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AROUND BOLZANO'S APPROACH TO REAL NUMBERS[1])

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First of all, I want to express my sincere thanks to academician Josef Novák, the president both of the Organizing committee of the 5-th Prague Topological Symposium and of the Committee for commemoration of the 200-th anniversary of birth of Bernard Bolzano, for the kind invitation to deliver a talk on the subject Around Bolzano's approach to real numbers.

I am very honoured and pleased to speak just here at the Universitas Carolinae, at this university of Jan Hus, Bernard Bolzano and other great spirits of Czechoslovakia.

0. Concerning Bolzano's approach to real numbers one has to bear in mind the following facts:

0 : 1. Bolzano dealt with real numbers in various of his investigations in mathematics, physics, philosophy. He accepted the edifice of these numbers and proved rigorously some fundamental facts, like his theorem that every continuous real map of a segment of real numbers yields again a convex set in the sense that any number, located between two values reached, cannot be dropped.

0 : 2. Bolzano thought very much on reals not only in connection with numbers but also in connection with time, the time being an excellent model of reals, used in this sense abundantly since Galilei and Newton.

0 : 3. Bolzano handled the decimal representations of real numbers as well as ratios of numbers.

0 : 3 : 1. This fact, the representation of reals in decimal system is one of the greatest human cultural acquisitions. Simon Stevin (1548 – 1620), from Brugge, has great merits in this field.

1. Real numbers (cf. Dhombres (1978)).

1 : 0. Every real number is like a building, a house. The set of all real numbers, called the continuum of real numbers is like a big town, an extraordinary one displayed in a *succession* in such a way that for any two buildings just one of them is

[1]) Presented August 24, 1981 at Charles University in Praha on the 200-th commemoration of birth of Bernard Bolzano (10 May 1781, Praha — 18 December 1848, Praha).

prior than the other one i.e. is located before the other one. What is the raw material, what are the bricks used to build each of these buildings?

1 : 1. Real numbers as ratios of segments were considered by the Greek mathematicians Eudoxus, Euclid and by R. Bombelli (16^{th} century) and especially by Isaac Barrow (1630–1677) and his pupil Isaac Newton (1642–1727). So in Newton's Arithmetica universalis published in 1707 one reads:

"Per numerum abstractum quantitatis cujusvis ad aliam ejusdem generis quantitatem, quae pro unitate habetur, rationem intellegimus."

Thus the number is tied with measuring of quantities of a same kind, and the results of such comparisons are real numbers.

1 : 2. A great step was performed by Simon Stevin (Brugge 1548 – Leyden or the Hague 1620) because he used a procedure to represent real numbers as decimal numbers with finite or infinite number of digits.

Both, for praxis and for theory, such an approach was good and understandable; in particular, for the infinitesimal analysis, developed stormily in 17^{th} and 18^{th} centuries such considerations of real numbers were sufficient.

1 : 3. The most important real numbers: 0 and 1 were not thought of as real numbers. For long time one had such a situation; so according to O. Gherli (Guastalla-Reggio 1730 – Parma 1780) "L'unitá non é numero" (p. 2, T. 1 of the compendium Odoardo Gherli, Gli elementi teoretico-practici delle mathematiche pure, Modena 1770–1777).

2. Bolzano's approach to real numbers (cf. Rychlík (1962)).

2 : 0. The starting position of Bolzano for his theory of real numbers is the set D of all rational integers, including in particular also the numbers 0, 1. He manipulates with such numbers in any way by performing infinitely many times the elementary operations: addition, substraction, multiplication, division. Such creations are called infinite magnitude's expressions (unendliche Grössensdrücke); Bolzano indicates following examples: $1 + 2 + 3 + \ldots$, $1/2 - 1/4 + 1/8 - \ldots + \ldots$, $(1 - 1/2) \cdot (1 - 1/4)(1 - 1/8)\ldots$

It is to be remarked that Bolzano did not indicate any infinite continuous fraction as example of number expressions, although even such creations are closely tied with measurings (otherwise, Bolzano considered continuous fractions, see e. g. Zahlenlehre I folio 76 – folio 79).

2 : 1. In Bolzano's manuscript Grössenlehre, deposited by a curious set of circumstances in the Nationalbibliothek in Vienna, a part of which was published in 1962 by the Czechoslovak Academy of Sciences as a book (cf. Rychlík, 1962) one has the following two fundamental items.

2 : 1 : 1. First item: general definition. Definition of infinite notion of magnitude (unendlicher Grössenbegriff) and infinite expresion of magnitude (unendlicher Grössenausdruck) as was described in 2 : 0. Bolzano's model works like a watch;

Bolzano is not worried to know the constituent parts of his device, and he writes: "Allein man muss sich erinnern, dass die Bestandteile, aus welchen der Gegenstand eines Begriffes zusammengesetzt ist, nicht notwendig in diesem Begriffe mit vorgestellt werden müssen; ja dass es oft möglich sei einen Begriff von einem Gegenstande – (ich meine immer nur einen sich auf ihn ausschliesslich beziehenden Begriff) – zu bilden, ohne nur einen einzigen der Bestandteile, aus welchen der Gegenstand besteht, in dem Begriffe zu denken. So ist z. B. der Begriff einer Taschenuhr nur der Begriff „einer Uhr, d. h. eines zum Messen der Zeit tauglichen Werkzeuges, welches so eingerichtet ist, dass man es in der Tasche bequem bei sich tragen kann"; und in diesem Begriffe kommt von den Rädern, Federn und anderen Teilen, aus welchen eine solche Uhr zusammengesetzt zu sein pflegt, gar keine Vorstellung vor. Auf ähnliche Art können wir uns denn auch leicht den Begriff von einer Zahl, die aus unendlich vielen Teilen zusammengesetzt ist, bilden, ohne dass diese Teile selbst und zwar alle, in dem Begriffe mit vorgestellt werden müssten. So enthält namentlich der Begriff "von einer Zahl, welche der Summe aller wirklichen Zahlen gleichet", – nur eine sehr geringe Anzahl von Bestandteilen und unter diesen befindet sich nicht eine einzige Vorstellung von einem der einzelnen Teile (1, 2, 3, ...), aus welchen jene Summe zusammengesetzt werden soll. Nenne ich also einen dergleichen Begriff unendlich; so lasse man sich durch diese nur uneigentliche Benennung nicht beirren. Sie wurde bloss gewählt, um zu verstehen zu geben, dass jene wirklichen oder nur eingebildeten Zahlen, welche der Gegenstand eines solchen Zahlenbegriffes sind, immer gedacht werden als durch eine unendliche Menge von Verrichtungen des Addierens, oder Subtrahierens, oder Multiplizierens, oder Dividierens entstanden."

2 : 1 : 2. The second fundamental item: Specification or particularisation of the first fundamental item.

Bolzano specifies his fundamental definition quoted as the first fundamental step and he presents his own procedure, a machinery, an algorithm built up of infinite number of integers considering a real number like a succession of intervals of rational numbers, length of these intervals being

$$1, 2^{-1}, 3^{-1}, \ldots \quad \text{respectively} .$$

Just such an infinite procedure was a real number for Bolzano. Let us consider the following comparison.

2 : 2. The most difficult of 15 Euclid's books is, probably, the fifth one; anyway this small book composed of 18 definitions and 25 theorems, is usually, not read by students nor by lovers of mathematics. The book is a Greek pearl of Greek ancient theory of real numbers in the form of ratios or *logos* – as the Greeks spoke. The question is on the ordering of ratios. In all schools we learn that

$$A : B = C : D \quad \text{if and only if} \quad AD = BC .$$

This is a pons asinorum (ass' bridge).

Now, just the above definition is the heart of Eudox's theory of what we call real numbers and what Euclid called logos, words.

Λογος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλἰκότητά ποια σχέσις

Logos ésti dúo megedon ómogenon ȇ katà pelikótetá poia schésis

i.e. *A* ratio is a sort of relation in respect to size between two magnitudes of the same kind.

In fact, this definition is not complete; consequently, Euclid did not give a working definition of ratio. The following definition of equality of ratios is more interesting:

The fifth definition in the fifth book says

$A/B = C/D$ if and only if for every oriented pair (m, n) of integers one has following implications

$$\begin{aligned} mA > nB &\quad \text{implies} \quad mC > nD \\ mA = nB &\quad \text{implies} \quad mC = nD \\ mA < nB &\quad \text{implies} \quad mC < nD \,, \end{aligned}$$

i.e. *if their multiples behave in a same way.*

2 : 2 : 1. So we are aware that the requested equality is tied with infinitely many implications concerning *multiples* of given magnitudes. Just this idea of multiples of quantities is the very basis of Bolzanos researches concerning real numbers (and only real numbers; by the way, he speaks of *measurable* numbers, messbare Zahlen, and not of real numbers).

2 : 3. The main idea in the positional system, e.g. for basis 10 consists in considering the set of all measure-standard-units

2 : 3 : 1. $$10^d \quad (d \in D)$$

ordered decreasingly. It is a huge storage of units. Then one measures a given magnitude x with the greatest fitted unit; the remainder with the next greatest etc.

Thus one measures a given magnitude and remainders by units of measuring in the set of all fundamental units 2 : 3 : 1.

2 : 3 : 2. Another idea, due to Bolzano, consists, for a given magnitude x, to consider like ancient Greeks the set

$$Nx = \{nx \;\therefore\; n \in N\}$$

of all multiples of x and to measure each of them with a *same unique standard* unit 1; in such a way one gets a sequence of results

2 : 3 : 3. $$x_1, x_2, \ldots, x_n, \ldots$$

by which x is determined.

For any magnitude x and any natural number n one has then

$$x_n \leqq nx < (x_n + 1)\,,$$

thus

2 : 3 : 4.
$$\frac{x_n}{n} \leqq x < \frac{x_n + 1}{n}.$$

The magnitude x is either x_n/n or is located between rational numbers x_n/n, $(x_n + 1)/n$ the distance of which is $1/n$.

2 : 3 : 5. Briefly, we have the following Bolzano's machine: To every natural number q corresponds a well-determined integer $p = p(q)$ such that for some *positive* expressions

2 : 3 : 6.
$$P_1 \geqq 0\,, \quad P_2 > 0 \quad \text{one has}$$
$$S = \frac{p}{q} + P_1 \quad \text{and} \quad S = \frac{p+1}{q} - P_2\,.$$

So one has to test, to check these relations for every natural number $q \in N$.

2 : 3 : 7. If one looks at this Bolzano's Empire building in detail, then one could think to be in a vicious circle because real number S is tied with real numbers $P_1 \geqq 0$, $P_2 > 0$.

Now, there is no vicious circle because the Bolzano's aim is not to define some particular real number but to define the set R of all real numbers as a structure. Particular and every real number was defined by its decimal representation; since Simon Stevin (1548 – 1620) this was known although not very diffused; Bolzano used such representation of real numbers (e.g. in Rein anal. Beweis ... § 8).

2 : 3 : 8. Bolzano's model of R works; Bolzano proved, in the period 1830 – 1835, in his Grössenlehre: his messbare Zahlen, thus real numbers, constitute what we could call in 20-th century, a complete ordered field; on the way of his proof he discovered and proved several mathematical statements well known in the mathematics of the 20-th century and which are labeled by great names as Cauchy, Weierstrass, Dedekind, Cantor; in each case the name of Bolzano should be placed on the first place.

2 : 3 : 9. Each of the known arithmetical theories of real numbers tied with names: Weierstrass, Méray, Cantor, Heine, Hilbert, Russell, Kolmogorov (1946), Kurepa (1953) is a specification or a variation and set up of Bolzano's general first approach 2 : 1 : 1. Great efforts were needed to prove all statements through Bolzano's mashinery; some proofs are quite complicated. But the existence of quoted approaches to define the linear continuum R are, at the same time, proving how the Bolzano's initial approach 2 : 1 : 1 was fruitful and farreaching.

In Bolzano's task of measurings ("Geschäft des Messens") and particularly in his performances (Verrichtungen) of used operations one is aware how his approach was general and fruitful.

2 : 3 : 10. Simplest formations are series of rational numbers; historically, precisely such an approach to R was given by Weierstrass in his lectures and quoted, partially textually, in Kossak (1872); one is astonished by the degree of similarity of the general Bolzano's approach 2 : 1 : 1 and Weierstrass' definition of real numbers.

2 : 3 : 11. On the other hand, since sequences and series are mutually organically linked, the Bolzano's creations are automatically interpretable in terms of sequences of rational numbers; in other words, the Bolzano's approach yields what was described and published much later by Charles Méray, 1868, 1872, Georg Cantor 1872 and Heine 1872 (Crelle's J. T. 74 p. 176). Kolmogorov's approach (1946) is practically the Bolzano's 2 : 3 : 6 (from my point of view, the impact of arithmetical "exact definitions" of real numbers (Weierstrass, Méray, Cantor, Heine) is over-estimated). Real numbers as decimal representations were also good and sufficient.

2 : 3 : 12. Objections were made to Bolzano's approach 2 : 3 : 5 in connection with numbers $P_1 \geqq 0$, $P_2 > 0$ occuring in 2 : 3 : 6. Now, it is easy and natural to remove this objection – simply by eliminating P_1, P_2 and writing

2 : 3 : 13. $$p/q \leqq S < p/q + 1/q$$

instead of 2 : 3 : 6. Systems 2 : 3 : 6, 2 : 3 : 13 are reducible one to another and describe a nesting of intervals, as much applied in Mathematics, especially in 19-th and 20-th centuries.

2 : 3 : 14. Such a nesting of intervals is particularly evident in 2 : 3 : 13 and is used as "Intervallschachtelung" e.g. in Knopp (1924) pp. 20–41. Just the first chapter in Knopp (1924) is a nice presentation of the Bolzano's particular approach 2 : 3 : 5 to real numbers; so is the Kolmogorov's one (1946), although this version is less intuitive.

2 : 4. Four important Bolzano's theorems concerning real numbers. Already in Bolzano (1817) § 12 we find the following pivotal theorem concerning R, with a strong and characteristic proof.

2 : 4 : 1. Theorem. "*If a property M does not belong to all values of a variable quantity x but does belong to all ones which are less than a particular value u; then there exists a quantity U, which is the greatest of those values for which we can maintain, that all inferior values have the property M*".

It is very useful to notify the use of quantors in the textual wording of the theorem. It is extremely instructive to read the Bolzano's proof bearing in mind that for him real numbers were defined by decimal representations or as sums of series of rational numbers. He applied his theorem in the proof of title theorem (Bolzano 1817) or rather of the following important statement.

2 : 4 : 2. Theorem. "*If two functions fx, φx are continuous either for all values of x or at least for those located between α and β, if further $f\alpha < \varphi\alpha$ and $f\beta > \varphi\beta$, then there exists always a certain value between α and β for which $fx = \varphi x$*" (Bolzano 1817 § 15 Lehrsatz). The title theorem of 1817 is obtained by specification

$\varphi x = 0$ for $a \leqq x \leqq b$. It is amazing that in his proof of 2 : 4 : 2 Bolzano starts with a text of 7 lines which should be a comment to the theorem; in the course of his proof he superfluously distinguished various cases concerning the signs of α, β, although in the proof such a differentiation was not applied at all.

On the other hand the wording of the Bolzano's theorem 2 : 4 : 2 is more flexible than the title theorem in Bolzano 1817. Precisely, in connection with the quoted Bolzano's comments p. 51_{10-4}, we have following two generalizations 2 : 4 : 2 : 1, 2 : 4 : 2 : 2 of 2 : 4 : 2 and which show that intuitively Bolzano saw a great scope of his statement 2 : 4 : 2.

2 : 4 : 2 : 1. Theorem. *Let (A, B) be a 2-un of ordered chains and $f, g : A \to B$ be continuous mappings such that for two distinct points $c, d \in A$ one has $fc < gc$, $fd > gd$. If the open interval $T := A(c, d)$ is order-dense and gapless then for an interior point $x \in A(c, d)$ one has $fx = gx$.*

Proof. The continuity of the functions f, g in I implies the existence of a subinterval $A[c, y) \subset I$ such that $fx < gx$ for every $x \in A(c, y)$. Let $M := \{y \therefore y \in A(c, d)$ and $fx < gx$ for every $x \in A(c, y)\}$; let $m := \sup M$. Since $A(c, d)$ has no gap the point m exists and satisfies $m \in A$, $m < d$. We claim that $fm = gm$. In opposite case there should be either $fm < gm$ or $fm > gm$.

Now, one has not $fm < gm$ because this relation and the continuity of f, g would imply the existence of an open interval $A(m_1 < m < m_2)$ in every point of which should be $f < g$ and consequently $m_2 \in M$, in contradiction with $m = \sup M$ and $m < m_2$. One has $fm > gz$ neither, because this relation and the continuity of f, g would imply that for some neighborhood Vm of m every $x \in Vm$ would satisfy $fx > gx$, in contradiction with the fact that $Vm \cap M \neq v$ (vacuous set) (because m is a left accumulation point of M) and that $fx < gx$ for every $x \in M$.

2 : 4 : 2 : 2. Theorem. 1. *Let (A, B) be a 2-un of topological spaces and f, g be continuous mappings of A into B. If the arrival space B has the T_2-separation property at least for two-point subsets of fA, then the functions f, g are equal on the boundary* FrM *of the set $M := \{x \therefore x \in A, fx \neq gx\}$, i.e. for every $y \in$* FrM *one has $fy = gy$.*

2. *In particular, if B is totally ordered and $b \in B$, then for the boundary* FrM *of the set $M := \{x \therefore x \in A, fx < b\}$ (resp. $M := \{x \therefore x \in A, fx > b\}$) one has $y \in \mathrm{Fr}M \Rightarrow fy = b$.*

3. *In both cases, if the departure space A is connected, then* Fr$M \neq v$; *this will happen e.g. if A is totally ordered order-dense, gapless and non void.* (Let us recall that the boundary or the frontier of a set X in a closure space S is defined as the set $\mathrm{Fr}X := \mathrm{Cl}X \cap \mathrm{Cl}(S - X)$ (Čech 1966 p. 358).

Proof. The set M is open. As a matter of fact, if $x \in M$ then fx, gx are two distinct points of B; B being T_2, there are disjoint open neighborhoods Vfx, Vgx of the points fx, gx. The assumed continuity of f, g in x implies the existence of neighborhoods O_1x, O_2x of x such that $fO_1x \subset Vfx$, $gO_2x \subset Vgx$; for the neighborhood $Ox := O_1x \cap O_2x$ one should have $y \in Ox \Rightarrow fy \neq fy$, i.e. $Ox \subset M$ which proves that M is open.

If $y \in \mathrm{Fr}M$, then $fy = gy$. In opposite case, there would be a $y \in \mathrm{Fr}M$ such that $fy \neq gy$; the continuity of f, g in y would imply that there exists a neighborhood Oy of y such that $fz \neq gz$ for every point z of Oy; by definition of M, Oy would be a subset of M; again, this property of Oy contradicts that, y being a point of $\mathrm{Fr}M$, one has $Vy \cap (A - M) \neq v$.

Let us still prove the second part of 2 : 4 : 2 : 2 Theorem. In opposite case, there would be a point $y \in \mathrm{Fr}M$ such that $fy < b$ (resp. $fy > b$); therefore because of the continuity of f at b there would be for every Vfy, and in particular for such one that (1) $Ofy < b$ (resp. $Ofy > b$) a neighborhood Oy of y such that $fOy \subset Ofy$. Now, since $y \in \mathrm{Fr}M$ there exists a point $z \in A - M$ such that $z \in Oy$, i.e. $fz \geqq b$ (resp. $fz \leqq b$), contrarily to (1). The third part of 2 : 4 : 2 : 2 Theorem is a well known fact in Topology (cf. Čech, p. 359).

2 : 4 : 3. Convergence criterion. It is remarkable that § 7 Lehrsatz in Bolzano 1817 is the sufficiency part of the convergence criterion for sequences. The proof is strong enough, the real numbers being considered by decimal representations.

2 : 4 : 4. An important Bolzano's theorem. Theorem 105 in Bolzano's Theory of magnitudes (Grössenlehre) is a real pearl, the specification of which yields the Dedekind's Schnittverfahren (Cut procedure) as well as the Cantor's Durchschnittssatz of intervals of the linear continuum:

Theorem. *If A, B are non void parts of the continuum R such that $A < B$ and neither A has the last nor B the first element, then there exists at least one member of R located between A and B; if the difference between $b \in B$ and $a \in A$ can't be arbitrarily small, there are infinitely many $r \in R$ between A, B. If $b - a$ can decrease arbitrarily, then there is just one $r \in R$ between A and B. If finally $b - a$ can decrease below any positive number and either A has a maximal or B has a minimal member, there is no $r \in R$ such that $A < r < B$.*

This wording characterizes the notion of a totally ordeied chain: to have neither gaps nor jumps, properties defining continuous chains. Consequently, in this field also Bernard Bolzano is not only a forerunner but an essential builder.

2 : 5. A quotation of Ladislav Rieger, editor of the book Rychlík (1962). Let us quote the following terminal part of the preface of the reference Rychlík 1962 (where TRZ (:= Theorie der reellen Zahlen) denotes the chapter Unendliche Zahlen – (Grössen –) Begriffe (– Ausdrücke) in Bolzano's book-manuscript Grösenlehre (written by Bolzano mainly in period 1830–1835).

"Es sei jedoch kurz auf eine andere Deutungsmöglichkeit für den nicht genügend klarem Grundbegriff der reellen Zahl der TRZ (d. h. für den Begriff des "unendlichen messbaren Grössenausdrucks" von Bolzano) aufmerksam gemacht. –

Als grundsätzliche Unklarheitsquelle der TRZ erscheint nämlich die (zu Bolzanos Zeit übliche und zuweilen noch immer überlebende) Undeutlichkeit des Unterschieds zwischen Gegenstand und Symbol in der Mathematik. Diese Undeutlichkeit ist aber nicht bloss als logischer Fehler abzufertigen; vielmehr wird sie begreiflich (wenn auch keineswegs annehmbar), wenn wir einsehen, dass das damalige mathematische Denken nicht im heutigen Sinne mengentheoretisch extensional, sondern vielmehr intensional war. Das heisst: die Definitionen der mathematischen Begriffe selbst, deren Konstruktionen und die zugehörigen Rechenverfahren (und schliesslich aber auch die zugehörigen symbolischen Ausdrücke) wurden meistens als primär und "mathematischwirklich" angesehen; erst viel später (durch die allgemeine Anerkennung des mengentheoretischen Denkens in der Mathematik) wurden eben diese Dinge als nicht eigentlich mathematisch (sondern als "formal") betrachtet (und es wurden nicht die Intensionen der Begriffe, sondern ihre Extensionen, d. h. Mengen, als mathematisch primär angesehen); schliesslich wurden die Intensionen der mathematischen Begriffe (als Objekte) der Metamathematik (in genügend weitem Sinne) zugewiesen, während die eigentlichen mathematischen Objekte als unabhängig von ihrer Definitions-, Konstruktions- und Berechnungsart bestehend angenommen wurden. (Diese Einstellung wird gegenwärtig, meines Erachtens nicht sehr glücklich, oft als mathematischer Platonismus bezeichnet).

Im Lichte dieser Bemerkung sollte nun klar sein, dass Bolzanos "unendliche Grössenausdrücke" von ihm auch als Symbole für effektiv angegebene unbegrenzt fortsetzbare Rechenverfahren (im Sinne der vier Spezies) mit rationalen Zahlen aufgefasst werden konnten – also für das, was wir heute etwa dem Sinne nach als Turingsche Maschine[1])) präzisieren möchten. Speziell die "messbaren" unendlichen Grössenausdrücke sind dann solche, deren (rationale) Teilergebnisse konvergieren (d. h. sich speziell benehmen). Die vier Spezies werden von den Teilergebnissen auf das ganze Rechnungsverfahren übertragen – und schliesslich wird ein besonderes Rechenverfahren angegeben, um Gleichheit und Anordnung zwischen zwei messbaren unendlichen Grössenausdrücken zu bestimmen.

Kurz und gut, man könnte auch daran denken den Bolzanoschen Begriff der reellen Zahl im Sinne der berechenbaren (oder allgemein rekursiven) Analysis[2])) zu deuten, da ja diese in gewisser Hinsicht eine neue Wiederkehr zum Nicht-extensionalen in der Analysis darstellt. – Da aber diese neue (konstruktive) Auffassung der Analysis heute noch gar nicht einheitlich und allgemein angenommen ist und auch nicht fertig vorliegt, so dürfte (schon aus diesem Grunde) die konstruktiv-intensionale Deutung des Begriffs der reellen Zahl in TRZ zur Zeit ausser Acht gelassen werden."

3. Bolzano's approach to non-standard numbers.

3 : 0. Bolzano's notion of "infinite magnitude's expression" – (unendlicher

[1]) Siehe z. B. Davis, Computability and Unsolvability, McGraw Hill, New York 1958.

[2]) Siehe sub 1), für weitere Literaturangaben.

Grössenausdruck) embraces all sequences of rational numbers; by his device he selects the measurable ones defining real numbers.

3 : 1. Bolzano was aware that there are also non real magnitudes and he considered, in particular, the magnitudes S satisfying one and only one of his two conditions:

$$S = \frac{p}{q} + P_1 , \quad S = \frac{p+1}{q} - P_2 .$$

3 : 2. If S satisfies only the first (second) condition, S is called a positive (negative) infinite number; e.g. $1 + 2 + \ldots$ is infinite positive.

3 : 3. He considered also infinitely small-positive and negative numbers; he established an arithmetic of his infinitely small (great) numbers and so he produced some non-standard real numbers.

3 : 4. Other outputs of Bolzano's approach.

3 : 4 : 1. Bolzano observed also that his general approach could yield something that is neither a real nor an infinite number. Such a feature is $1 - 1 + 1 - 1 + \ldots$ $\ldots - \ldots$

In 20-th century one learnt to study such features as well, in particular, by summability procedures one clarified such indeterminations.

3 : 4 : 2. Historically we could mention still another fact concerning the question: *How many real numbers there are*? Cantor put this question and answered that this number should be $\aleph_1$ – the next – one to the cardinality $\aleph_0$ of the set of all natural numbers. About eighty years mathematicians tried to prove it. In 1953 in Paris Academy of Science I published that $2^{\aleph_0}$ should be not only $\aleph_1$ but any cardinal number greater than $\aleph_0$ and non cofinal to $\aleph_0$; this extraordinary statement was proved, as exact, by Eastman in 1970, and so was established also an indecidability concerning the whole real continuum, for which Bolzano was so worried.

3 : 5. Final remarks on non standard magnitudes. Independently of Bolzano, non standard magnitudes appeared as a by product in some other considerations: so in D. Kurepa's doctoral Thesis (Paris 1935) and developed as semireals by M. Krasner and Dokas; in 1948 Edwin Hewitt introduced hyperreal fields that could serve as non standard models of analysis.

In 1955 arose the notion of ultraproduct (Łos), one considered again infinitely small numbers with applications (1958 Schmieden-Laugwitz); non-standard model $*N$ of the set N of all natural numbers was established as early as 1934 (Skolem); corresponding phenomena established for the continuum R of real numbers yielded a non standard model $*R$ of R and provoked corresponding non standard considerations in Analysis, Probability, Physics, ...; this was magnificiently shown by Abraham Robinson after 1960.

So, on this day August 24, 1981 when we are commemorating the 200-th anniversary of birth of Bernard Bolzano in his birth town Praha we can frankly say that Bolzano's contribution around his approach to real numbers was tremendously fruitful and that standard mathematics, non standard mathematics, constructive mathematics and applications are firmly established, greatly in the spirit forcasted by Bolzano; Bolzano's critical minds would surely agree with such results.

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