mu_0, x_0) \in R^1 \times R^n$ for which the matrix $L_{x_0}(g_{\mu_0})$ has an eigenvalue on S_1 consists of isolated points.*

In this paper we generically classify the set of all two-parametric systems of matrices by their eigenvalues on the unit circle and we apply the results obtained to two-parametric systems of diffeomorphisms.

The topological structure of trajectories of a vector field near its critical point essentially depends on the fact, whether the matrix of its linear part, computed at this critical point, has an eigenvalue with zero real part, and on the number of such eigenvalues (see [1], [3]). For the generic classification of k-parametric systems of matrices and vector fields ($k = 1, 2$) from this point of view, we refer to the papers [2]–[6], [10], [11], [13], [14]. The equations for eigenvalues defining various classes of diffeomorphisms are more complicated than those corresponding to the case of vector fields. We use a special method developed for this purpose by P. Brunovský (see [8]) and we extend its practicability to a wider class of parametric systems of matrices and diffeomorphisms.

1. TWO-PARAMETRIC SYSTEMS OF DIFFEOMORPHISMS

Let $H^r = H^r(R^2, R^n)$ be the set of all two-parametric systems of diffeomorphisms on R^n with values of parameter in R^2, of class C^r and endowed with the C^r Whitney topology. For $h \in H^r(R^2, R^n)$ denote by $Z_k = Z_k(h)$ the set of all $(\mu, x) \in R^2 \times R^n$, for which x is a k-periodic point of h_μ.

Lemma 1. *There exists a residual subset $H_1 \subset H^r$ such that for $h \in H_1$ the set $Z_k(h)$ is a 2-dimensional submanifold of $R^2 \times R^n$, $Z_1(h)$ is closed.*

The proof of this lemma nearly coincides with the proof of the similar assertion for one-parametric systems of diffeomorphisms (see [7, Theorem 1, (i)]). To demonstrate the idea of the proof, we prove the lemma for $k = 1$.

It is clear that the set $Z_1(h)$ is closed for $h \in H^r$. Define the mapping $\varrho : H^r \to C^r(R^2 \times R^n, R^n \times R^n)$, $\varrho(h)(\mu, x) = (x, h(\mu, x))$. This mapping is a C^r representation (see [1]). The set $\Delta = \{(x, x) \mid x \in R^n\}$ is a closed submanifold of $R^n \times R^n$ of codimension n. The mapping $ev_\varrho : H^r \times R^2 \times R^n \to R^n \times R^n$, $ev_\varrho(h, \mu, x) = \varrho(h)(\mu, x)$, transversally intersects the submanifold Δ. Therefore by [1, Theorem 19. 1] the set $H_1 = \{h \in H^r \mid \varrho(h) \mathrel{\overline{\pitchfork}} \Delta\}$ ($\overline{\pitchfork}$ is the symbol for the transversal intersection) is residual in H^r. By [1, Corollary 17. 1], if $h \in H_1$, then $Z_1(h) = [\varrho(h)]^{-1}(\Delta)$ is a 2-dimensional submanifold of $R^2 \times R^n$.

Since we are interested in the generic classification of the set of two-parametric systems of diffeomorphisms by their linear parts, it suffices to give the generic classification of the set of all two-parametric systems of matrices. For this purpose we need to stratify some algebraic and semialgebraic varieties in the spaces $M_n \times R^2$, $M_n \times R^4$, $M_n \times R^6$, where M_n is the set of all $n \times n$ matrices, and to estimate the codimension of their strata.

2. STRATIFICATIONS OF SEMIALGEBRAIC VARIETIES

A semialgebraic variety in R^m is the set of all $x = (x_1, x_2, \ldots, x_m) \in R^m$ satisfying the inequalities $P_i(x_1, x_2, \ldots, x_m) \geqq 0$, $i = 1, 2, \ldots, n$. (An algebraic variety is given by the equalities $P_i(x) = 0$.) We shall be frequently refering to the following Whitney's theorem on stratification of semialgebraic varieties into smooth manifolds.

Lemma 2 ([15], § 11]). *Let W be a semialgebraic variety. Then $W = W_1 \cup W_2 \cup \ldots \cup W_p$, where W_j $(j = 1, 2, \ldots, p)$ are smooth manifolds with a finite number of components, $\bigcup_{j=\varrho}^{p} W_j$ is closed for all $1 \leqq \varrho \leqq p$,* codim $W_j <$ codim W_{j+1} *for $j = 1, 2, \ldots, p - 1$.*

A decomposition of a semialgebraic variety W into a finite union of smooth manifolds with the properties mentioned in Lemma 2 is called *the stratification of W* and the manifolds W_j are called *the strata of W.*

We can give a smooth structure on the set M_n induced by the natural identification of M_n with R^{n^2}. Now we shall define some subsets of M_n by the properties introduced in the following table. We write $m(\lambda)$ instead of the multiplicity of an eigenvalue. Let Z be the set of all natural numbers.

notation	eigenvalues on S_1	$m(\lambda)$	$m(\mu)$	other properties
A_1	λ	$\geqq 1$	–	–
A_2	λ	$\geqq 2$	–	–
$A_2(+1)$	$\lambda = +1$	$\geqq 2$	–	–
$A_2(-1)$	$\lambda = -1$	$\geqq 2$	–	–
A_{31}	λ	$\geqq 1$	–	$\lambda \neq \pm 1,\ \lambda^l = 1,\ l \in Z$
A_3	λ	$\geqq 1$	–	$\lambda \neq \pm 1,\ \exists k \in Z : \lambda^k = 1$
A_4	λ	$\geqq 3$	–	–
$B_1(+1, c)$	$\lambda = +1;\ \mu$	$\geqq 1$	$\geqq 1$	$\mu \neq \pm 1$
$B_1(-1, c)$	$\lambda = -1;\ \mu$	$\geqq 1$	$\geqq 1$	$\mu \neq \pm 1$
$B_1(+1, -1)$	$\lambda = +1;\ \mu = -1$	$\geqq 1$	$\geqq 1$	–
B_1	$\lambda;\ \mu$	$\geqq 1$	$\geqq 1$	$\lambda, \mu \neq \pm 1$
$B_2(+1, c)$	$\lambda;\ \mu = +1$	$\geqq 2$	$\geqq 1$	$\lambda \neq \pm 1$
$B_2(-1, c)$	$\lambda;\ \mu = -1$	$\geqq 2$	$\geqq 1$	$\lambda \neq \pm 1$
$B_{2l}(c, \mathrm{i})$	$\lambda = \mathrm{i};\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \mu^l = 1,\ l \in Z$
$B_{2l}(c, +1)$	$\lambda = +1;\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \mu^l = 1$
$B_{2l}(c, -1)$	$\lambda = -1;\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \mu^l = 1$
$B_2(c, \mathrm{i})$	$\lambda = \mathrm{i};\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \exists k \in Z : \mu^k = 1$
$B_2(c, +1)$	$\lambda = +1;\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \exists k \in Z : \mu^k = 1$
$B_2(c, -1)$	$\lambda = -1;\ \mu$	$\geqq 2$	$\geqq 1$	$\mu \neq \pm 1,\ \exists k \in Z : \mu^k = 1$
B_2	$\lambda;\ \mu$	$\geqq 2$	$\geqq 1$	$\lambda, \mu \neq \pm 1$
B_{3l}	$\lambda;\ \mu$	$\geqq 1$	$\geqq 1$	$\lambda, \mu \neq \pm 1,\ \mu^l = 1$
B_3	$\lambda;\ \mu$	$\geqq 1$	$\geqq 1$	$\lambda, \mu \neq \pm 1,\ \exists k \in Z : \mu^k = 1$
C	λ, μ, ν	all of the mult. $\geqq 1$		$\lambda, \mu, \nu \neq \pm 1$

Now for the sets of the type A, B, C we can define the corresponding semialgebraic varieties in $M_n \times R^2$, $M_n \times R^4$, $M_n \times R^6$, respectively, for which we shall use the same symbols as for the sets of the type A, B, and C, but with a tilde. If $\pi_k : M_n \times R^k \to M_n$ is the natural projection, then $\pi_2(\tilde{A}) = A$, $\pi_4(\tilde{B}) = B$ and $\pi_6(\tilde{C}) = C$. We shall use $P(\lambda) = P_1(\lambda_1, \lambda_2) + \mathrm{i}P_2(\lambda_1, \lambda_2)$ to denote the characteristic polynomial of a matrix $A \in M_n$ and

$$P' = \frac{\mathrm{d}P}{\mathrm{d}\lambda} = P'_1 + \mathrm{i}P'_2\,, \quad P'' = \frac{\mathrm{d}^2P}{\mathrm{d}\lambda^2} = P''_1 + \mathrm{i}P''_2\,.$$

notation	equations defining the set	
$\tilde{A}_1$	$P_i(\lambda_1, \lambda_2) = 0\ (i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0$
$\tilde{A}_2$	$P_i(\lambda_1, \lambda_2) = P'_i(\lambda_1, \lambda_2) = 0$ $(i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0$
$\tilde{A}_2(+1)$	–,,–	$\lambda_1 - 1 = 0,\ \lambda_2 = 0$
$\tilde{A}_2(-1)$	–,,–	$\lambda_1 + 1 = 0,\ \lambda_2 = 0$
$\tilde{A}_3(\lambda_{10}, \lambda_{20})$	$P_i(\lambda_1, \lambda_2) = 0\ (i = 1, 2)$	$\lambda_1 - \lambda_{10} = 0,$ $\lambda_2 - \lambda_{20} = 0$
$\tilde{A}_4$	$P_i(\lambda_1, \lambda_2) = P'_i(\lambda_1, \lambda_2) =$ $= P''_i(\lambda_1, \lambda_2) = 0\ (i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0$
$\tilde{B}_1(+1, c)$	$P_i(\lambda_1, \lambda_2) = 0,\ \tilde{P}_i = P_i(\mu_1, \mu_2) = 0$ $(i = 1, 2)$	$\lambda_1 - 1 = 0,\ \lambda_2 = 0$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_2(-1, c)$	–,,–	$\lambda_1 + 1 = 0,\ \lambda_2 = 0$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_1(+1, -1)$	–,,–	$\lambda_1 - 1 = 0,\ \lambda_2 = 0$ $\mu_1 + 1 = 0,\ \mu_2 = 0$
$\tilde{B}_1$	–,,–	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1,$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$ $\lambda_1 - \mu_1 \neq 0$
$\tilde{B}_2(+1, c)$	$P_i(\lambda_1, \lambda_2) = P'_i(\lambda_1, \lambda_2) = 0,$ $\tilde{P}_i = P_i(\mu_1, \mu_2) = 0\ (i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1$ $\mu_1 - 1 = 0,\ \mu_2 = 0$
$\tilde{B}_2(-1, c)$	$P_i(\lambda_1, \lambda_2) = P'_i(\lambda_1, \lambda_2) = 0, \lambda_2) = 0,$ $\tilde{P}_i = P_i(\mu_1, \mu_2) = 0\ (i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1$ $\mu_1 + 1 = 0,\ \mu_2 = 0$
$\tilde{B}_2(\mu_{10}, \mu_{20}, \mathrm{i})$	–,,–	$\lambda_1 = 0,\ \lambda_2 - 1 = 0$ $\mu_1 - \mu_{10} = 0,$ $\mu_2 - \mu_{20} = 0$

notation	equations defining the set	
$\tilde{B}_2(\mu_{10}, \mu_{20}, +1)$	–„–	$\lambda_1 - 1 = 0,\ \lambda_2 = 0$ $\mu_1 - \mu_{10} = 0,$ $\mu_2 - \mu_{20} = 0$
$\tilde{B}_2(\mu_{10}, \mu_{20}, -1)$	–„–	$\lambda_1 + 1 = 0,\ \lambda_2 = 0$ $\mu_1 - \mu_{10} = 0,$ $\mu_2 - \mu_{20} = 0$
$\tilde{B}_2(c, \mathrm{i})$	–„–	$\lambda_1 = 0,\ \lambda_2 - 1 = 0$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_2(c, +1)$	–„–	$\lambda_1 - 1 = 0,\ \lambda_2 = 0$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_2(c, -1)$	–„–	$\lambda_1 + 1 = 0,\ \lambda_2 = 0$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_2$	–„–	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$
$\tilde{B}_3(\mu_{10}, \mu_{20})$	$P_i(\lambda_1, \lambda_2) = 0,\ \tilde{P}_i = P_i(\mu_1, \mu_2) = 0$ $(i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1$ $\mu_1 - \mu_{10} = 0,$ $\mu_2 - \mu_{20} = 0$
$\tilde{C}$	$P_i(\lambda_1, \lambda_2) = 0,\ \tilde{P}_i = P_i(\mu_1, \mu_2) = 0,$ $\hat{P}_i = P_i(\nu_1, \nu_2) = 0\ (i = 1, 2)$	$\lambda_1^2 + \lambda_2^2 - 1 = 0,$ $\lambda_1 \neq \pm 1$ $\mu_1^2 + \mu_2^2 - 1 = 0,$ $\mu_1 \neq \pm 1$ $\nu_1^2 + \nu_2^2 - 1 = 0,$ $\nu_1 \neq \pm 1$

We shall denote by $W_I^{a,j}$, $W_I^{b,j}$, $W_I^{c,j}$ or $W_I^{a,j}(J)$, $W_I^{b,j}(J)$, $W_I^{c,j}(J)$ the strata of sets of the type A, B, C, respectively. For instance, $\tilde{A}_2 = \bigcup_j W_2^{a,j}$, $\tilde{B}_2(+1, c) = \bigcup_j W_2^{b,j}(+1, c)$ are stratifications.

Lemma 3 (Brunovský [8, Lemmas 1, 4, 5]).

(a) codim $W_2^{a,1} \geqq 4$,

(b) codim $W_1^{a,1} = 3$.

(c) *If* $\lambda_{20} \neq 0$, *then* codim $W_3^{a,1}(\lambda_{10}, \lambda_{20}) \geqq 4$.

Lemma 4.
(a) codim $W_2^{a,1} = 4$,
(b) codim $W_3^{a,1}(\lambda_{10}, \lambda_{20}) = 4$.

Proof. Assume $I = [0, 2] \times [0, 2]$ and consider the mapping $F : I \times R^2 \to M_n \times \times R^2$, $F(t, s, \lambda_1, \lambda_2) = (A(t, s), \lambda_1, \lambda_2)$, where

$$\text{(a) } A(t, s) = \operatorname{diag}\left\{\begin{pmatrix} t & 1+t & 1 & 0 \\ -1-t & t & 0 & 1 \\ 0 & 0 & s & 1+s \\ 0 & 0 & -1-s & s \end{pmatrix}, \quad 0, \ldots, 0\right\},$$

$$\text{(b) } A(t, s) = \operatorname{diag}\left\{\begin{pmatrix} t + \sin\frac{2\pi}{k} & s + \cos\frac{2\pi}{k} \\ -s - \cos\frac{2\pi}{k} & t + \sin\frac{2\pi}{k} \end{pmatrix}, \quad 0, \ldots, 0\right\}.$$

By Lemma 3, codim $W_2^{a,1} \geqq 4$, codim $W_3^{a,1}(\lambda_{10}, \lambda_{20}) \geqq 4$. If codim $W_2^{a,1} > 4$, then it would follow from [1, Corollary 17. 2] that there exists a small C^r perturbation $\tilde{A}$ of A such that no value of it has the eigenvalue $\lambda \in S_1$ of multiplicity 2. This, however, is obviously impossible and hence codim $W_2^{a,1} = 4$. By the same argument the asertion (b) is valid.

Lemma 5. codim $W_4^{a,1} \geqq 5$.

For the proof of this lemma as well as for the estimation of codimensions of the other above defined algebraic sets, we need the following lemma.

Lemma 6 ([8, Lemma 2]). *For any $A \in M_n$ the set of all matrices similar to A is an immersed submanifold of M_n of codimension $\geqq n$.*

Corollary. *Let $p : M_n \to R^n$ be defined as $p(A) = (a_1, a_2, \ldots, a_n)$, where $P(y) = y^n + a_1 y^{n-1} + \ldots + a_n$ is the characteristic polynomial of the matrix A, and let $p_k : M_n \times R^k \to R^{n+k}$ be defined as $p_k = p \times \mathrm{id}_{R^k}$. Then for any point $x \in R^{n+k}$, $p_k^{-1}(x)$ is a finite disjoint union of immersed submanifolds of $M_n \times R^k$ of codimension $\geqq n$.*

Denote by $V \subset R^{n+2}$ the set of points $(a_1, a_2, \ldots, a_n, \lambda_1, \lambda_2)$ such that $\lambda = \lambda_1 + i\lambda_2 \in S_1$ ($\lambda_1 \neq \pm 1$) is a root of the polynomial $P(y)$ of multiplicity $\geqq 3$. Obviously $P_2(\tilde{A}_4) = V$.

Lemma 7. *The mapping $p_2/\tilde{A}_4 : \tilde{A}_4 \to V$ is open (in the topologies on $\tilde{A}_4$ and V, induced by their imbeddings into M_n and R^{n+2}, respectively).*

The proof of this lemma coincides with the proof of [8, Lemma 3].

Proof of Lemma 5. V is a semialgebraic variety in R^{n+2} defined by the polynomials $P_i(\lambda_1, \lambda_2) = P_i'(\lambda_1, \lambda_2) = P_i''(\lambda_1, \lambda_2) = 0$, $\lambda_1^2 + \lambda_2^2 - 1 = 0$, $\lambda_1 \neq \pm 1$. Let

$V = \bigcup_i V_i$ be its stratification. The definition of the stratification implies that the set V_1 is open in V. First we shall prove that $\dim V_1 \leqq n - 3$. By [15] $\operatorname{codim} V_1 \geqq \operatorname{rank}_x V$ for any $x \in V_1$, where $\operatorname{rank}_x V$ is the dimension of the linear space spanned by the differential at x of the polynomials of the ideal associated with V.

$$P_1(\lambda_1, \lambda_2) = \ldots + a_{n-2}(\lambda_1^2 - \lambda_2^2) + a_{n-1}\lambda_1 + a_n\,,$$
$$P_2(\lambda_1, \lambda_2) = \ldots + a_{n-2}(2\lambda_1\lambda_2) + a_{n-1}\lambda_2\,,$$
$$P_1(\lambda_1, \lambda_2) = \ldots + 2a_{n-2}\lambda_1 + a_{n-1}\,,$$
$$P_2(\lambda_1, \lambda_2) = \ldots + 2a_{n-2}\lambda_2\,,$$
$$P_1(\lambda_1, \lambda_2) = \ldots + 6a_{n-3}\lambda_1 + 2a_{n-2}\,,$$
$$P_2''(\lambda_1, \lambda_2) = \ldots + 6a_{n-3}\lambda_2\,.$$

Then for $x \in V$ we have

$$\mathrm{d}P_1 = (\ldots, \lambda_1^2 - \lambda_2^2,\ \lambda_1, 1, 0, 0)\,,$$
$$\mathrm{d}P_2 = (\ldots, 2\lambda_1\lambda_2, \lambda_2, 0, 0, 0)\,,$$
$$\mathrm{d}P_1' = (\ldots, \lambda_1, 1, 0, 0, 0)\,,$$
$$\mathrm{d}P_2' = (\ldots, 2\lambda_2, 0, 0, 0, 0)\,,$$
$$\mathrm{d}P_1'' = (\ldots, 2, 0, 0, \partial P_1''/\partial\lambda_1,\ \partial P_1''/\partial\lambda_2)\,,$$
$$\mathrm{d}P_2'' = (\ldots, 0, 0, 0, \partial P_2''/\partial\lambda_1,\ \partial P_2''/\partial\lambda_2)\,,$$
$$\mathrm{d}(\lambda_1^2 + \lambda_2^2 - 1) = (\ldots, 0, 0, 0, 2\lambda_1, 2\lambda_2)\,.$$

Now we introduce the following notation. Given a system of vectors

$$v_1 = (v_{1,m}, v_{1,m-1}, \ldots, v_{11}), \ldots, v_k = (v_{k,m}, v_{k,m-1}, \ldots, v_{k,1})\,,$$

where $k \leqq m$, denote

$$D[v_1, \ldots, v_k] = \det \begin{bmatrix} v_{1,k};\ v_{1,k-1};\ \ldots;\ v_{11} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ v_{k,k};\ v_{k,k-1};\ \ldots;\ v_{k1} \end{bmatrix}.$$

By simple computations, it is possible to show that

$$D(x) = D[\mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}P_1'', \mathrm{d}P_2'', \mathrm{d}(\lambda_1^2 + \lambda_2^2 - 1)] =$$
$$= 4\left(\lambda_2 \frac{\partial P_2}{\partial\lambda_1} - \lambda_1 \frac{\partial P_2}{\partial\lambda_2}\right) = 4(-1)^n \operatorname{Re} \lambda P'''(\lambda)\,.$$

It suffices to prove that the set $\{x \in V \mid D(x) \neq 0\}$ is dense in V.

It is clear that the set of $x \in V$ for which $P'''(\lambda) \neq 0$ is dense in V. If $\lambda \in S_1$ is real, then $\lambda P'''(\lambda) = \operatorname{Re} \lambda P'''(\lambda) \neq 0$ if $P'''(\lambda) \neq 0$. Assume that $\lambda \in S_1$ is not real and $\operatorname{Re} \lambda P'''(\lambda) = 0$. For a nonzero real ε define the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y) = y^n + a_{1\varepsilon}y^{n-1} + \ldots + a_{n\varepsilon}$, where $\varphi(y) = (y - \lambda)^3\,(y - \bar{\lambda})^3$. Obviously $x_\varepsilon = (a_{1\varepsilon}, \ldots, a_{n\varepsilon}, \lambda_1, \lambda_2) \in V$ $(\lambda = \lambda_1 + \mathrm{i}\lambda_2)$; $\lambda P_\varepsilon'''(\lambda) = \lambda P'''(\lambda) + 6\varepsilon\lambda(\lambda - \bar{\lambda})^3$, i.e.

$\operatorname{Re} \lambda P_\varepsilon'''(\lambda) = 6\varepsilon \operatorname{Re} \lambda(\lambda - \bar{\lambda})^3 = 12\varepsilon\lambda_2^4 \neq 0$ since λ is not real. We have proved that for any neighbourhood $U(x)$ of the point x there is an $\varepsilon \neq 0$ such that $x_\varepsilon \in U(x) \subset V$ and $D(x_\varepsilon) \neq 0$, which we were to prove. This implies that $\operatorname{codim} V_1 \geqq 5$, i.e. $\dim V_1 = n + 2 - \operatorname{codim} V_1 \leqq n - 3$.

Now we shall prove that $\operatorname{codim} W_4^{a,1} \geqq 5$. The set $W_1 = W_4^{a,1}$ is open in $\tilde{A}_4$ and by Lemma 7 the set $p_2(W_1)$ is open in V. Therefore there exists integer i such that $p_2(W_1) \cap V_i \neq \emptyset$. Let i be the first index for which the set $p_2(W_1) \cap V_i$ is nonempty. Since $\bigcup_{j=1}^{i} V_j$ is open in V, then $M_i = p_2^{-1}(\bigcup_{j=1}^{i} V_j) = p_2^{-1}(V_i)$ is open in W_1 and thus $p_2(M_i)$ is open in V_i. By Sard's theorem (see [1, Theorem 15.1]) there exists an $\tilde{A} \in M_i$ such that the mapping p_2 is regular at $\tilde{A}$. Therefore $p_2^{-1}(p_2(\tilde{A})) =$ is a submanifold of W_1, where $\dim p_2^{-1}(p_2(A)) = \dim W_1 - \dim V_i \geqq \dim W_1 - n + 3$. By Corollary of Lemma 6 we have $\operatorname{codim} p_2^{-1}(p_2(\tilde{A})) \geqq n$, i.e. $\dim p_2^{-1}(p_2(\tilde{A})) \leqq n^2 - n$. Therefore $n^2 - n \geqq \dim W_1 - n + 3$, i.e. $\dim W_1 \leqq n^2 - 3$. This implies that $\operatorname{codim} W_1 \geqq n^2 + 2 - (n^2 - 3) = 5$, and the proof of Lemma 5 is complete.

Lemma 8.
(a) $\operatorname{codim} W_1^{b,1}(+1, c) \geqq 6$,
(b) $\operatorname{codim} W_1^{b,1}(-1, c) \geqq 6$,
(c) $\operatorname{codim} W_1^{b,1}(+1, -1) \geqq 6$,
(d) $\operatorname{codim} W_1^{b,1} \geqq 6$.

Proof. (a) The proof is similar to that of Lemma 5, but instead of the space $M_2 \times R^2$ we deal with the space $M_2 \times R^4$. Similarly as above, denote by $V \subset R^{n+4}$ the set of points $(a_1, a_2, \ldots, a_n, \lambda_1, \lambda_2, \mu_1, \mu_2)$ such that $\lambda = \lambda_1 + i\lambda_2 = 1$, $\mu = \mu_1 + i\mu_2 \in S_1$, $\mu_1 \neq \pm 1$ are zeros of the polynomial $P(y)$ of multiplicity $\geqq 1$. Obviously $p_4(W_1^{b,1}(+1, c)) = V$. The set V is a semialgebraic variety in R^{n+4} defined by the equations $P_i(\lambda_1, \lambda_2) = 0$, $\tilde{P}_i = P_i(\mu_1, \mu_2) = 0$ $(i = 1, 2)$, $\lambda_1 - 1 = 0$, $\lambda_2 = 0$, $\mu_1^2 + \mu_2^2 - 1 = 0$, $\mu_1 \neq \pm 1$. Let $V = \bigcup_i V_i$ be its stratification. It suffices to prove that $\operatorname{codim} V_1 \geqq 6$. Indeed, if this inequality is satisfied, then $\dim V_1 = n + 4 - \operatorname{codim} V_1 \leqq n - 2$. Similarly as above, it is possible to prove that there is a point $\tilde{A} \in W_1 = W_1^{b,1}$ such that the mapping p_4 is regular at this point and hence $Q = p_4^{-1}(p_4(\tilde{A}))$ is a submanifold of W_1, where $\dim Q \leqq n^2 - n$ (see the proof of Lemma 5). This implies that $\dim W_1 \geqq n^2 - 2$ and therefore $\operatorname{codim} W_1 \geqq n^2 + 4 - (n^2 - 2) = 6$.

Now, we prove that $\operatorname{codim} V_1 \geqq 6$. Let $x = (a_1, a_2, \ldots, a_n, 1, 0, \mu_1, \mu_2) \in V$. Denote $D(x) = D[dB_1, dP_2, d(\mu_1^2 + \mu_2^2 - 1), dP_1, d(\lambda_1 - 1), d\lambda_2]$. Then

$$D(x) = (\mu_1 - 1) \frac{\partial \tilde{P}_1}{\partial \mu_1} + \mu_2 \frac{\partial \tilde{P}_1}{\partial \mu_2}$$

and it suffices to prove that the set $H = \{x \in V \mid D(x) \neq 0\}$ is dense in V.

Assume that $x = (a_1, a_2, \ldots, a_n, 1, 0, \mu_1, \mu_2) \in V$ and $D(x) = 0$. For a nonzero real ε define the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y) = y^n + a_{1\varepsilon}y^{n-1} + \ldots + a_{n\varepsilon}$, where $\varphi(y) = (y-1)(y-\mu)(y-\bar{\mu}) = y^3 - (2\mu_1 + 1)\,y^2 + (2\mu_1 + 1)\,y - 1$. Obviously $x_\varepsilon = (a_{1\varepsilon}, \ldots, a_{n\varepsilon}, 1, 0, \mu_1, \mu_2) \in V$. Let $\tilde{P}_\varepsilon(\mu) = \tilde{P}_{\varepsilon 1}(\mu_1, \mu_2) + i\tilde{P}_{\varepsilon 2}(\mu_1, \mu_2)$, $a(\mu_1, \mu_2) = \partial\psi(\mu_1, \mu_2)/\partial y_1$, $b(\mu_1, \mu_2) = \partial\psi(\mu_1, \mu_2)/\partial y_2$, $y = y_1 + iy_2$, $\psi(y_1, y_2) = \operatorname{Re}\varphi(y) = (y_1 - 1)(y_1^2 - y_2^2 - 2\mu_1 y_1 + 1) + 2\mu_1 y_2^2$. $D(x_\varepsilon) = D(x) + \varepsilon(\mu_1 - 1)\,a(\mu_1, \mu_2) + \mu_2 b(\mu_1, \mu_2) = D(x) + 2\varepsilon\mu_2^2(\mu_1 + 1)$. Therefore if $D(x) = 0$, then in any neighbourhood of the origin there is a number ε such that $D(x_\varepsilon) \neq 0$ and hence the set H is dense in V.

The proof of (b) proceeds in the same way as the proof of (a). Proof of (c). Let $V \subset R^{n+4}$ be the set of all points $(a_1, a_2, \ldots, a_n, \lambda_1, \lambda_2, \mu_1, \mu_2)$ such that $\lambda = \lambda_1 + i\lambda_2 = 1$, $\mu = \mu_1 + i\mu_2 = -1$ are the roots of the polynomial $P(y)$ of multiplicity $\geqq 1$. It suffices to show that the set $H = \{x \in V \mid D(x) \neq 0\}$ is dense in V, where $D(x) = [\mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}(\lambda_1 - 1), \mathrm{d}\lambda_2, \mathrm{d}(\mu_1 + 1), \mathrm{d}\mu_2]$. However, $D(x) = 2$ for all $x \in V$ and thus the proof is complete. Proof of (d). Let $W(c, i) = \{(A, \lambda_1, \lambda_2, \mu_1, \mu_2) \in M_n \times R^4 \mid P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = 0,\ \tilde{P}_1(\mu_1, \mu_2) = \tilde{P}_2(\mu_1, \mu_2) = 0,\ \lambda_1^2 + \lambda_2^2 = 1,\ \mu_1 = 0,\ \mu_2 = 1\}$ and let V be the corresponding semialgebraic variety in R^{n+4} defined similarly as above. Let $W(c, i) = \bigcup_{j=1}^{r} W_j$ be a stratification. It is easy to show that $D[\mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}\tilde{P}_1, \mathrm{d}\tilde{P}_2, \mathrm{d}(\mu_1), \mathrm{d}(\mu_2 - 1)] = -\,|P'(\lambda)|^2 \neq 0$ and therefore $\operatorname{codim} W_1 \geqq 6$. (We shall even prove that $\operatorname{codim} W_1 \geqq 7$, see Lemma 10, (d).) Let $\tilde{B}_2 \setminus W(c, i) = \bigcup_{j=1}^{s} W_j$ be a stratification. It suffices to show that $\operatorname{codim} W_1 \geqq 6$. Let U be the semialgebraic variety in R^{n+4} defined by the polynomials $P_i(\lambda_1, \lambda_2)$, $\tilde{P}_i(\mu_1, \mu_2)$, $i = 1, 2$, $\lambda_1^2 + \lambda_2^2 = 1$, $\mu_1^2 + \mu_2^2 = 1$, $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\mu_1 \neq 0$, $\mu_2 \neq 0$, which corresponds to the set $\tilde{B}_2 \setminus W(c, i)$. For $x \in U$ we have

$$\begin{aligned} D(x) = D[\mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}(\lambda_1^2 + \lambda_2^2 - 1), \mathrm{d}\tilde{P}_1, \mathrm{d}\tilde{P}_2, \mathrm{d}(\mu_1^2 + \mu_2^2 - 1)] = \\ = 4[(\lambda_1 - \mu_1)(\mu_1 B_{11} + \mu_2 B_{12})(\lambda_1 A_{11} + \lambda_2 A_{12}) - \\ - \lambda_2(\mu_1 B_{11} + \mu_2 B_{12})(\lambda_1 A_{12} - \lambda_2 A_{11}) + \\ + \mu_2(\mu_1 B_{12} - \mu_2 B_{11})(\lambda_1 A_{11} + \lambda_2 A_{12}), \end{aligned}$$

where

$$A_{11} = \frac{\partial P_1}{\partial \lambda_1}, \quad A_{12} = \frac{\partial P_1}{\partial \lambda_2}, \quad B_{11} = \frac{\partial \tilde{P}_1}{\partial \mu_1}, \quad B_{12} = \frac{\partial \tilde{P}_1}{\partial \mu_2}.$$

It suffices to prove that the set $H = \{x \in U \mid D(x) \neq 0\}$ is dense in V. Let $D(x) = 0$ for some $x \in V$ and let ε be a nonzero real number. Define $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y)$, where $\varphi(y) = (y - \lambda)(y - \bar{\lambda})(y - \mu)(y - \bar{\mu})$ and let $\psi(y_1, y_2) = \operatorname{Re}\varphi(y) = (y_1^2 - y_2^2 - 2\lambda_1 y_1 + 1)(y_1^2 - y_2^2 - 2\mu_1 y_1 + 1) - 4(y_1 - \lambda_1)(y_1 - \mu_1)\,y_2^2$, $y = y_1 + iy_2$. Let $P(y) = P_1(y_1, y_2) + iP_2(y_1, y_2) = y^n + a_{\varepsilon 1}y^{n-1} + \ldots + a_{\varepsilon n}$, $A_{11}(\varepsilon) = \partial P_{\varepsilon 1}(\lambda_1, \lambda_2)/\partial y_1$, $A_{12}(\varepsilon) = \partial P_{\varepsilon 2}(\lambda_1, \lambda_2)/\partial y_2$, $B_{11}(\varepsilon) = \partial P_{\varepsilon 1}(\mu_1, \mu_2)/\partial y_1$, $B_{12}(\varepsilon) = \partial P_{\varepsilon 2}(\mu_1, \mu_2)/\partial y_2$, $x_\varepsilon = (a_{\varepsilon 1}, a_{\varepsilon 2}, \ldots, a_{\varepsilon n}, \lambda_1, \lambda_2, \mu_1, \mu_2)$. $D(x_\varepsilon) = D[\mathrm{d}P_{\varepsilon 1},$

$dP_{\varepsilon 2}, d(\lambda_1^2 + \lambda_2^2 - 1), d\tilde{P}_{\varepsilon 1}, d\tilde{P}_2, d(\mu_1^2 + \mu_2^2 - 1)] (P_i = P_i(\lambda_1, \lambda_2), \tilde{P}_i = P_i(\mu_1, \mu_2), i = 1, 2)$ has the same form as $D(x)$, but instead of A_{ij}, B_{ij} it involves $A_{ij}(\varepsilon), B_{ij}(\varepsilon)$, respectively. From the form of $D(x)$ we have that $D(x_\varepsilon) = D(x) + \alpha_1\varepsilon + \alpha_2\varepsilon^2$, where $\alpha_j = d^j D(x_\varepsilon)/d\varepsilon^j|_{\varepsilon=0}$. If $D(x) = 0$ and $\alpha_1 \neq 0$, then obviously $D(x_\varepsilon) \neq 0$ for sufficiently small $\varepsilon \neq 0$. If $D(x) = 0$ and $\alpha_1 = 0$, then $D(x_\varepsilon) = \varepsilon^2\alpha_2$, where α_2 has the same form as $D(x)$, but with $\tilde{A}_{ij}, \tilde{B}_{ij}$ instead of A_{ij}, B_{ij}, respectively; $\tilde{A}_{11} = -4\lambda_2^2(\lambda_1 - \mu_1)$, $A_{12} = -4\lambda_1\lambda_2(\lambda_1 - \mu_1)$, $B_{11} = 4\mu_2^2(\lambda_1 - \mu_1)$, $B_{12} = 4\mu_1\mu_2(\lambda_1 - \mu_1)$. Using the expression for $D(x)$ we obtain that $\alpha_2 = -512(\lambda_1 - \mu_1)\lambda_1\lambda_2\mu_1\mu_2 \neq 0$ and so the lemma is proved.

Lemma 9.

(a) codim $W_1^{b,1}(+1, c) = 6$,
(b) codim $W_1^{b,1}(-1, c) = 6$,
(c) codim $W_1^{b,1}(+1, -1) = 6$,
(d) codim $W_1^{b,1} = 6$.

Proof. Assume $I = [0, 2] \times [0, 2]$ and consider the mapping $F : I \times R^4 \to M_n \times R^4$, $F(t, s, \lambda_1, \ldots, \lambda_4) = (A(t, s), \lambda_1, \ldots, \lambda_4)$, where

$$A(t, s) = \operatorname{diag}\left\{\begin{pmatrix} t & 1 + t \\ -1 - t & t \end{pmatrix}, \quad 1 + s, 0, \ldots, 0\right\}.$$

By Lemma 8, codim $W_1^{b,1}(+1, c) \geqq 6$. If codim $W_1^{b,1}(+1, c) > 6$, then it would follow from [1, Corollary 17. 2] that there exists a small C^r perturbation $\tilde{A}$ of A such that no value of ithas the eigenvalue $\lambda = +1$ and a comlex eigenvalue $\mu \in S_1$. This, however, is obviously impossible and hence codim $W_1^{b,1} = 6$. By the same argument it is possible to prove (b)–(d), where the mapping $F : I \times R^4 \to M_n \times R^4$ is defined as follows: $F(t, s, \lambda_1, \ldots, \lambda_4) = (A(t, s), \lambda_1, \ldots, \lambda_4)$,

(b) $A(t, s) = \operatorname{diag}\left\{\begin{pmatrix} t & 1 + t \\ -1 - t & t \end{pmatrix}, \quad -1 + s, 0, \ldots, 0\right\}$,

(c) $A(t, s) = \operatorname{diag}\{1 + t, -1 + s, 0, \ldots, 0\}$,

(d) $A(t, s) = \operatorname{diag}\left\{\begin{pmatrix} t & 1 + t \\ -1 - t & t \end{pmatrix}, \begin{pmatrix} s & 1, 1 + s \\ -1, 1 - s & s \end{pmatrix}, \quad 0, \ldots, 0\right\}$.

Lemma 10.

(a) codim $W_2^{b,1}(+1, c) \geqq 7$, codim $W_2^{b,1}(-1, c) \geqq 7$,
(b) codim $W_2^{b,1}(\mu_{10}, \mu_{20}, \mathrm{i}) \geqq 7$,
(c) codim $W_2^{b,1}(\mu_{10}, \mu_{20}, +1) \geqq 7$, codim $W_2^{b,1}(\mu_{10}, \mu_{20}, -1) \geqq 7$,
(d) codim $W_2^{b,1}(c, \mathrm{i}) \geqq 7$,
(e) codim $W_2^{b,1}(c, +1) \geqq 7$, codim $W_2^{b,1}(c, -1) \geqq 7$,
(f) codim $W_2^{b,1} \geqq 7$,
(g) codim $W_3^{b,1}(\mu_{10}, \mu_{20}) \geqq 7$,
(h) codim $W^{c,1} \geqq 9$.

Proof. (a)

$$D(x) = D[\mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}P_1, \mathrm{d}P_2, \mathrm{d}(\mu_1 - 1), \mathrm{d}\mu_2, \mathrm{d}(\lambda_1^2 + \lambda_2^2 - 1)] =$$

$$= 4\left[\lambda_2 \frac{\partial P_2'}{\partial \lambda_1} + \lambda_1 \frac{\partial P_1'}{\partial \lambda_1}\right] = 4 \operatorname{Re}(\lambda^{-1}P''(\lambda)).$$

It suffices to prove that the set $H = \{x \in V \mid D(x) \neq 0\}$ is dense in V, where $V \subset R^{n+4}$ is defined similarly as in the proof of Lemma 8. Suppose that for some $x \in V$, $\operatorname{Re}(\lambda^{-1} P''(\lambda)) = 0$. For a nonzero real number ε defined the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y)$, where $\varphi(y) = (y-1)(y-\lambda)^2(y-\bar{\lambda})^2$. Since $\operatorname{Re}(\lambda^{-1} P_\varepsilon''(\lambda)) = \varepsilon \operatorname{Re}(\bar{\lambda}(\lambda - 1)(\lambda - \bar{\lambda})^2) = -4(1-\lambda_1)\lambda_2^2 \neq 0$ the density of the set H is proved. The other part of the proof of (a) proceeds as in the proof of Lemma 8.

The proofs of the inequalities (b) and (c) are easy.

(d) Let us stratify the set $\tilde{B}_2(c, i)$ in the following way: Denote $\tilde{B}_2(k) = \bigcup \tilde{B}_2(\mu_{10}, \mu_{20}, \mathrm{i})$, where the union is taken over all μ_{10}, μ_{20} such that $(\mu_{10} + \mathrm{i}\mu_{20})^l \neq 1$, $l = 3, 4, \ldots, k$, $\mu_{20} = 0$. Rewrite the set $\tilde{B}_2(c, \mathrm{i})$ in the form $\tilde{B}_2(c, \mathrm{i}) = \tilde{B}_2(k) \cup (\tilde{B}_2(c, \mathrm{i}) \setminus \tilde{B}_2(k))$, where $k > 6$. By part (b) the set $\tilde{B}_2(k)$ has a stratification $\bigcup W_2^j(k)$, where $\operatorname{codim} W_2^1(k) \geqq 7$. Therefore it suffices to prove that if $\hat{B}_2(k) = \tilde{B}_2(c, \mathrm{i}) \setminus \tilde{B}_2(k) = \bigcup W^j(k)$ is a stratification, then $\operatorname{codim} W^1(k) \geqq 7$.

The set $\hat{B}_2(k)$ is defined by the polynomials $P_1, P_2, P_1', P_2', \lambda_1, \lambda_2 - 1, \tilde{P}_1, \tilde{P}_2, \mu_1^2 + \mu_2^2 - 1$ and by some inequalities which express the fact that $\hat{B}_2(k) \cap \tilde{B}_2(k) = \emptyset$.

$$D(x) = D[\mathrm{d}P_1, \mathrm{d}P_1', \mathrm{d}P_2', \mathrm{d}\lambda_1, \mathrm{d}(\lambda_2 - 1), \mathrm{d}\tilde{P}_2, \mathrm{d}(\mu_1^2 + \mu_2^2 - 1)] =$$

$$= 4\lambda_2\left[\mu_1 \frac{\partial \tilde{P}_2}{\partial \mu_2} - \mu_2 \frac{\partial \tilde{P}_2}{\partial \mu_1}\right] = 4 \operatorname{Re} \mu\tilde{P}'(\mu).$$

We shall prove that the set $H = \{x \in V \mid \operatorname{Re} \mu\tilde{P}'(\mu) \neq 0\}$ is dense in V, where $V \subset R^{n+4}$ is the set of such points $(a_1, a_2, \ldots, a_n, \lambda_1, \lambda_2, \mu_1, \mu_2) \in R^{n+4}$ for which $\lambda = \lambda_1 + \mathrm{i}\lambda_2 = \mathrm{i}$ is the zero of the polynomial $P(y) = y^n + a_1 y^{n+1} + \ldots + a_n$ of multiplicity $\geqq 2$ and $\mu = \mu_1 + \mathrm{i}\mu_2 \in S_1$ is the zero of the polynomial $P(y)$ of multiplicity $\geqq 1$, such that $\mu^l \neq 1$ for $l = 3, 4, \ldots, k$. Assume that for some $x \in V$, $\operatorname{Re} \mu\,\tilde{P}'(\mu) = 0$. For a nonzero real ε defined the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y)$, where $\varphi(y) = (y^2 + 1)^2 (y - \mu)(y - \bar{\mu})$. It is clear that the corresponding point $(a_{1\varepsilon}, \ldots, a_{n\varepsilon}, 0, 1, \mu_1, \mu_2) \in V$, $\operatorname{Re} \mu\, P_\varepsilon'(\mu) = \varepsilon \operatorname{Re} \mu(2i\mu_2)(1 + \mu^2)^2 = 2\varepsilon\mu_2 \operatorname{Im} \mu(1 + \mu^2)^2$. We have to prove that $\operatorname{Im} \mu(1 + \mu^2)^2 \neq 0$.

Since $\mu \in S_1$, we have $\mu = \mathrm{e}^{\mathrm{i}\alpha}$ for some α. Then $\mu(1 + \mu^2)^2 = \mathrm{e}^{\mathrm{i}\alpha}(1 + \mathrm{e}^{2\mathrm{i}\alpha})^2 = (\mathrm{e}^{\mathrm{i}\alpha/2} + \mathrm{e}^{5\mathrm{i}\alpha/2})^2$. For a complex number $a + ib$, $\operatorname{Im}(a + ib)^2 = 2ab$ and therefore it suffices to prove that $\cos\frac{1}{2}\alpha + \cos\frac{5}{2}\alpha \neq 0$ and $\sin\frac{1}{2}\alpha + \sin\frac{5}{2}\alpha \neq 0$ for $x = (a_1, \ldots, a_n, 0, 1, \mu_1, \mu_2) \in V$.

The following identities are valid:

$$\cos\frac{\alpha}{2} + \cos\frac{5}{2}\alpha = 2\cos\frac{\alpha}{2}\left(1 - 4\sin^2\frac{\alpha}{2}\right);$$

$$\sin\frac{\alpha}{2} + \sin\frac{5}{2}\alpha = -2\sin\frac{\alpha}{2}\left(1 - 4\cos^2\frac{\alpha}{2}\right).$$

(1) Assuming $\cos\frac{1}{2}\alpha = 0$ we have $\alpha = \pi + 2s\pi$, $s \in Z$, i.e. $\mu = e^{(\pi+2s\pi)i} = -1$, but this is impossible.
(2) If $\cos\alpha = 0$, then $\mu = e^{(\pi/2+s\pi)i}$, i.e. $\mu^4 = 1$. This is impossible because we have assumed $k > 6$ and therefore $\mu^l \neq 1$ for $l = 3, 4, \ldots, 6$.
(3) If $\sin\frac{1}{2}\alpha = \pm\frac{1}{2}$, then $= e^{(\pm\pi/3+4s\pi)i}$ and $\mu^6 = 1$, which is again impossible.

Thus, we have proved that $\cos\frac{1}{2}\alpha + \cos\frac{5}{2}\alpha \neq 0$ for $x \in V$. The proof of the second inequality is similar. This implies density of the set H and so codim $W_2^{b,1} \geqq 7$.

The proof of (e) follows the same lines as the proof of (d).

(f) Define the set $C(k) = \tilde{B}_2(+1, c) \cup \tilde{B}_2(-1, c) \cup \tilde{B}_2(c, i) \cup \tilde{B}_2(c, +1) \cup \cup \tilde{B}_2(c, -1) \cup (\bigcup_{j=1}^{3} C_j(k))$, where $C_1(k) = \bigcup\tilde{B}_2(\mu_{10}, \mu_{20}, i)$, $C_2(k) = \bigcup\tilde{B}_2(\mu_{10}, \mu_{20}, +1)$, $C_3(k) = \bigcup\tilde{B}_2(\mu_{10}, \mu_{20}, -1)$ and the unions are taken over all μ_{10}, μ_{20} such that $(\mu_{10} + i\mu_{20})^l = 1$ for $l = 3, 4, \ldots, k$, $\mu_{20} \neq 0$.

Now rewrite the set $\tilde{B}_2$ as $\tilde{B}_2 = C(k) \cup \tilde{C}(k)$, where $\tilde{C}(k) = \tilde{B}_2 \setminus C(k)$. By the previous parts of Lemma 10 all strata of the set $C(k)$ have codimensions $\geqq 7$ and therefore it suffices to prove that the set $\tilde{C}(k)$ also has such a stratification.

$\tilde{C}(k)$ is the set of $(A, \lambda_1, \lambda_2, \mu_1, \mu_2) \in M_n \times R^4$ for which $P_i(\lambda_1, \lambda_2) = P_i'(\lambda_1, \lambda_2) = 0$, $\tilde{P}_i = P_i(\mu_1, \mu_2) = 0$ $(i = 1, 2)$, $\lambda_1^2 + \lambda_2^2 - 1 = 0$, $\mu_1^2 + \mu_2^2 - 1 = 0$, $\lambda_1 \neq \mu_1$, $\lambda_2 \neq 0$, $\mu_1 \neq \pm 1$, $\lambda_2 \neq \pm 1$, $\mu^l = 1$ for $l = 1, 2, \ldots, k$.

Suppose that $k > 6$.

$$D(x) = D[dP_2, dP_1', dP_2', d(\lambda_1^2 + \lambda_2^2 - 1), d\tilde{P}_1, d\tilde{P}_2, d(\mu_1^2 + \mu_2^2 - 1)] =$$
$$= 4\lambda_2^2\left(\mu_2\frac{\partial\tilde{P}_2}{\partial\mu_1} - \mu_1\frac{\partial\tilde{P}_2}{\partial\mu_2}\right)\left(\lambda_2\frac{\partial P_1'}{\partial\lambda_1} - \lambda_1\frac{\partial P_1'}{\partial\lambda_2}\right) =$$
$$= 4\lambda_2^2(\operatorname{Re}\mu\,\tilde{P}(\mu))(\operatorname{Im}\lambda\,P''(\lambda)).$$

We shall show that the set $H = \{x \in V \mid \operatorname{Re}\mu\,\tilde{P}'(\mu) \neq 0,\ \operatorname{Im}\lambda\,P''(\lambda) \neq 0\}$ is dense in V, where the set V corresponds to the set $\tilde{C}(k)$ such that $p_4(\tilde{C}(k)) = V$. (I) We shall show that $\operatorname{Re}\mu\,\tilde{P}'(\mu) \neq 0$ for a dense subset of V. Assume that $\operatorname{Re}\mu\,\tilde{P}'(\mu) = 0$ for some $x \in V$. Similarly as above, define the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\,\varphi(y) = y^n + a_{1\varepsilon}y^{n-1} + \ldots + a_{n\varepsilon}$, where ε is a nonzero real number and $\varphi(y) = (y - \mu)\,.\,(y - \bar{\mu})(y - \lambda)^2(y - \bar{\lambda})^2$. Obviously $x_\varepsilon = (a_{1\varepsilon}, \ldots, \mu_2) \in V$. $\operatorname{Re}\mu\,P_\varepsilon'(\mu) = 2\varepsilon\mu_2\,.\,\operatorname{Im}\mu(\mu - \lambda)^2(\mu - \bar{\lambda})^2$. Since $\mu, \lambda \in S_1$, there are α, β such that $\mu = e^{i\alpha}$, $\lambda = e^{i\beta}$. Then we have

$$\mu(\mu - \lambda)^2(\mu - \bar{\lambda})^2 = e^{i\alpha}(e^{i\alpha} - e^{i\beta})(e^{i\alpha} - e^{i\beta}) =$$
$$= [e^{i\alpha/2}(^{i\alpha} - e^{i\beta})(e^{i\alpha} - e^{i\beta})]^2 =$$
$$= [e^{i\alpha/2}(e^{2\alpha i} - e^{(\alpha+\beta)i} - e^{(\alpha-\beta)i} + 1)]^2 = [Q_1(\alpha, \beta) + iQ_2(\alpha, \beta)]^2,$$

where

$$Q_1(\alpha, \beta) = \cos \tfrac{5}{2}\alpha - \cos\left(\tfrac{3}{2}\alpha + \beta\right) - \cos\left(\tfrac{3}{2}\alpha - \beta\right) + \cos \tfrac{1}{2}\alpha\,,$$

$$Q_2(\alpha, \beta) = \sin \tfrac{5}{2}\alpha - \sin\left(\tfrac{3}{2}\alpha + \beta\right) - \sin\left(\tfrac{3}{2}\alpha - \beta\right) + \sin \tfrac{1}{2}\alpha\,.$$

Since $\operatorname{Im} \mu(\mu - \lambda)^2 (\mu - \bar{\lambda})^2 = 2Q_1(\alpha, \beta)\, Q_2(\alpha, \beta)$, it suffices to prove that for $x \in V$, $Q_1(\alpha, \beta) \neq 0$ and $Q_2(\alpha, \beta) \neq 0$. The following equalities are valid:

$$Q_1(\alpha, \beta) = 2 \cos \tfrac{3}{2}\alpha(\cos \alpha - \cos \beta)\,, \quad Q_2(\alpha, \beta) = 2 \sin \tfrac{3}{2}\alpha(\cos \alpha - \cos \beta)\,.$$

(1) If $\cos \tfrac{3}{2}\alpha = 0$, then $\mu = e^{i\alpha} = e^{(\pi/3 + 2s\pi/3)i}$ and so $\mu^6 = 1$, which is impossible because $k > 6$.

(2) If $\sin \tfrac{3}{2}\alpha = 0$, then $\mu = e^{2/3s\pi i}$ and so $\mu^3 = 1$, which is impossible as well.

(3) Due to the definition of the set $\tilde{C}(k)$ the equality $\mu_1 = \cos \alpha = \cos \beta = \lambda_1$ does not hold for $x \in V$.

Thus we have proved the density of those points $v \in V$ for which $\operatorname{Re} \mu\, \tilde{P}'(\mu) \neq 0$.

(II) We shall show that $\operatorname{Im} \lambda\, P''(\lambda) \neq 0$ for a dense subset of the set V. Assume $\operatorname{Im} \lambda\, P''(\lambda) = 0$ for some $x \in V$. Let $P_\varepsilon(y)$ be the polynomial as above. Since $\lambda\, P''_\varepsilon(\lambda) = \lambda\, P''(\lambda) + 2\varepsilon(\lambda - \mu)(\lambda - \bar{\mu})(\lambda - \bar{\lambda})^2 \lambda$ we have $\operatorname{Im} \lambda\, P''_\varepsilon(\lambda) = -32\lambda_2^2\lambda_1(\lambda_1 - \mu_1)\, \varepsilon \neq 0$ for all $x \in V$ and the proof is complete.

(g) $D(x) = D[dP_1, dP_2, d(\lambda_1^2 + \lambda_2^2 - 1), d\tilde{P}_1, d\tilde{P}_2, d(\mu_1 - \mu_{20}), d(\mu_2 - \mu_{20}) = 4\mu_2(\mu_1 - \lambda_1)\, D_1(x)$, where

$$D_1(x) = (\lambda_1^2 - \lambda_2^2) \frac{\partial P_1(\lambda_1, \lambda_2)}{\partial y_1} - 2\lambda_1\lambda_2 \frac{\partial P_1(\lambda_1, \lambda_2)}{\partial y_2}\,,$$

$$P(y) = P_1(y_1, y_2) + iP_2(y_1, y_2) = y^n + a_1 y^{n-1} + \ldots + a_n\,,$$

$$y = y_1 + iy_2\,, \quad \tilde{P}_i = P_i(\mu_1, \mu)\,, \quad i = 1, 2\,.$$

It suffices to prove that the set $H = \{x \in V \mid D_1(x) \neq 0\}$ is dense in V, where $V = \{x\} = (a_1, \ldots, a_n, \lambda_1, \lambda_2, \mu_1, \mu_2) \in R^{n+4} \mid \lambda = \lambda_1 + i\lambda_2 \in S_1$, $\mu = \mu_1 + i\mu_2 \in S_1$ are roots of the polynomial $P(y)$ and $\mu^k = 1\}$. Let $x = (a_1, \ldots, a_n, \lambda_1, \lambda_2, \mu_{10}, \mu_{20}) \in V$ and $D_1(x) = 0$. For a nonzero real number ε define the polynomial $P_\varepsilon(y) = P(y) + \varepsilon\, \varphi(y) = y^n + a_{1\varepsilon} y^{n-1} + \ldots + a_{n\varepsilon} = P_{1\varepsilon}(y_1, y_2) + iP_{2\varepsilon}(y_1, y_2)$, where $\varphi(y) = (y - \lambda)(y - \bar{\lambda})(y - \mu)(y - \bar{\mu})$. Then obviously $x_\varepsilon = (a_1, \ldots, a_n, \lambda_1, \lambda_2, \mu_{10}, \mu_{20}) \in V$ and

$$D_1(x_\varepsilon) = (\lambda_1^2 - \lambda_2^2) \frac{\partial P_{1\varepsilon}(\lambda_1, \lambda_2)}{\partial y_1} - 2\lambda_1\lambda_2 \frac{\partial P_{1\varepsilon}(\lambda_1, \lambda_2)}{\partial y_2} =$$

$$= D_1(x) + \varepsilon\left[(\lambda_1^2 - \lambda_2^2) \frac{\partial \psi(\lambda_1, \lambda_2)}{\partial y_1} - 2\lambda_1\lambda_2 \frac{\partial \psi(\lambda_1, \lambda_2)}{\partial y_2}\right],$$

where $\psi(y_1, y_2) = \operatorname{Re} \varphi(y) = (y_1^2 - y_2^2 - 2\lambda_1 y_1 + 1)(y_1^2 - y_2^2 - 2\mu_{10} y_1 + 1) - 4(y_1 - \lambda_1)(y_1 - \mu_{10})\, y_2^2$. Therefore if $D_1(x) = 0$, then $D_1(x_\varepsilon) = 4\varepsilon\lambda_2^2(\lambda_1 - \mu_{10}) \neq 0$ and hence the set H is dense in V.

(h) Let $V \subset R^{n+6}$ be the set of all points $x = (a_1, a_2, \dots, a_n, \lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2)$ such that $\lambda = \lambda_1 + i\lambda_2 \in S_1$, $\mu = \mu_1 + i\mu_2 \in S_1$, $\nu = \nu_1 + i\nu_2 \in S_1$ are roots of the polynomial $P(y) = P_1(y_1, y_2) + iP_2(y_1, y_2) = y^n + a_1 y^{n-1} + \dots + a_n$ of multiplicity $\geqq 1$.

$$D(x) = D[dP_1, dP_2, d(\lambda_1^2 + \lambda_2^2 - 1), d\tilde{P}_1, d\tilde{P}_2, d(\mu_1^2 + \mu_2^2 - 1),$$
$$d\hat{P}_1, d\hat{P}_2, d(\nu_1^2 + \nu_2^2 - 1)] = A - B + C,$$

where

$$\tilde{P}_i = P_i(\mu_1, \mu_2), \quad \hat{P}_i = P_i(\nu_1, \nu_2), \quad i = 1, 2,$$

$$A = \left[\left(\frac{\partial \tilde{P}_1}{\partial \mu_1}\right)\left(\nu_2 \frac{\partial \hat{P}_1}{\partial \nu_2} - \nu_1 \frac{\partial \hat{P}_1}{\partial \nu_1}\right) + \left(\frac{\partial \hat{P}_1}{\partial \nu_1}\right)\left(\mu_2 \frac{\partial \tilde{P}_1}{\mu_2} - \mu_1 \frac{\partial \tilde{P}_1}{\partial \mu_1}\right)\right]\left(\lambda_2 \frac{\partial P_1}{\partial \lambda_2} - \lambda_1 \frac{\partial P_1}{\partial \lambda_1}\right),$$

$$B = \left[\left(\frac{\partial \hat{P}_1}{\partial \nu_1}\right)\left(\lambda_2 \frac{\partial P_1}{\partial \lambda_2} - \lambda_1 \frac{\partial P_1}{\partial \lambda_1}\right) + \left(\frac{\partial P_1}{\partial \lambda_1}\right)\left(\nu_2 \frac{\partial \hat{P}_1}{\partial \nu_2} - \nu_1 \frac{\partial \hat{P}_1}{\partial \nu_1}\right)\right]\left(\mu_2 \frac{\partial \tilde{P}_1}{\partial \mu_2} - \mu_1 \frac{\partial \tilde{P}_1}{\partial \mu_1}\right),$$

$$C = \left[\left(\frac{\partial \tilde{P}_1}{\partial \mu_1}\right)\left(\lambda_2 \frac{\partial P_1}{\partial \lambda_2} - \lambda_1 \frac{\partial P_1}{\partial \lambda_1}\right) + \left(\frac{\partial P_1}{\partial \lambda_1}\right)\left(\mu_2 \frac{\partial \tilde{P}_1}{\partial \mu_2} - \mu_1 \frac{\partial \tilde{P}_1}{\partial \mu_1}\right)\right]\left(\nu_2 \frac{\partial \hat{P}_1}{\partial \nu_2} - \nu_1 \frac{\partial \hat{P}_1}{\partial \nu_1}\right).$$

It suffices to prove that the set $\{x \in V \mid D(x) \neq 0\}$ si dense in V. For a nonzero real number ε define the polynomial $P_\varepsilon(y) = y^n + a_{1\varepsilon} y^{n-1} + \dots + a_{n\varepsilon}$, $\varphi(y) = (y - \lambda) \cdot (y - \bar{\lambda})(y - \mu)(y - \bar{\mu})(y - \nu)(y - \bar{\nu})$. Obviously $x_\varepsilon = (a_{1\varepsilon}, \dots, a_{n\varepsilon}, \lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2) \in V$ and $D(x_\varepsilon) = A(\varepsilon) - B(\varepsilon) + C(\varepsilon)$, where $A(\varepsilon), B(\varepsilon), C(\varepsilon)$ have the same form as A, B, C, respectively, but instead of the partial derivatives of $P_1, \tilde{P}_1, \hat{P}_1$ they involve the corresponding partial derivatives of $P_{1\varepsilon}, \tilde{P}_{1\varepsilon}, \hat{P}_{1\varepsilon}$, respectively. $D(x_\varepsilon) = D(x) + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3$, where $\alpha_j = d^j D(x_\varepsilon)/d\varepsilon^j|_{\varepsilon=0}$, $j = 1, 2, 3$. If $D(x) = 0$, $\alpha_1 \neq 0$, or $D(x) = \alpha_1 = 0$, $\alpha_2 \neq 0$, then obviously $D(x_\varepsilon) \neq 0$ for $\varepsilon \neq 0$ sufficiently small. If $D(x) = \alpha_1 = \alpha_2 = 0$, then $D(x_\varepsilon) = \varepsilon^3\alpha_3$, where $\alpha_3 = \alpha - \beta + \gamma$, where α, β, γ have the same form as A, B, C, respectively, but instead of the partial derivatives of $P_1, \tilde{P}_1, \hat{P}_1$ they involve the corresponding partial derivatives of $\psi = \psi(\lambda_1, \lambda_2)$, $\tilde{\psi} = \psi(\mu_1, \mu_2)$, $\hat{\psi} = \psi(\nu_1, \nu_2)$, respectively, where $\psi(y_1, y_2) = \operatorname{Re} \varphi(y) = (y_1^2 - y_2^2 - 2\lambda_1 y_1 + 1)(y_1^2 - y_2^2 - 2\mu_1 y_1 + 1)(y_1^2 - y_2^2 - 2\nu_1 y_1 + 1) - 4y_2^2(y_1^2 - y_2^2 - 2\nu_1 y_1 + 1)(y_1 - \lambda_1)(y_1 - \mu_1) - 4y_2^2(y_1^2 - y_2^2 - 2\lambda_1 y_1 + 1)(y_1 - \mu_1)(y_1 - \nu_1) - 4y_2^2(y_1^2 - y_2^2 - 2\mu_1 y_1 + 1)(y_1 - \lambda_1)(y_1 - \nu_1)$. Therefore $D(x_\varepsilon) = 64\varepsilon^3 \mu_1 \lambda_2^2 \mu_2^2 \nu_2^2 (\lambda_1 - \mu_1)(\lambda_1 - \nu_1)(\mu_1 - \nu_1)$. Let $U = \{x \in V \mid \mu_1 = 0\}$. Since $\lambda_2 \neq 0$, $\mu_2 \neq 0$, $\nu_2 \neq 0$, $\lambda_1 \neq \mu_1$, $\lambda_1 \neq \nu_1$, $\mu_1 \neq \nu_1$, we have shown that the set $H = \{x \in V \setminus U \mid D(x) \neq 0\}$ is dense in $V \setminus U$ and hence if $V \setminus U = \bigcup_{i=1}^{r} \tilde{V}_i$ is a stratification, then $\operatorname{codim} \tilde{V}_1 \geqq 9$. Therefore it suffices to show that if $\bigcup_{j=1}^{s} U_j$ is a stratification of the set U, then $\operatorname{codim} U_1 \geqq 9$. Let $x = (a_1, \dots, a_n, \lambda_1, \lambda_2, 0, 1, \nu_1, \nu_2) \in U$, $\tilde{D}(x) = D[dP_1, dP_2, d(\lambda_1^2 + \lambda_2^2 - 1), d\tilde{P}_1, d\tilde{P}_2, d(\mu_1), d(\mu_2 - 1), d\hat{P}_1, d\hat{P}_2, d(\nu_1^2 + \nu_2^2 - 1)]$ and let $\varphi(y), \psi(y), x_\varepsilon$ be as above. Then $\tilde{D}(x)$ has the same form as

$D(x)$, but with 1, 1 instead of μ_1, μ_2, respectively. Therefore $\tilde{D}(x_\varepsilon) = \tilde{D}(x) + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \alpha_3\varepsilon^3$, where $\alpha_3 = 64\lambda_2^2 v_2^2(\lambda_1 - 1)(\lambda_1 - v_1)(v_1 - 1) \neq 0$ and hence the proof is complete.

3. TWO-PARAMETRIC SYSTEMS OF MATRICES

Let $I \subset R^2$ be a compact interval and $F^r = F^r(I)$ the set of C^r mappings of I into M_n, endowed with the usual C^r topology. Let $T_1 = A_1$, T_2 be the union of the sets A_2, $A_2(-1)$, A_3, B_1, $A_2(+1)$, $B_1(+1, -1)$, $B_1(+1, c)$, $B_1(-1, c)$ and $T_3 = M_n \setminus (T_1 \cup T_2)$.

Theorem 1. *There exists an open dense set F_1 in $F^r(I)$ such that if $A \in F^r(I)$, then*
(1) $A(I) \cap T_3 = \emptyset$.
(2) *The set $X_1(A) = \{\mu \in I \mid A(\mu) \in T_1\}$ has a codimension $\geqq 1$. The set $X_{11}(A) \subset X_1(A)$ of all $\mu \in I$, for which the matrix $A(\mu)$ has an eigenvalue on the unit circle S_1 of multiplicity 1, is a one-dimensional submanifold of I.*
(3) *The set $X_2(A) = \{\mu \in I \mid A(\mu) \in T_2\}$ consists of isolated points.*

Remark. Theorem 1 says that generically the following holds:

(I) There is a one-dimensional submanifold $X_{11}(A)$ of I such that if $\mu \in X_{11}(A)$, then the matrix $A(\mu)$ has one of the following simple eigenvalues on S_1:

(1) $\lambda = +1$;
(2) $\lambda = -1$;
(3) $\lambda, \bar{\lambda} \in S_1$, $\lambda^k \neq \pm 1$, $k = 1, 2, 3, \ldots$.

There are no other eigenvalues on S_1.

(II) There is a set $X_2(A)$ consisting of isolated points and such that if $\mu \in X_2(A)$, then the matrix $A(\mu)$ has one of the following eigenvalues on S_1:

(1) $\lambda = +1$ of multiplicity 2;
(2) $\lambda = -1$ of multiplicity 2;
(3) $\lambda \in S_1$ of multiplicity 2, $\lambda \neq \pm 1$;
(4) $\lambda = +1$, $v = -1$ both of multiplicity 1;
(5) $\lambda \in S_1$ of multiplicity 1, $\lambda \neq \pm 1$, $\lambda^k = 1$ for some $k \in Z$;
(6) $\lambda = +1$, $v \in S_1$ both of multiplicity 1, $v^k \neq \pm 1$, $k = 1, 2, 3, \ldots$;
(7) $\lambda = -1$, $v \in S_1$ both of multiplicity 1, $v^k \neq \pm 1$, $k = 1, 2, 3, \ldots$;
(8) $\lambda \in S_1$, $v \in S_1$ both of multiplicity 1, $\lambda, v \neq \pm 1$, $\operatorname{Re} \lambda \neq \operatorname{Re} v$.

There are no other eigenvalues on S_1 (except for complex conjugate ones).

(III) For $\mu \in I \setminus (X_{11}(A) \cup X_2(A))$ the matrix $A(\mu)$ has no eigenvalue on S_1.

Proof of Theorem 1. Let $J \subset R^2$ be an open interval, $I \subset J$. Denote by $\tilde{F}^r$ the space of all mappings $\tilde{F} : J \times R^k \to M_n \times R^k$, $\tilde{F} = F \times \mathrm{id}_{R^k}$, $F \in R^r(J)$, endowed with the C^r Whitney topology. The mapping $\varrho : F^r \to \tilde{F}^r$, $\varrho(F) = \tilde{F}$ is a C^r representation and the mapping $ev_\varrho : F^r \times J \times R^k \to M_n \times R^k$ transversally intersects every submanifold of $M_n \times R^k$. (For the definition of the evaluation mapping ev and of the C^r representation see [1].)

$T_1 = T_{1a}$, $T_2 = T_{2a} \cup T_{2b}$, $T_3 = T_{3a} \cup T_{3c}$, where T_{ia}, T_{ib}, T_{ic} $(i = 1, 2, 3)$ are sets of type A, B and C, respectively. Let $\tilde{T}_{ia} \subset R^{n+2}$, $\tilde{T}_{ib} \subset R^{n+4}$, $\tilde{T}_{ic} \subset R^{n+6}$ be the corresponding semialgebraic varieties and let

$$T_{ia} = \bigcup_{j=1}^{r_i(a)} G_i^{a,j}, \quad T_{ib} = \bigcup_{j=1}^{r_i(b)} G_i^{b,j}, \quad T_{ic} = \bigcup_{j=1}^{r_i(c)} G_i^{c,j}$$

$(i = 1, 2, 3)$ be stratifications. By Lemma 3, (b) codim $G_1^{a,1} = 3$, by Lemma 4 codim $G_2^{a,1} = 4$, by Lemma 5 codim $G_3^{a,1} \geqq 5$. By Lemma 9 codim $G_2^{b,1} = 6$, by Lemma 10 codim $G_3^{b,1} \geqq 7$ and codim $G_3^{c,1} \geqq 9$.

Denote $\tilde{\psi}_s(\tilde{T}_{i\alpha}) = \{F \in F^r(J) \mid \varrho(F) \mathrel{\bar{\cap}} \bigcup_{j=r_i(\alpha)-s+1}^{r_i(\alpha)} G_i^{\alpha,j}\}$, $1 \leqq s \leqq r_i(\alpha)$, $i = 1, 2, 3$, $\alpha = a, b, c$. By [1, Theorem 19. 1] all sets $\tilde{\psi}_s(\tilde{T}_{i\alpha})$ are dense in $F^r(J)$. Therefore if $\psi_s(\tilde{T}_{i\alpha}) = \{G \in F^r(I) \mid G = F/I$ for some $F \in \tilde{\psi}_s(\tilde{T}_{i\alpha})\}$, then the set $F_1 = \bigcap_{\substack{i,s \\ \alpha=a,b,c}} \psi_s(\tilde{T}_{i\alpha})$ is dense in $F^r(I)$. Since I is compact, the openess of F_1 follows from [1, Theorem 18. 2]. From the above equalities and inequalities for the codimensions of the strata and from [1, Corollary 17. 1] we obtain that if $A \in F_1$, then the assertions (1)–(3) hold.

4. APPLICATIONS TO ONE-PARAMETRIC SYSTEMS OF DIFFEOMORPHISMS

Theorem 2. *There exists a residual set H_1 in $H^r(R^2, R^n)$ such that if $h \in H_1$, $(\mu_0, x_0) \in Z_k(h)$ and $h_\mu(x) = A(\mu)x + R(\mu, x)$, $R(\mu, x) = o(\|x\|)$ for $\mu \in I$, $x \in U$ (I is an open interval in R^2 containing the point μ_0 and U is a neighbourhood of x_0 in R^n), then*

(1) *there is a one-dimensional submanifold $X_{11}(A)$ of $R^2 \cap I$*, for which the assertion (I) from Section 3 holds;

(2) there is a set $X_1(A) \subset R^2 \cap I$, consisting of isolated points, for which the assertion (II) from Section 3 holds.

Moreover, for the matrix $A(\mu)$ the assertion (III) from Section 3 holds.

The idea of proof of Theorem 2 is precisely the same as the idea of proof of the results on generic properties of one-parametric systems of diffeomorphisms studied in [7, 8] and we omit it. We only remark that for a given $g \in H^r(R^2, R^n)$ the method of bump functions can be used for the construction of an $h \in H^r(R^2, R^n)$ sufficiently C^r close to g, whose linearization at a k-periodic point has the properties (1), (2) from Theorem 2.

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