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ABSOLUTELY CONVERGENT EXPANSIONS ASSOCIATED WITH A BOUNDARY-VALUE PROBLEM WITH THE EIGENVALUE PARAMETER CONTAINED IN ONE BOUNDARY CONDITION

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1. INTRODUCTION

Consider the differential equation

$$y''(x) - \{\lambda - q(x)\}\, y(x) = 0\,, \tag{1,1}$$

where x runs over the compact interval $[a, b]$, λ is a complex parameter and $q \in L[a, b]$ is real. The boundary conditions

$$y(a) \cos \alpha + y'(a) \sin \alpha = 0\,, \tag{1,2}$$

$$-\{\beta_1\, y(b) - \beta_2\, y'(b)\} = \lambda\{\beta'_1\, y(b) - \beta'_2\, y'(b)\}\,, \tag{1,3}$$

where $\alpha, \beta_1, \beta_2, \beta'_1, \beta'_2$ are all real, are imposed.

The boundary value problem thus defined has been studied extensively in recent years. Walter [4] has given an operator-theoretic formulation of the problem, and it has turned out to be necessary to impose the determinantal condition

$$\varrho = \begin{vmatrix} \beta'_1 & \beta_1 \\ \beta'_2 & \beta_2 \end{vmatrix} > 0\,,$$

which will be assumed to hold here. C. T. Fulton [1] has carried over the methods of Titchmarsh [3, ch. 1] to this problem, obtaining L^2 and pointwise convergence results for the associated eigenfunction expansions. We shall adopt his formulation of the theory. Various other expansion results and the theory of what is essentially the domain of the square-root of the self-adjoint operator associated with this problem are dealt with in the paper [2] by D. B. Hinton.

Fulton [loc. cit.] has established first-order asymptotic formulae for the first components of the eigenfunctions and, where q has bounded variation over $[a, b]$, second-order asymptotic formulae for the eigenvalues. Second-order asymptotic formulae for the eigenvalues and for both components of the eigenfunctions are derived in section 3 of this paper, valid for any $q \in L[a, b]$. These are used in section 4 to prove a theorem (Theorem 3) relating the eigenfunction expansions to trigonometric expansions. Analogues of the classical Wiener-Lévy theorem are then considered.

2. THE EIGENVALUES AND EIGENFUNCTIONS

The theory of the boundary value problem is set in the Hilbert space $H = L^2[a, b] \oplus \mathbb{C}$ of two-component elements, which has inner product defined by

$$\langle F, G\rangle = \int_a^b F_1(x)\, \overline{G_1(x)}\, dx + \frac{1}{\varrho} F_2 \bar{G}_2$$

for any $F(x) = (F_1(x), F_2)$, $G(x) = (G_1(x), G_2)$ in H. $\mathbb{C}$ denotes the complex field.

As in Sturm-Liouville theory we introduce the solution $\phi(x, \lambda)$ of (1,1) which satisfies

$$\phi(a, \lambda) = \sin\alpha\,, \quad \phi'(a, \lambda) = -\cos\alpha\,.$$

It is shown in [1, p. 296] that the eigenvalues λ_n, $n = 0, 1, 2, \ldots$, are precisely the zeros of the integral function ω of λ defined by

$$\omega(\lambda) = (\beta_1'\lambda + \beta_1)\, \phi(b, \lambda) - (\beta_2'\lambda + \beta_2)\, \phi'(b, \lambda)\,.$$

These eigenvalues are real, tend to ∞ and have no finite cluster point.

Let us write $R_b'(y) = \beta_1'\, y(b) - \beta_2'\, y'(b)$, for any differentiable function y. The normalised eigenvectors are

$$\Psi_n(x) = (\psi_n(x), R_b'(\psi_n))\,,$$

where

$$\psi_n(x) = \|\Phi_n\|^{-1}\, \phi(x, \lambda_n)\,,$$

$$\Phi_n(x) = (\phi(x, \lambda_n),\ \beta_1'\phi(b, \lambda_n) - \beta_2'\phi'(b, \lambda_n))$$

and

$$\|\Phi_n\| = \langle \Phi_n, \Phi_n\rangle^{1/2}\,.$$

The eigenvectors satisfy the orthogonality condition

$$\langle \Psi_m, \Psi_n\rangle = 0\,, \quad m \neq n\,,$$

and it should be noted that the functions ψ_n are not orthogonal in $L^2[a, b]$.

The eigenfunction expansion of $F(x) = (F_1(x), F_2)$ is formally

$$F(x) = \sum_{n=0}^{\infty} \langle F, \Psi_n\rangle\, \Psi_n(x)\,, \tag{2,1}$$

with

$$\langle F, \Psi_n\rangle = \int_a^b F_1(x)\, \psi_n(x)\, dx + \frac{1}{\varrho} F_2\, R_b'(\psi_n)\,, \tag{2,2}$$

where we assume that $F_1 \in L[a, b]$ and $F_2 \in \mathbb{C}$.

3. ASYMPTOTIC FORMULAE

Define $s = \sqrt{\lambda} = \sigma + it$ to mean

$$s = 2^{-1/2}\{\pm \sqrt{(|\lambda| + \mathrm{re}\,\lambda)} + \mathrm{i}\sqrt{(|\lambda| - \mathrm{re}\,\lambda)}\},$$

the plus sign being taken when $0 \leqq \arg\lambda \leqq \pi$, the minus sign when $\pi < \arg\lambda < 2\pi$; then s is real when $\lambda > 0$, and $t \geqq 0$ always.

The following lemmas show how $\phi(x, \lambda)$ behaves as $|\lambda| \to \infty$.

Lemma 1. *If* $\lambda \neq 0$ *we have*

$$\phi(x, \lambda) = \cos\{s(x-a)\}\sin\alpha - s^{-1}\sin\{s(x-a)\}\cos\alpha + \\ + s^{-1}\int_a^x \sin\{s(x-w)\}\, q(w)\,\phi(w, \lambda)\,\mathrm{d}w\,.$$

Proof. See [3, p. 9].

Lemma 2. *Let* $s_0 > 0$ *be given. Then for* $|s| \geqq s_0$,

$$\phi(x, \lambda) = O(\mathrm{e}^{t(x-a)}), \quad \phi'(x, \lambda) = O(|s|\,\mathrm{e}^{t(x-a)})$$

when $\sin\alpha \neq 0$, *while if* $\sin\alpha = 0$

$$\phi(x, \lambda) = O(|s|^{-1}\,\mathrm{e}^{t(x-a)}), \quad \phi'(x, \lambda) = O(\mathrm{e}^{t(x-a)})\,.$$

These results hold uniformly in x *over* $[a, b]$.

Proof. See [3, p. 10].

Lemma 3. *Let* $s_0 > 0$ *be given. Then for* $|s| \geqq s_0$,

$$\phi(x, \lambda) = \cos\{s(x-a)\}\sin\alpha - s^{-1}\sin\{s(x-a)\}\cos\alpha + \\ + s^{-1}\sin\alpha\int_a^x q(w)\sin\{s(x-w)\}\cos\{s(w-a)\}\,\mathrm{d}w + O(|\lambda|^{-1}\,\mathrm{e}^{t(x-a)})$$

when $\sin\alpha \neq 0$, *and when* $\sin\alpha = 0$

$$\phi(x, \lambda) = -s^{-1}\cos\alpha\sin\{s(x-a)\} - \\ - \lambda^{-1}\cos\alpha\int_a^x q(w)\sin\{s(x-w)\}\sin\{s(w-a)\}\,\mathrm{d}w + O(|s|^{-3}\,\mathrm{e}^{t(x-a)})\,.$$

Formulae for $\phi'(x, \lambda)$ *are obtained by formal differentiation, the new order terms being*

$$O(|s|^{-1}\,\mathrm{e}^{t(x-a)}), \quad O(|\lambda|^{-1}\,\mathrm{e}^{t(x-a)})$$

respectively. These results hold uniformly in x *over* $[a, b]$.

Proof. By Lemma 1,

$$\phi(x, \lambda) - \cos\{s(x-a)\} \sin\alpha + s^{-1} \sin\{s(x-a)\} \cos\alpha =$$

$$= s^{-1} \int_a^x \sin\{s(x-w)\}\, q(w) \left[\sin\alpha \cos\{s(w-a)\} - s^{-1}\cos\alpha \sin\{s(w-a)\}\right] dw +$$

$$+ \lambda^{-1} \int_a^x \sin\{s(x-w)\}\, q(w)\, dw \int_a^w \sin\{s(w-u)\}\, q(u)\, \phi(u, \lambda)\, du \cdot$$

The integrand in the last term is of the order of

$$|q(w)|\, |q(u)| \exp\left[t\{|x-w| + |w-u| + u - a\}\right] =$$

$$= |q(w)|\, |q(u)| \exp\{t(x-a)\}$$

when $\sin\alpha \neq 0$, with an additional factor $|s|^{-1}$ when $\sin\alpha = 0$, by Lemma 2. Hence the term is of the order of $\exp\{t(x-a)\}$ multiplied by

$$|\lambda|^{-1} \left\{\int_a^b |q(w)|\, dw\right\}^2, \quad |s|^{-3} \left\{\int_a^b |q(w)|\, dw\right\}^2$$

respectively. This proves the formula for ϕ when $\sin\alpha \neq 0$. When $\sin\alpha = 0$ we also have the term

$$-\lambda^{-1} \cos\alpha \int_a^x \sin\{s(x-w)\} \sin\{s(w-a)\}\, q(w)\, dw$$

in which the product of sines is of the order of

$$\exp\left[t\{|x-w| + w - a\}\right] = \exp\{t(x-a)\}$$

as before. The results for ϕ' are proved similarly, using the differentiated form of Lemma 1. *Q.E.D.*

In what follows it is necessary to consider the four cases (see (1,2) and (1,3))

I: $\beta_2' \sin\alpha \neq 0$,

II: $\sin\alpha \neq 0$, $\beta_2' = 0$,

III: $\sin\alpha = 0$, $\beta_2' \neq 0$,

IV: $\sin\alpha = \beta_2' = 0$.

We also write $\varkappa = b - a$.

It is shown by Fulton [1, p. 300] that the eigenvalues λ_n satisfy $\lambda_n^{1/2} = p\pi\varkappa^{-1} + O(n^{-1})$ with p given by $n-1$, $n-\frac{1}{2}$, $n-\frac{1}{2}$ and n respectively in the four cases. We now extend these results.

Theorem 1. *The eigenvalues* λ_n *satisfy the following asymptotic formulae. In case* I

$$\lambda_n^{1/2} = (n-1)\pi\varkappa^{-1} + (n-1)^{-1}\pi^{-1}\left\{-\cot\alpha - \beta_1'/\beta_2' + \int_a^b q(w)\cos^2\{(n-1)\pi\varkappa^{-1}(w-a)\}\,dw\right\} + O(n^{-2}).$$

In case II, $\lambda_n^{1/2}$ *is as before, with* $n-1$ *replaced by* $n-\frac{1}{2}$ *and* $-\beta_1'/\beta_2'$ *replaced by* β_2/β_1'. *In case* III

$$\lambda_n^{1/2} = (n-\tfrac{1}{2})\pi\varkappa^{-1} + (n-\tfrac{1}{2})^{-1}\pi^{-1}\left\{-\beta_1'/\beta_2' + \int_a^b q(w)\sin^2\{(n-\tfrac{1}{2})\pi\varkappa^{-1}(w-a)\}\,dw\right\} + O(n^{-2}).$$

In case IV, $\lambda_n^{1/2}$ *is as in case* III *with* $n-\frac{1}{2}$ *replaced by* n *and* $-\beta_1'/\beta_2'$ *replaced by* β_2/β_1'.

Proof. Suppose that $\beta_2'\sin\alpha \neq 0$. The eigenvalues satisfy $\omega(\lambda_n) = 0$ and by Lemma 3 we have

$$\begin{aligned}\omega(\lambda) &= (\beta_1'\lambda + \beta_1)\,\phi(b,\lambda) - (\beta_2'\lambda + \beta_2)\,\phi'(b,\lambda) = \\ &= (\beta_1'\lambda + \beta_1)\Big\{\sin\alpha\cos\varkappa s - s^{-1}\cos\alpha\sin\varkappa s + \\ &\quad + s^{-1}\sin\alpha\int_a^b q(w)\sin\{s(b-w)\}\cos\{s(w-a)\}\,dw + O(\lambda^{-1})\Big\} - \\ &\quad - (\beta_2'\lambda + \beta_2)\Big\{-s\sin\alpha\sin\varkappa s - \cos\alpha\cos\varkappa s + \\ &\quad + \sin\alpha\int_a^b q(w)\cos\{s(b-w)\}\cos\{s(w-a)\}\,dw + O(s^{-1})\Big\}\end{aligned}$$

for sufficiently large positive λ. Now put $\lambda = \lambda_n$, $s_n = \lambda_n^{1/2} = \varkappa^{-1}\{(n-1)\pi + \varepsilon_n\}$, where $\varepsilon_n = O(n^{-1})$. We have

$$s_n\sin\varkappa s_n = (-1)^{n-1}s_n\sin\varepsilon_n = (-1)^{n-1}s_n\varepsilon_n + O(n^{-1})$$

and

$$\cos\varkappa s_n = (-1)^{n-1}\cos\varepsilon_n = (-1)^{n-1} + O(n^{-2}).$$

Also

$$\begin{aligned}\cos\{s_n(b-w)\} &= \cos\left[\varkappa^{-1}\{(n-1)\pi + \varepsilon_n\}(b-w)\right] = \\ &= \cos\{(n-1)\pi - \varkappa^{-1}(n-1)\pi(w-a) + \varkappa^{-1}\varepsilon_n(b-w)\} = \\ &= (-1)^{n-1}\left[\cos\{\varkappa^{-1}(n-1)\pi(w-a)\}\cos\{\varkappa^{-1}\varepsilon_n(b-w)\} + \right. \\ &\quad \left. + \sin\{\varkappa^{-1}(n-1)\pi(w-a)\}\sin\{\varkappa^{-1}\varepsilon_n(b-w)\}\right]\end{aligned}$$

and

$$\cos\{s_n(w-a)\} \approx \cos[\varkappa^{-1}\{(n-1)\pi+\varepsilon_n\}(w-a)] =$$
$$= \cos\{\varkappa^{-1}(n-1)\pi(w-a)\}\cos\{\varkappa^{-1}\varepsilon_n(w-a)\} -$$
$$- \sin\{\varkappa^{-1}(n-1)\pi(w-a)\}\sin\{\varkappa^{-1}\varepsilon_n(w-a)\}.$$

Clearly both terms $\sin\{\varkappa^{-1}\varepsilon_n(b-w)\}$ and $\sin\{\varkappa^{-1}\varepsilon_n(w-a)\}$ are $O(n^{-1})$, and $\cos\{\varkappa^{-1}\varepsilon_n(b-w)\}$ and $\cos\{\varkappa^{-1}\varepsilon_n(w-a)\}$ are both $1+O(n^{-2})$, all uniformly in w. This shows that

$$\int_a^b q(w)\cos\{s(b-w)\}\cos\{s(w-a)\}\,\mathrm{d}w =$$
$$= (-1)^{n-1}\int_a^b q(w)\cos^2\{\varkappa^{-1}(n-1)\pi(w-a)\}\,\mathrm{d}w + O(n^{-1}).$$

Since also $s_n^{-1} = O(n^{-1})$ and $\lambda_n^{-1} = O(n^{-2})$ it follows from the above expression for $\omega(\lambda)$ that

$$0 = (\beta_1'\lambda_n + \beta_1)\{-\sin\alpha + O(n^{-1})\} -$$
$$- (\beta_2'\lambda_n + \beta_2)\{s_n\varepsilon_n\sin\alpha + \cos\alpha - I_n\sin\alpha + O(n^{-1})\},$$

where

$$I_n = \int_a^b q(w)\cos^2\{\varkappa^{-1}(n-1)\pi(w-a)\}\,\mathrm{d}w.$$

On division by λ_n we obtain

$$s_n\varepsilon_n\sin\alpha + \cos\alpha - \sin\alpha I_n + (\beta_1'/\beta_2')\sin\alpha = O(n^{-1}) \tag{3,1}$$

and so

$$\varepsilon_n = \varkappa(n-1)^{-1}\pi^{-1}\{-\cot\alpha - \beta_1'/\beta_2 + I_n\} + O(n^{-2}).$$

The analysis is similar in the other 3 cases. *Q.E.D.*

Remark. If we make the additional assumption that q has bounded variation over $[a, b]$ then in the above formulae for $\lambda_n^{1/2}$ we may replace the integral in each case by

$$\frac{1}{2}\int_a^b q(w)\,\mathrm{d}w.$$

These are then the formulae obtained by Fulton [1, pp. 300–301]. (Note that the first + sign in Fulton's formula $(4.14)_2$ should be a – sign.)

With the aid of the above formulae for the eigenvalues we can now establish formulae for the eigenvectors.

Theorem 2. (i) *In case* I *we have*

$$\psi_n(x) = \sqrt{\left(\frac{2}{\varkappa}\right)} \frac{\sin \alpha}{|\sin \alpha|} [A_n(x) \cos \{\varkappa^{-1}(n-1)\pi(x-a)\} -$$
$$- B_n(x) \sin \{\varkappa^{-1}(n-1)\pi(x-a)\}] + O(n^{-2})$$

uniformly in x over $[a, b]$, *where*

$$A_n(x) = 1 + \frac{1}{2\pi(n-1)} \int_a^b q(w)(b-w) \sin \{2\varkappa^{-1}(n-1)\pi(w-a)\} \,dw -$$
$$- \frac{\varkappa}{2\pi(n-1)} \int_a^x q(w) \sin \{2\varkappa^{-1}(n-1)\pi(w-a)\} \,dw$$

and

$$B_n(x) = \pi^{-1}(n-1)^{-1} \Big\{(b-x)\cot\alpha - (x-a)\beta_1'/\beta_2' +$$
$$+ (x-a)\int_a^b q(w)\cos^2\{\varkappa^{-1}(n-1)\pi(w-a)\}\,dw -$$
$$- \varkappa \int_a^x q(w)\cos^2\{\varkappa^{-1}(n-1)\pi(w-a)\}\,dw\Big\}.$$

The second component of Ψ_n *is*

$$R_b'(\psi_n) = \sqrt{\left(\frac{2}{\varkappa}\right)} \frac{(-1)^{n+1}\varrho\varkappa^2 \sin\alpha}{\beta_2'\pi^2(n-1)^2|\sin\alpha|} + O(n^{-3}).$$

(ii) *In case* II, $\psi_n(x)$ *is as in* (i), *with* $n-1$ *replaced throughout by* $n-\frac{1}{2}$ *and* $-\beta_1'/\beta_2'$ *replaced by* β_2/β_1'. *The second component is*

$$R_b'(\psi_n) = \sqrt{\left(\frac{2}{\varkappa}\right)} \frac{(-1)^n \beta_2\varkappa \sin\alpha}{\pi(n-1)|\sin\alpha|} + O(n^{-2}).$$

(iii) *In case* III *we have*

$$\psi_n(x) = -\sqrt{\left(\frac{2}{\varkappa}\right)} \cos\alpha[A_n(x)\sin\{\varkappa^{-1}(n-\tfrac{1}{2})\pi(x-a)\} +$$
$$+ B_n(x)\cos\{\varkappa^{-1}(n-\tfrac{1}{2})\pi(x-a)\}] + O(n^{-2})$$

uniformly in x over $[a, b]$, *where*

$$A_n(x) = 1 - \frac{1}{2(n-\frac{1}{2})\pi}\int_a^b q(w)(b-w)\sin\{2\varkappa^{-1}(n-\tfrac{1}{2})\pi(w-a)\}\,dw +$$
$$+ \frac{\varkappa}{2(n-\frac{1}{2})\pi}\int_a^x q(w)\sin\{2\varkappa^{-1}(n-\tfrac{1}{2})\pi(w-a)\}\,dw$$

and

$$B_n(x) = (n - \tfrac{1}{2})^{-1}\,\pi^{-1}\Big\{-(x - a)\,\beta_1'/\beta_2' +$$

$$+ (x - a)\int_a^b q(w)\sin^2\{\varkappa^{-1}(n - \tfrac{1}{2})\,\pi(w - a)\}\,\mathrm{d}w -$$

$$- \varkappa\int_a^x q(w)\sin^2\{\varkappa^{-1}(n - \tfrac{1}{2})\,\pi(w - a)\}\,\mathrm{d}w\Big\}\,.$$

The second component is

$$R_b'(\psi_n) = -\sqrt{\left(\frac{2}{\varkappa}\right)}\cos\alpha\,\frac{(-1)^{n+1}\,\varrho\varkappa^2}{\beta_2'\pi^2(n - \frac{1}{2})^2} + O(n^{-3})\,.$$

(iv) *In case* IV, $\psi_n(x)$ *is as in* (iii), *with* $n - \frac{1}{2}$ *replaced throughout by* n *and* $-\beta_1'/\beta_2'$ *replaced by* β_2/β_1'. *The second component is*

$$R_b'(\psi_n) = -\sqrt{\left(\frac{2}{\varkappa}\right)}\cos\alpha\,\frac{(-1)^n\,\beta_2\varkappa}{n\pi} + O(n^{-2})\,.$$

Proof. In case I, we have

$$\phi(x, \lambda_n) = \sin\alpha\cos\{s_n(x - a)\} - s_n^{-1}\cos\alpha\sin\{s_n(x - a)\} +$$

$$+ s_n^{-1}\sin\alpha\int_a^x q(w)\sin\{s_n(x - w)\}\cos\{s_n(w - a)\}\,\mathrm{d}w + O(n^{-2})\,,$$

and, with obvious abbreviations,

$$s_n = \varkappa^{-1}(n - 1)\,\pi + (n - 1)^{-1}\,(c + J_n) + O(n^{-2}) = \varkappa^{-1}(n - 1)\,\pi + \delta_n\,.$$

Next

$$\cos\{s_n(x - a)\} = \cos[\{\varkappa^{-1}(n - 1)\,\pi + \delta_n\}\,(x - a)] =$$

$$= \cos\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} - \delta_n(x - a)\sin\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} + O(n^{-2}) =$$

$$= \cos\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} - (n - 1)^{-1}\,(x - a)\,.$$

$$.\,(c + J_n)\sin\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} + O(n^{-2})$$

and

$$\sin\{s_n(x - a)\} = \sin\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} + O(n^{-1})\,.$$

Hence

$$\phi(x, \lambda_n) = \sin\alpha[\cos\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} - \tag{3,2}$$

$$- (n - 1)^{-1}\,(x - a)\,(c + J_n)\sin\{\varkappa^{-1}(n - 1)\,\pi(x - a)\}] -$$

$$- \varkappa(n - 1)^{-1}\,\pi^{-1}\cos\alpha\sin\{\varkappa^{-1}(n - 1)\,\pi(x - a)\} +$$

$$+ \varkappa(n-1)^{-1}\,\pi^{-1}\sin\alpha\int_a^x q(w)\sin\{\varkappa^{-1}(n-1)\,\pi(x-w)\}\,.$$

$$.\cos\{\varkappa^{-1}(n-1)\,\pi(w-a)\}\,\mathrm{d}w + O(n^{-2})\,.$$

The next step is to obtain an asymptotic formula for $\|\Phi_n\|^{-1}$, where

$$\|\Phi_n\|^2 = \int_a^b \phi^2(x,\lambda_n)\,\mathrm{d}x + \frac{1}{\varrho}\{\beta_1'\phi(b,\lambda_n) - \beta_2'\phi'(b,\lambda_n)\}^2\,.$$

It follows from Lemma 3 that

$$\int_a^b \phi^2(x,\lambda)\,\mathrm{d}x = \tfrac{1}{2}\sin^2\alpha(\varkappa + \tfrac{1}{2}s^{-1}\sin 2\varkappa s) -$$

$$-\tfrac{1}{2}s^{-1}\sin^2\alpha\int_a^b q(w)\,(b-w)\sin\{2s(w-a)\}\,\mathrm{d}w + O(\lambda^{-1})$$

if $\lambda > 0$ and so, as $\sin 2\varkappa s_n = O(n^{-1})$, we have

$$\int_a^b \phi^2(x,\lambda_n)\,\mathrm{d}x = \tfrac{1}{2}\varkappa\sin^2\alpha -$$

$$-\tfrac{1}{2}\varkappa\pi^{-1}(n-1)^{-1}\sin^2\alpha\int_a^b q(w)\,(b-w)\sin\{2\varkappa^{-1}(n-1)\,\pi(w-a)\}\,\mathrm{d}w + O(n^{-2})\,,$$

since $\lambda_n^{-1} = O(n^{-2})$. Now from the boundary conditions (1,3) we have

$$(3,3)\qquad \beta_1'\phi(b,\lambda_n) - \beta_2'\phi'(b,\lambda_n) = -\lambda_n^{-1}\{\beta_1\phi(b,\lambda_n) - \beta_2\phi'(b,\lambda_n)\}\,,$$

where we again use Lemma 3 to obtain $\phi(b,\lambda_n) = O(1)$ and $\phi'(b,\lambda_n) = O(n)$; it then follows that the expression (3,3) is $O(n^{-1})$. Hence we obtain the result

$$\|\Phi_n\|^2 = \tfrac{1}{2}\varkappa\sin^2\alpha\Big\{1 - \pi^{-1}(n-1)^{-1}\,.$$

$$.\int_a^b q(w)\,(b-w)\sin\{2\varkappa^{-1}\pi(n-1)\,(w-a)\}\,\mathrm{d}w\Big\} + O(n^{-2})\,,$$

whence

$$(3,4)\qquad \|\Phi_n\|^{-1} = \sqrt{(2\varkappa^{-1})}\,|\mathrm{cosec}\,\alpha|\Big\{1 + \tfrac{1}{2}\pi^{-1}(n-1)^{-1}\,.$$

$$.\int_a^b q(w)\,(b-w)\sin\{2\varkappa^{-1}\pi(n-1)\,(w-a)\}\,\mathrm{d}w\Big\} + O(n^{-2})\,.$$

Accordingly,

$$\psi_n(x) = \|\Phi_n\|^{-1} \phi(x, \lambda_n) =$$

$$= \sqrt{(2\varkappa^{-1})} \sin \alpha \, |\operatorname{cosec} \alpha| \Big\{ \cos \{\varkappa^{-1} \pi(n-1)(x-a)\} -$$

$$- (n-1)^{-1} (x-a)(c + I_n) \sin \{\varkappa^{-1} \pi(n-1)(x-a)\} -$$

$$- \varkappa \pi^{-1}(n-1)^{-1} \cot \alpha \{\sin \varkappa^{-1} \pi(n-1)(x-a)\} +$$

$$+ \varkappa \pi^{-1}(n-1)^{-1} \int_a^x q(w) \sin \{\varkappa^{-1} \pi(n-1)(x-w)\} \, .$$

$$. \cos \{\varkappa^{-1} \pi(n-1)(w-a)\} \, dw + \tfrac{1}{2}\pi^{-1}(n-1)^{-1} \cos \{\varkappa^{-1} \pi(n-1)(x-a)\} \, .$$

$$. \int_a^b q(w)(b-w) \sin \{2\varkappa^{-1} \pi(n-1)(w-a)\} \, dw \Big\} + O(n^{-2}) \, ,$$

and the stated formula for $\psi_n(x)$ in case I follows from this.

The second component of the eigenfunction Ψ_n is

$$\|\Phi_n\|^{-1} \{\beta_1' \phi(b, \lambda_n) - \beta_2' \phi'(b, \lambda_n)\} \, .$$

Now

$$\phi(b, \lambda_n) = -(-1)^n \sin \alpha + O(n^{-1})$$

from (3,2), and from Lemma 3 we deduce that

$$\phi'(b, \lambda_n) = -s_n \sin \alpha \sin \varkappa s_n - \cos \alpha \cos \varkappa s_n +$$

$$+ \sin \alpha \int_a^b q(w) \cos \{s_n(b-w)\} \cos \{s_n(w-a)\} \, dw + O(n^{-1}) =$$

$$= (-1)^n \Big\{ s_n \varepsilon_n \sin \alpha + \cos \alpha - \sin \alpha \int_a^b q(w) \cos^2 \{\varkappa^{-1} \pi(n-1)(w-a)\} \, dw \Big\} +$$

$$+ O(n^{-1}) = -(-1)^n \beta_1' \sin \alpha / \beta_2' + O(n^{-1}) \, ,$$

where we use results established and notation used in the proof of Theorem 1 (in particular the result (3,1)). Thus

$$\beta_1 \phi(b, \lambda_n) - \beta_2 \phi'(b, \lambda_n) = (-1)^n \sin \alpha (-\beta_1 + \beta_2 \beta_1' / \beta_2') + O(n^{-1}) = \tag{3,5}$$

$$= (-1)^n \varrho \sin \alpha / \beta_2' + O(n^{-1}) \, .$$

Also,

$$\lambda_n^{-1} = (s_n^{-1})^2 = \{\varkappa \pi^{-1}(n-1)^{-1} + O(n^{-3})\}^2 = \tag{3,6}$$

$$= \varkappa^2 \pi^{-2}(n-1)^{-2} + O(n^{-4}) \, .$$

The formula for the second component, $R_b'(\psi_n)$, of Ψ_n in case I is now obtained from (3,3)–(3,6).

The analysis is similar in the other three cases. Note that in cases II and IV one obtains the formulae for $R_b'(\psi_n)$ directly from those for $\psi_n(x)$, since then $R_b'(\psi_n) = \beta_1' \psi_n(b)$. Q.E.D.

Lemma 4. *In all four cases we have* $\psi_n'(x) = O(n)$, *uniformly in* x *over* $[a, b]$.

Proof. This follows readily from Lemma 2 and the proof of Theorem 2. In the proof of the latter it is seen that $\|\Phi_n\|^{-1} = O(1)$ in case I; this is also true in case II, while in cases III and IV one establishes similarly that $\|\Phi_n\|^{-1} = O(n)$. Q.E.D.

4. RELATIONS WITH TRIGONOMETRIC SERIES

In sections 3 and 5 of [1], Fulton examines the convergence of the eigenfunction expansion (2,1) and also equiconvergence with Fourier series. Here we consider results analogous to those in sections 1 and 2 of this author's paper [5] on Sturm-Liouville theory. We shall be dealing with functions $F(x) = (F_1(x), F_2)$ in the following Wiener-type space

$$\mathscr{W} = \{F \mid F_1 \in L[a, b] \text{ and } \sum_{n=0}^{\infty} |f_n| < \infty\},$$

where

$$f_n = \int_a^b F_1(x)\, \psi_n(x)\, dx . \tag{4,1}$$

Functions in $\mathscr{W}$ have the following property.

Lemma 5. *If* $F \in \mathscr{W}$ (F_2 *being arbitrary*) *then almost everywhere we have*

$$F(x) = \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle\, \Psi_n(x)$$

and

$$F_1(x) = \sum_{n=0}^{\infty} f_n\, \psi_n(x) .$$

Proof. Since $\sum_{n=0}^{\infty} |f_n| < \infty$ we have $\sum_{n=0}^{\infty} |f_n|^2 < \infty$.

Also $\sum_{n=0}^{\infty} |\varrho^{-1} F_2\, R_b'(\psi_n)|^2 < \infty$, since $R_b'(\psi_n) = O(n^{-1})$ by Theorem 2. Hence, from (2,2),

$$\sum_{n=0}^{\infty} |\langle F, \Psi_n \rangle|^2 \leqq 2 \sum_{n=0}^{\infty} |f_n|^2 + 2 \sum_{n=0}^{\infty} |\varrho^{-1} F_2\, R_b'(\psi_n)|^2 < \infty .$$

Thus $F \in L^2[a, b] \oplus \mathbb{C}$ and so by the Proof of Corollary 1.1 of [1]

$$F_1(x) = \sum_{n=0}^{\infty} \{f_n + \varrho^{-1} F_2 R_b'(\psi_n)\} \psi_n(x) \tag{4,2}$$

holds, with convergence in the $L^2[a, b]$ norm. Since $\sum_{n=0}^{\infty} |f_n| < \infty$ it follows from Theorem 2 of [1] that the series (4,2) converges pointwise almost everywhere. Hence the sum of the series is equal almost everywhere to $F_1(x)$. This deals with the first component of the eigenfunction expansion (2,1); the convergence of the second component to F_2 follows from (3.29) and (3.31) of [1], since $F_1 \in L^2[a, b]$. Again using Theorem 2 of [1] we obtain $F_1(x) = \sum_{n=0}^{\infty} f_n \psi_n(x)$ almost everywhere. *Q.E.D.*

Now we relate the coefficients (4,1) to trigonometric coefficients. From Theorem 2 we obtain for the first component of the eigenfunction Ψ_n the formula

$$\psi_n(x) = c\, T_n(x) + O(n^{-1})$$

uniformly in x, where $c = \pm 1$ and

$$T_n(x) = \sqrt{\frac{2}{\varkappa}}\, A\{p\pi\varkappa^{-1}(x - a)\}$$

with $A(y) = \cos y$ (cases I and II) or $\sin y$ (cases III and IV) and p given in the four cases by $n - 1$, $n - \frac{1}{2}$, $n - \frac{1}{2}$ and n respectively. We introduce the trigonometric coefficients

$$a_n = \int_a^b F_1(x)\, T_n(x)\, dx\,, \quad n \geqq 1\,, \tag{4,3}$$

for any $F_1 \in L[a, b]$.

The argument of [5, section 2] may now be applied to establish the following result, making use of Theorems 1 and 2 and Lemmas 4 and 5.

Theorem 3. *For any $F_1 \in L[a, b]$ the series $\sum_{n=0}^{\infty} |f_n|$ and $\sum_{n=1}^{\infty} |a_n|$ converge and diverge together, where f_n and a_n are given by* (4,1) *and* (4,3) *respectively.*

It may now be shown that in cases I and IV, in which ψ_n is asymptotically like a cosine or sine, there is an analogue of the trigonometric Wiener-Lévy theorem (see [6], p. 245, Theorem 5.2), as follows. To make available the notions of *oddness* and *even-ness*, we let $[a, b]$ be $[0, \pi]$.

Theorem 4. *If $F \in \mathscr{W}$ and $\xi(z)$ is a function of the complex variable z which is analytic in a set*

$$\bigcup_{x \in [0, \pi]} \{z \mid |z - F_1(x)| < \gamma\}\,, \tag{4,4}$$

for some $\gamma > 0$, then in case I $G \in \mathscr{W}$, *where* $G(x) = (G_1(x), G_2)$, $G_1(x) = \xi(F_1(x))$ *and G_2 is arbitrary. In case* IV *the same is true, provided also that ξ is an odd function.*

Proof. Let $F \in \mathscr{W}$. Then as $\sum_{n=0}^{\infty} |f_n| < \infty$ and (by Theorem 2) the functions ψ_n are uniformly bounded, the series $\sum_{n=0}^{\infty} f_n \psi_n(x)$ is uniformly and absolutely convergent to a continuous sum. We have seen that this sum is equal almost everywhere to $F_1(x)$ (cf. Lemma 5). If we re-define $F_1(x)$, if necessary, to be the sum of this series then the set (4,4) has a meaning.

Suppose that case I applies. Then

$$a_n = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\pi} F_1(x) \cos(n - 1)\, x\, dx$$

and, by the previous theorem, $\sum_{n=1}^{\infty} |a_n| < \infty$. Give $F_1(x)$ an *even* extension to the interval $[-\pi, \pi]$ and let

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_1(x)\, e^{-inx}\, dx$$

for any integer n. (These are of course the classical Fourier coefficients of F_1.) It is easily seen that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ (because F_1 is even) and so by the Wiener-Lévy theorem we have $\sum_{n=-\infty}^{\infty} |d_n| < \infty$, where

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(F_1(x))\, e^{-inx}\, dx\,.$$

Again using the even-ness of F_1 we see that $\sum_{n=1}^{\infty} |e_n| < \infty$, where

$$e_n = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\pi} \xi(F_1(x)) \cos(n - 1)\, x\, dx$$

and so $G \in \mathscr{W}$, by Theorem 3.

The method is very similar in case IV. Now we have

$$a_n = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\pi} F_1(x) \sin nx\, dx$$

and we give $F_1(x)$ an *odd* extension to $[-\pi, \pi]$. With c_n and d_n as before one has again $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ and hence $\sum_{n=-\infty}^{\infty} |d_n| < \infty$, since ξ is odd. At this point we use the oddness of both F_1 and ξ to obtain $\sum_{n=1}^{\infty} |g_n| < \infty$, where

$$g_n = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\pi} \xi(F_1(x)) \sin nx \, dx \, ,$$

and we may now apply Theorem 3 to obtain $G \in \mathscr{W}$. *Q.E.D.*

Remark. If $F(x) = (F_1(x), F_2)$ and $F_1 \in L[a, b]$ then by (2,2) and (4,1)

$$\langle F, \Psi_n \rangle = f_n + \varrho^{-1} F_2 \, R_b'(\psi_n) \, .$$

If $\beta_2' = 0$ (cases II and IV) and $F_2 \neq 0$ it follows from the asymptotic formulae for $R_b'(\psi_n)$ (cf. Theorem 2) that if $\sum_{n=0}^{\infty} |\langle F, \Psi_n \rangle| < \infty$ then $\sum_{n=0}^{\infty} |f_n|$ must diverge, and so the series $\sum_{n=1}^{\infty} |a_n|$ diverges, by Theorem 3. Thus, the above definition of $\mathscr{W}$ is preferable to the use of the set

$$\{F \mid F_1 \in L[a, b] \text{ and } \sum_{n=0}^{\infty} |\langle F, \Psi_n \rangle| < \infty\} \, ,$$

since if we used this instead of $\mathscr{W}$ we would lose the case IV version of Theorem 4. In cases I and III however ($\beta_2' \neq 0$), we have $R_b'(\psi_n) = O(n^{-2})$ and so the conditions $\sum_{n=0}^{\infty} |f_n| < \infty$ and $\sum_{n=0}^{\infty} |\langle F, \Psi_n \rangle| < \infty$ are equivalent.

The above comments notwithstanding, in all four cases if $F_1 \in L[a, b]$ and $\sum_{n=0}^{\infty} |\langle F, \Psi_n \rangle| < \infty$ then $F(x) = \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle \, \Psi_n(x)$ and $F_1(x) = \sum_{n=0}^{\infty} f_n \, \psi_n(x)$ still hold almost everywhere. To see this, let $G(x) = \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle \, \Psi_n(x)$. (The series converges uniformly and absolutely because of the asymptotic formulae in Theorem 2.) Then for any p we have

$$\langle G, \Psi_p \rangle = \int_a^b \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle \, \psi_n(x) \, \psi_p(x) \, dx + \varrho^{-1} \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle \, R_b'(\psi_n) \, R_b'(\psi_p) =$$

$$= \sum_{n=0}^{\infty} \langle F, \Psi_n \rangle \left\{ \int_a^b \psi_n(x) \, \psi_p(x) \, dx + \varrho^{-1} \, R_b'(\psi_n) \, R_b'(\psi_p) \right\} = \langle F, \Psi_p \rangle \, ,$$

and so $G(x) = F(x)$ almost everywhere. The inversion above is justified by the uniform boundedness of the functions ψ_n. It then follows that $F_1(x) = \sum_{n=0}^{\infty} f_n \, \psi_n(x)$ almost everywhere because of Theorem 2 of [1].

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