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FACTORIZATIONS OF MATRICES AND FUNCTIONS OF TWO VARIABLES

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In this paper we shall give a characterization of functions and matrices that can be decomposed in the forms

$$h(x, t) = \sum_{k=1}^{n} f_k(x)\, g_k(t)\,, \quad \text{and} \quad (a_{ij}) = \big(\sum_{k=1}^{n} b_k(i)\, c_k(j)\big)\,.$$

In the case when h is sufficiently many times differentiable we get a characterization and a construction of f_k and g_k from h in terms of partial and ordinary differential equations.

Without regularity conditions on the function h and for matrices, we give a characterization and even explicit formulas for evaluating f_k, g_k, and $b_k(i)$, $c_k(j)$. These formulas enable us to perform efficient computer computations, because the values of $f_k(x)$, $g_k(t)$, as well as $b_k(i)$, $c_k(j)$ can be evaluated parallelly for different (x, t), and (i, j) by pointwise multiplications only.

Moreover, if h is continuous or of a class C^d on $I \times J$, then the same kind of regularity holds for f_k on I and g_k on J for all $k = 1, \ldots, n$. This means that we also have explicit formulas for solutions of the partial and ordinary differential equations mentioned above.

We write

$$D_m(h) := \begin{Vmatrix} h & h_t & h_{tt} & \ldots & h_{t^m} \\ h_x & h_{xt} & h_{xtt} & \ldots & h_{xt^m} \\ \ldots \\ h_{x^m} & h_{x^m t} & h_{x^m tt} & \ldots & h_{x^m t^m} \end{Vmatrix}$$

for a function h of x and t with continuous $\partial^{i+j}h/\partial^i x\, \partial^j t = h_{x^i t^j}$ on $I \times J \subset R^2$, $i, j \leqq m$. (Here I and J are unions of intervals.)

Theorem 1. *If a function $h : I \times J \to R$, having continuous derivatives $h_{x^i t^j}$ for $i, j \leqq n$, can be written in the form*

$$(1) \qquad h(x, t) = \sum_{k=1}^{n} f_k(x)\, g_k(t) \quad \text{on} \quad I \times J\,,$$

then

(2) $$\det D_n(h) \equiv 0 \quad on \quad I \times J .$$

If, moreover, $f_k \in C^n(I)$, $g_k \in C^n(J)$ *and*

(2_1) $$\det (f_k^{(j)}(x)) \neq 0 \quad for\ all \quad x \in I \quad and \quad \det (g_k^{(j)}(t)) \neq 0 \quad for\ all \quad t \in J ,$$

then also

(2_2) $$\det D_{n-1}(h) \neq 0 \quad for\ all \quad (x, t) \in I \times J$$

holds.

If h *satisfies* (2) *and* (2_2) *then there exist* $f_k \in C^n(I)$ *and* $g_k \in C^n(J)$, $k = 1, \ldots, n$, *such that* (1) *and* (2_1) *hold (and thus* f_k *and* g_k *are linearly independent). All decompositions of* h *of the form*

$$h(x, t) = \sum_{k=1}^{n} \bar{f}_k(x)\, \bar{g}_k(t)$$

are exactly those for which

$$(\bar{f}_1, \ldots, \bar{f}_n) = (f_1, \ldots, f_n) \,.\, C^{\mathrm{T}} , \quad and \quad (\bar{g}_1, \ldots, \bar{g}_n) = (g_1, \ldots, g_n) \,.\, C^{-1} ,$$

where C *is an arbitrary* n *by* n *nonsingular constant matrix,* C^{T} *and* C^{-1} *being its transpose and inverse, respectively.*

Remark 1. The functions f_k and g_k in (1) can be constructed from an h satisfying (2) and (2_2) as solutions of two ordinary linear differential equations with coefficients evaluated from h (see (4_1) below for f_k and its tianspose anologue for g_k). For constructing f_k and g_k from h satisfying (2) and (2_1), see also Theorem 3 and Remark 5 below.

Proof of Theorem 1. If h is of the form (1), then $h(x, t_0), h_t(x, t_0), \ldots, h_{t^n}(x, t_0)$ are $n + 1$ functions, each of them is a linear combination with constant coefficients of n functions $f_1, \ldots, f_n$, so $h, h_t, \ldots, h_{t^n}$ are linearly dependent. Hence their Wronski determinant of order $n + 1$ is zero, i.e. $\det D_n(h) = 0$. We also have

(3)
$$D_{n-1}(h) = \begin{Vmatrix} \sum f_k g_k & \sum f_k g_k' & \ldots & \sum f_k g_k^{(n-1)} \\ \sum f_k' g_k & \sum f_k' g_k' & \ldots & \sum f_k' g_k^{(n-1)} \\ \ldots & & & \\ \sum f_k^{(n-1)} g_k & \sum f_k^{(n-1)} g_k' & \ldots & \sum f_k^{(n-1)} g_k^{(n-1)} \end{Vmatrix} =$$
$$= \begin{Vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \ldots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{Vmatrix} . \begin{Vmatrix} g_1 & g_1' & \ldots & g_1^{(n-1)} \\ g_2 & g_2' & \ldots & g_2^{(n-1)} \\ \ldots & & & \\ g_n & g_n' & \ldots & g_n^{(n-1)} \end{Vmatrix} ,$$

hence (2_2) holds because of (2_1).

If h satisfies (2) and (2_2), then there exist $A_i = A_i(x, t)$, $i = 0, \ldots, n - 1$, such that

$$(4) \qquad \begin{aligned} h_{t^n} &= A_0 h + A_1 h_t + \ldots + A_{n-1} h_{t^{n-1}} \\ h_{t^n x} &= A_0 h_x + A_1 h_{tx} + \ldots + A_{n-1} h_{t^{n-1}x} \\ &\ldots \\ h_{t^n x^n} &= A_0 h_{x^n} + A_1 h_{tx^n} + \ldots + A_{n-1} h_{t^{n-1}x^n} \end{aligned}$$

holds. These A_i are differentiable with respect to x, because the system of the first n equations of (4) has a (unique) system of solutions $A_0, \ldots, A_{n-1}$ which are quotients of polynomials in $h_{x^i t^j}$, $i \leqq n - 1$, $j \leqq n$. By differentiating (4_1) with respect to x and subtracting (4_2) we get

$$\frac{\partial A_0}{\partial x} h + \frac{\partial A_1}{\partial x} h_t + \ldots + \frac{\partial A_{n-1}}{\partial x} h_{t^{n-1}} = 0 .$$

Similarly (4_2) and (4_3) give

$$\frac{\partial A_0}{\partial x} h_x + \frac{\partial A_1}{\partial x} h_{xt} + \ldots + \frac{\partial A_{n-1}}{\partial x} h_{xt^{n-1}} = 0$$

and analogously up to

$$\frac{\partial A_0}{\partial x} h_{x^{n-1}} + \frac{\partial A_1}{\partial x} h_{x^{n-1}t} + \ldots + \frac{\partial A_{n-1}}{\partial x} h_{x^{n-1}t^{n-1}} = 0 .$$

Since $\det D_{n-1}(h) \neq 0$, all $\partial A_i/\partial x = 0$. Hence A_i are functions of t only.

Thus h satisfies (4_1) for all $x_0 \in I$ and it can be written as

$$(5) \qquad h(x_0, t) = \sum_{k=1}^{n} f_k(x_0)\, g_k(t) ,$$

where g_k are independent solutions of (4_1); hence also $g_k \in C^n(J)$ and $\det (g_k^{(j)}) \neq 0$ on J, cf. (2_1). Any other set $\bar{g}_k$ of independent solutions of (4_1) satisfies

$$(6) \qquad (\bar{g}_1, \ldots, \bar{g}_n) = (g_1, \ldots, g_n)\, C^{-1}$$

with a nonsingular constant n by n matrix C.

Now, from (5) we have $f_k \in C^n(I)$ and due to (3), where $\det D_{n-1}(h) \neq 0$, the first part of (2_1) holds too and f_k are independent (since they also satisfy a linear differential equation). Moreover, after the g_k were chosen, the f_k are uniquely determined for a given h in order that (3) be satisfied. That also gives $(\bar{f}_1, \ldots, \bar{f}) = (f_1, \ldots, f_n) \,.\, C^{\mathrm{T}}$, if the $\bar{g}_k$ are chosen as in (6). Q.E.D.

Theorem 2. *A matrix* $H = (H_{ij})$, $i = 1, \ldots, r$; $j = 1, \ldots, s$, *can be written in the form*

$$(7) \qquad (H_{ij}) = (\sum_{k=1}^{n} F_k(i)\, G_k(j))$$

with n independent vectors $F_1(i), \ldots, F_n(i)$, *and n independent vectors* $G_1(j), \ldots$ $\ldots, G_n(j)$, *if and only if*

$$\operatorname{rank} H = n\,, \quad n \leqq \min(r, s)\,.$$

If this assumption is satisfied, then all decompositions of H in the form

$$(H_{ij}) = \left(\sum_{k=1}^{n} \bar{F}_k(i)\, \bar{G}_k(j)\right)$$

are exactly those for which

$$\bar{F} = F\,.\,C\,, \quad \bar{G} = C^{-1}\,.\,G\,,$$

C being a nonsingular real n by n matrix,

$$\bar{F} = (\bar{F}_{ik}) := (\bar{F}_k(i))\,, \quad F = (F_{ik}) := (F_k(i))\,,$$

$$\bar{G} = (\bar{G}_{kj}) := (\bar{G}_k(j))\,, \quad G = (G_{kj}) := (G_k(j))\,.$$

Remark 2. If the linear independence of F_k and G_k is not supposed, then rank $H \leqq$ $\leqq n \leqq \min(r, s)$.

Remark 3. If the assumption of Theorem 2 is satisfied, then all the decompositions (7) can be constructed from H by using (8) below.

Proof of Theorem 2. 1° If (7) is satisfied, i.e., $H = F\,.\,G$, then rank $F \leqq n$, $n \leqq r$, rank $G \leqq n$, $n \leqq s$, hence rank $H \leqq n$, $n \leqq \min(r, s)$, cf. Remark 2. If F_k and G_k are linearly independent, then rank $F = n$, rank $G = n$, and rank $H = n$.

2° Let rank $H = n$. Then H can be reindexed in the form

$$H = \begin{Vmatrix} H^* & H^*\,.\,M \\ P\,.\,H^* & P\,.\,H^*\,.\,M \end{Vmatrix}$$

where H^* is a nonsingular n by n matrix, M and P are suitable n by $(s - n)$ and $(r - n)$ by n matrices, respectively.

Choose any nonsingular n by n matrix F^* and put $G^* := F^{*-1}\,.\,H^*$. Evidently $H^* = F^*\,.\,G^*$ and any such relation for a given H^* can be established exactly by taking $F^*\,.\,C$ and $C^{-1}\,.\,G^*$ instead of F^* and G^*, respectively, for any nonsingular n by n matrix C. Then we can write

(8) $$H = \begin{Vmatrix} F^* \\ P\,.\,F^* \end{Vmatrix} \,.\, \|G^*, G^*\,.\,M\|\,,$$

or

$$H = F\,.\,G \quad \text{for} \quad F := \begin{Vmatrix} F^* \\ P\,.\,F^* \end{Vmatrix}, \quad G := \|G^*, G^*\,.\,M\|\,.$$

All factorizations of H into $\bar{F} \,.\, \bar{G}$ are exactly those where

$$\bar{F} = \left\| \begin{matrix} F^* \,.\, C \\ P \,.\, F^* \,.\, C \end{matrix} \right\| = F \,.\, C \,, \quad \text{and} \quad \bar{G} = \left\| C^{-1} \,.\, G^*, \; C^{-1} \,.\, G^* \,.\, M \right\| = C^{-1} \,.\, G \,,$$

Q.E.D.

Now we shall apply Theorem 2 to get a characterization of functions h satisfying (1) without requiring the differentiability of h.

According to [1, Sec. 4.2.5], n functions $\phi_k : S \to R$, (S is a subset of R, S need not be an interval; $k = 1, \ldots, n$) are linearly independent on $S \subset R$, if there exist n points x_k, $k = 1, \ldots, n$, in S such that the matrix

$$W[\phi_k, x_k]_{k=1}^n = \left\| \begin{matrix} \phi_1(x_1), \ldots, \phi_n(x_1) \\ \phi_1(x_n), \ldots, \phi_n(x_n) \end{matrix} \right\|$$

is regular.

Theorem 3. *Let I and J be arbitrary nonempty sets.*

A function $h : I \times J \to R$ can be written in the form (1) with linearly independent f_k and g_k if and only if the maximum of the rank of the matrices

$$\left(h(x_i, t_j)\right), \quad i = 1, \ldots, r\,; \quad j = 1, \ldots, s\,,$$

is n when $x_i \in I$, $t_j \in J$, r and s being arbitrary integers.

If, in addition, I and J are intervals, $h \in C^d(I \times J)$, $d \geqq 0$, then $f_k \in C^d(I)$ and $g_k \in C^d(J)$ for all $k = 1, \ldots, n$.

Remark 4. The explicit formula for evaluating f_k and g_k from a given h will be given in (10) below. From the point of view of computation it is important that, when constant matrices H^* and G^* are chosen, then the values of f_k at x and g_k at t depend only on the values of h at (x_k, t) and at (x, t_k), $k = 1, \ldots, n$, respectively. Hence the f_k and g_k can be evaluated separately and at the same time for different arguments, see (10).

Remark 5. Note that the formula (10) below also gives in a constructive way the functions f_k and g_k by which the solutions h of the nonlinear partial differential equation $\det D_n(h) = 0$ can be decomposed in the sense of (1) without the necessity of solving linear differential equations as mentioned in Remark 1.

Proof of Theorem 3. 1° If $h(x, t) = \sum_{k=1}^{n} f_k(x)\, g_k(t)$ on $I \times J$, then (7) i satisfied for

$$H_{ij} := h(x_i, t_j)\,, \quad F_k(i) := f_k(x_i)\,, \quad G_k(j) := g_k(t_j)$$

and any r-tuple of $x_i \in I$ and s-tuple of $t_j \in J$. In view of Remark 2, rank $(H_{ij}) \leqq n$. Since f_k and g_k are linearly independent, there exist an n-tuple of $x_i \in I$ and an n-tuple of $t_j \in J$ such that both $W[f_k, x_k]$ and $W[g_k, t_k]$ are nonsingular. For these n-tuples, $(H_{ij}) = W[f_k, x_k] . W^T[g_k, t_k]$, hence the rank n is achieved.

2° Let $H^* := (H^*_{ij}) = (h(x_i, t_j))$, $i = 1, \dots, n$: $j = 1, \dots, n$, be a nonsingular n by n matrix. Consider an $(n + 1)$ by $(n + 1)$ matrix

$$\begin{Vmatrix} h(x_1, t_1), \dots, h(x_1, t_n), h(x_1, t) \\ h(x_2, t_1), \dots, h(x_2, t_n), h(x_2, t) \\ \dots \\ h(x_n, t_1), \dots, h(x_n, t_n), h(x_n, t) \\ h(x, t_1), \dots, h(x, t_n), \quad h(x, t) \end{Vmatrix} = \begin{Vmatrix} H^* & a \\ b & c \end{Vmatrix},$$

where a is an n by 1 vector, and b is a 1 by n vector. The rank of the matrix is n.

Choose any nonsingular n by n matrix C. Then

$$(9) \qquad \begin{Vmatrix} H^* . C \\ b . C \end{Vmatrix} . \| C^{-1}, C^{-1} . H^{*-1} . a \| = \begin{Vmatrix} H^* & a \\ b & c \end{Vmatrix},$$

see also (8). Define

$$(f_1(x), \dots, f_n(x)) := (h(x, t_1), \dots, h(x, t_n)) . C = b . C,$$

$$(10) \qquad \begin{Vmatrix} g_1(t) \\ \dots \\ g_n(t) \end{Vmatrix} := C^{-1} . H^{*-1} . \begin{Vmatrix} h(x_1, t) \\ \dots \\ h(x_n, t) \end{Vmatrix} = C^{-1} . H^{*-1} . a$$

$x \in I$, $t \in J$. Due to (9),

$$c = b . C . C^{-1} . H^{*-1} . a = b . H^{*-1} . a,$$

and we have

$$h(x, t) = \sum_{k=1}^{n} f_k(x)\, g_k(t).$$

The matrices C and H^* in (10) are constant, hence, if $h \in C^d(I \times J)$, $d \geqq 0$, I and J are intervals, then
$x \mapsto h(x, t) \in C^d(I)$ and $t \mapsto h(x, t) \in C^d(J)$, and also $f_k \in C^d(I)$, $g_k \in C^d(J)$ for all $k = 1, \dots, n$. Q.E.D.

A program for evaluating f_k and g_k from a given h may be constructed as follows (dot denotes matrix multiplication):

STEP 1 For a sufficiently large or dense set $\{(x_i, t_j); i = 1, \dots, s; j = 1, \dots, r\}$, determine the rank of the matrix $H = (h(x_i, t_j))$ as n, find a regular n by n submatrix H^* of H, the corresponding indices forming the sets K and L (each of the cardinality n).

STEP 2 Choose a regular n by n matrix C.

STEP 3(i) Take the row vector $\{h(x_i, t_l); l \in L\}$ from H^* and determine the vector $(f_1(x_i), \dots, f_n(x_i)) := \{h(x_i, t_l); l \in L\} . C$ for $i = 1, 2, \dots, s$.

STEP 4(j) Take the column vector $\{h(x_k, t_j);\ k \in K\}$ from H^* and determine the

$$\text{vector} \begin{Vmatrix} g_1(t_j) \\ \dots \\ g_n(t_j) \end{Vmatrix} := C^{-1} \,.\, H^{*-1} \,.\, \{h(x_k, t_j);\ k \in K\} \text{ for } j = 1, 2, \dots, r\,.$$

STEP 5 Check the relation

$$(f_1(x_i), \dots, f_n(x_i)) \,.\, \begin{Vmatrix} g_1(t_j) \\ \dots \\ g_n(t_j) \end{Vmatrix} = h(x_i, t_j)\,.$$

For $i = 1, \dots s$ and $j = 1, \dots, r$ it should be satisfied, otherwise there is an error in the computation.

STEP 6 Now, we may enlarge the initial set $\{(x_i, t_j);\ i = 1, \dots, s;\ j = 1, \dots, r\}$ by adding (x_i, t_j), $i > s$ and/or $j > r$. Go to STEP 3(i), $i > s$, and STEP 4(j), $j > r$. Then do STEP 5 for $i > s$, $j > r$. If all the relations are satisfied, then the extensions of $f_k(x_i)$ for $i > s$ and $g_k(t_j)$ for $j > r$ form the factorization (1). If there is an error here, we may either accept the extended f_k and g_k as approximations of our factorization, or we may change the initial set of (x_i, t_j), or enlarge it by adding points with $i > s, j > r$.

Remark 6. STEP 3 and STEP 4 are independent of each other, and also STEP 3(i) and STEP 3(i') as well as STEP 4(j) and STEP 4(j') are independent, hence they can be performed simultaneously.

Remark 7. All arithmetic operations are expressible by pointwise multiplications only.

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