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THE LEAST SEPARATIVE CONGRUENCE ON A WEAKLY COMMUTATIVE SEMIGROUP

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In [1], a relation π on an arbitrary semigroup S has been defined. For elements a and b of S, $a \pi b$ if and only if $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer n. It has been proved that, if S is a weakly commutative semigroup, then π is a separative congruence on S. The author has proved that, if S is a duo semigroup (i.e. every one-sided ideal of S is two-sided), then S/π is a maximal separative homomorphic image of S. See Theorem 5 of [1].

In this note we shall extend this result on duo semigroups to weakly commutative semigroups.

Definition 1. A semigroup S is called *weakly commutative if*, for any $a, b \in S$, we have $(ab)^k = xa = by$ for some $x, y \in S$ and a positive integer k. See Definition 6.4 of [2].

Definition 2. We define a relation π on a semigroup S as follows: $a \pi b$ if and only if $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer n. See [1].

Remark 1. *Let S be a semigroup, a and $b \in S$ and ϱ a congruence on S. If $ab^{n+1} \varrho b^{n+2}$ and $(ab^n)^m \varrho (b^{n+1})^m$ for positive integers n and m, then $(ab^n)^M \varrho (b^{n+1})^M$ for any positive integer $M > m$.*

Similarly, if $b^{n+1}a \varrho b^{n+2}$ and $(b^n a)^m \varrho (b^{n+1})^m$ for positive integers n and m, then $(b^n a)^M \varrho (b^{n+1})^M$ for any positive integer $M > m$.

Proof. We prove only the first part of the remark, because the second part can be proved in a similar way. Let us suppose $ab^{n+1} \varrho b^{n+2}$, $(ab^n)^m \varrho (b^{n+1})^m$ for some $a, b \in S$ and positive integers n and m. Let M be an arbitrary positive integer with $M > m$. Then $(ab^n)^M = (ab^n)^{M-m} (ab^n)^m \varrho (ab^n)^{M-m} (b^{n+1})^m =$
$= (ab^n)^{M-m-1} ab^n b^{n+1} (b^{n+1})^{m-1} \varrho (ab^n)^{M-m-1} (b^{n+1})^{m+1} \varrho \ldots \varrho (b^{n+1})^{m+M-m} =$
$= (b^{n+1})^M$.

Lemma 1. (B. Pondělíček [1]). *If S is a weakly commutative semigroup, then π is a separative congruence on S.*

Theorem 1. *If S is a weakly commutative semigroup, then S/π is a maximal separative homomorphic image of S.*

Lemma 2. *Let S be a weakly commutative semigroup and ϱ a separative congruence on S. Let $a, b \in S$. If $ab^n \varrho b^{n+1} \varrho b^n a$ and $ba^n \varrho a^{n+1} \varrho a^n b$ for a positive integer n, then $a \varrho b$.*

Proof. Since ϱ is a separative congruence, the result is true for $n = 1$. Assume now that the assertion holds for $n \geqq 1$. Let $ab^{n+1} \varrho b^{n+2} \varrho b^{n+1}a$ and $ba^{n+1} \varrho a^{n+2} \varrho a^{n+1}b$. Since S is weakly commutative, $(ab^n)^k = by$ for some $y \in S$ and a positive integer k. Thus $(ab^n)^{k+1} = ab^{n+1}y \varrho b^{n+2}y = b^{n+1}(ab^n)^k = b^{n+1}ab^n(ab^n)^{k-1} \varrho \varrho (b^{n+1})^2 (ab^n)^{k-1} \varrho \dots \varrho (b^{n+1})^{k+1}$. Similarly, $(b^n a)^t = ub$ for some $u \in S$ and a positive integer t. Thus

$$(b^n a)^{t+1} = ub^{n+1}a \varrho ub^{n+2} = (b^n a)^t b^{n+1} = (b^n a)^{t-1} b^n a b^{n+1} \varrho \varrho (b^n a)^{t-1} (b^{n+1})^2 \varrho \dots \varrho (b^{n+1})^{t+1}.$$

Consequently, $(ab^n)^{k+1} \varrho (b^{n+1})^{k+1}$ and $(b^{n+1})^{t+1} \varrho (b^n a)^{t+1}$. By Remark 1, it follows that

$$(ab^n)^m \varrho (b^{n+1})^m \varrho (b^n a)^m \quad \text{for a positive integer } m.$$

Let $m_1 = \min\{m: (ab^n)^m \varrho (b^{n+1})^m \varrho (b^n a)^m\}$. We prove that $m_1 = 1$. Let us suppose that $m_1 \neq 1$ and let

$$m_2 = \begin{cases} m_1 & \text{if } m_1 \text{ is an even number}, \\ m_1 + 1 & \text{if } m_1 \text{ is an odd number}. \end{cases}$$

Then, by Remark 1,

$$(ab^n)^{m_2} \varrho (b^{n+1})^{m_2} \varrho (b^n a)^{m_2}.$$

Let $m_3 = m_2/2$. Then $m_3 > m_1$ and

$$((ab^n)^{m_3})^2 = (ab^n)^{2m_3} = (ab^n)^{m_2} \varrho (b^{n+1})^{m_2} = ((b^{n+1})^{m_3})^2 = = (b^{n+1})^{m_2} \varrho (b^n a)^{m_2} = (b^n a)^{2m_3} = ((b^n a)^{m_3})^2.$$

Moreover,

$$(ab^n)^{m_3} (b^{n+1})^{m_3} = (ab^n)^{m_3-1} ab^n b^{n+1} (b^{n+1})^{m_3-1} \varrho$$
$$\varrho (ab^n)^{m_3-1} (b^{n+1})^2 (b^{n+1})^{m_3-1} = (ab^n)^{m_3-1} (b^{n+1})^{m_3+1} \varrho \dots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2$$

and

$$(b^n a)^{m_3} (b^{n+1})^{m_3} = (b^n a)^{m_3-1} b^n a b^{n+1} (b^{n+1})^{m_3-1} \varrho$$
$$\varrho (b^n a)^{m_3-1} (b^{n+1})^2 (b^{n+1})^{m_3-1} = (b^n a)^{m_3-1} (b^{n+1})^{m_3+1} \varrho \dots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2.$$

Thus we have $((ab^n)^{m_3})^2 \varrho (ab^n)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2$ and

$$((b^n a)^{m_3})^2 \varrho (b^n a)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2.$$

Since ϱ is a separative congruence, it follows that

$$(ab^n)^{m_3} \varrho (b^{n+1})^{m_3} \varrho (b^n a)^{m_3}.$$

Since this result contradicts $m_3 < m_1$, we have $m_1 = 1$. Consequently, $ab^n \varrho b^{n+1} \varrho \varrho b^n a$. We can prove $ba^n \varrho a^{n+1} \varrho a^n b$ in a similar way. Hence we get $a \varrho b$. The result therefore follows by induction. Thus the lemma is proved.

The proof of Theorem 1. Let ϱ be an arbitrary separative congruence on a weakly commutative semigroup S. If $a \pi b$ $(a, b \in S)$, then $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer n. Thus $ab^n \varrho b^{n+1} \varrho b^n a$ and $ba^n \varrho a^{n+1} \varrho \varrho a^n b$. By Lemma 2, it follows that $a \varrho b$. Consequently $\pi \subseteq \varrho$.

Corollary 1. *If S is a duo semigroup, then S/π is a maximal separative homomorphic image of S.*

Corollary 2. *If S is a normal semigroup (i.e. $aS = Sa$ for any $a \in S$), then S/π is a maximal separative homomorphic image of S.*

Corollary 3. *If S is a quasicommutative semigroup (i.e. for any $a, b \in S$, we have $ab = b^r a$ for a positive integer r), then S/π is a maximal separative homomorphic image of S.*

References

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