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CYCLICALLY ORDERED SETS

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It is well known that it is impossible to define an orientation of a circle by means of a binary relation, but it is sufficient to use a ternary relation. Such a relation, the so called cyclic order relation, must be asymmetric, transitive, cyclic and complete (definition of these concepts are below). The aim of this paper is to derive properties of those relations which are not necessarily complete — an analogue of order relations in connection with linear order relations.

1. TERNARY RELATIONS

1.1. Definition. Let G be a set. A *ternary relation* T on the set G is any subset of the 3[rd] cartesian power $G^3 : T \subseteq G^3$.

1.2. Definition. Let G be a set, T a ternary relation on G. This relation is called:
asymmetric, iff $(x, y, z) \in T \Rightarrow (z, y, x) \bar{\in} T$
transitive, iff $(x, y, z) \in T$, $(x, z, u) \in T \Rightarrow (x, y, u) \in T$
cyclic, iff $(x, y, z) \in T \Rightarrow (y, z, x) \in T$
complete, iff $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow$ there exists a permutation (u, v, w) of the sequence (x, y, z) such that $(u, v, w) \in T$

1.3. Lemma. *Let G be a set, T a ternary relation on the set G which is asymmetric and cyclic. Let $x, y, z \in G$, $(x, y, z) \in T$. Then $(y, z, x) \in T$, $(z, x, y) \in T$, $(z, y, x) \bar{\in} T$, $(y, x, z) \bar{\in} T$, $(x, z, y) \bar{\in} T$.*

Proof. $(y, z, x) \in T$, $(z, x, y) \in T$ follows from the assumption that T is cyclic, $(z, y, x) \bar{\in} T$ follows from the asymmetry of T. If $(y, x, z) \in T$, then $(z, y, x) \in T$ because T is cyclic and this is a contradiction. Analogously $(x, z, y) \in T$ implies $(z, y, x) \in T$, a contradiction.

1.4. Lemma. *Let G be a set, T a ternary relation on G which is asymmetric and cyclic. Let $x, y, z \in G$, $(x, y, z) \in T$. Then $x \neq y \neq z \neq x$.*

Proof. If $x = z$, then $(x, y, x) \in T$ which contradicts the asymmetry of T. If

$x = y$, then (x, x, z), thus $(x, z, x) \in T$ because T is cyclic and this contradicts the asymmetry. Analogously $y = z$ implies $(x, y, y) \in T$, thus $(y, x, y) \in T$, a contradiction.

1.5. Lemma. *Let G be a set, T a ternary relation on G which is asymmetric, cyclic and complete. Let $x, y, z \in G$, $x \neq y \neq z \neq x$. Then just one from the possibilities $(x, y, z) \in T$, $(z, y, x) \in T$ holds.*

Proof. At most one of the possibilities $(x, y, z) \in T$, $(z, y, x) \in T$ holds, because T is asymmetric. As T is complete, there exists a permutation (u, v, w) of the sequence (x, y, z) such that $(u, v, w) \in T$. If this permutation is even, then $(x, y, z) \in T$ because T is cyclic; if this permutation is odd, then $(z, y, x) \in T$.

1.6. Theorem. *Let G be a set, T a cyclic ternary relation on G. T is transitive if and only if one of the following equivalent conditions holds:*

(1) $(x, y, z) \in T,\ (x, u, y) \in T \Rightarrow (x, u, z) \in T$,
(2) $(x, y, z) \in T,\ (x, u, y) \in T \Rightarrow (u, y, z) \in T$,
(3) $(x, y, z) \in T,\ (y, u, z) \in T \Rightarrow (x, y, u) \in T$,
(4) $(x, y, z) \in T,\ (y, u, z) \in T \Rightarrow (x, u, z) \in T$.

Proof. The transitivity of T is equivalent to (1) according to the definition.

(1) $\Rightarrow$ (2): Let (1) hold and let $x, y, z, u \in G$, $(x, y, z) \in T$, $(x, u, y) \in T$. As T is cyclic, $(y, x, u) \in T$, $(y, z, x) \in T$ hold and thus $(y, z, u) \in T$ according to (1). The cyclicity of T then implies $(u, y, z) \in T$ and (2) holds.

(2) $\Rightarrow$ (3): Let (2) hold and let $x, y, z, u \in G$, $(x, y, z) \in T$, $(y, u, z) \in T$. Then $(z, y, u) \in T$, $(z, x, y) \in T$ and according to (2), $(x, y, u) \in T$. Hence (3) holds.

(3) $\Rightarrow$ (4): Let (3) hold and let $x, y, z, u \in G$, $(x, y, z) \in T$, $(y, u, z) \in T$. Then $(u, z, y) \in T$, $(z, x, y) \in T$ and according to (3), $(u, z, x) \in T$. As T is cyclic, $(x, u, z) \in T$ and, therefore, (4) holds.

(4) $\Rightarrow$ (1): Let (4) hold and let $x, y, z, u \in G$, $(x, y, z) \in T$, $(x, u, y) \in T$. Then $(u, y, x) \in T$, $(y, z, x) \in T$ and according to (4), $(u, z, x) \in T$. Hence $(x, u, z) \in T$ and (1) holds.

1.7. Definition. Let G be a set, T a ternary relation on G. The ternary relation T^* on G defined by

$$(x, y, z) \in T^* \Leftrightarrow (z, y, x) \in T$$

is called the *dual relation* to T.

1.8. Lemma. *Let G be a set, T a ternary relation on G. Then:*

(1) *If T is asymmetric, then T^* is asymmetric,*
(2) *if T is cyclic, then T^* is cyclic,*
(3) *if T is complete, then T^* is complete,*
(4) *if T is transitive and cyclic, then T^* is transitive.*

Proof. (1) and (3) are trivial.

(2): Let T be cyclic and $x, y, z \in G$, $(x, y, z) \in T^*$. Then $(z, y, x) \in T$, thus $(x, z, y) \in T$ and $(y, z, x) \in T^*$, i.e. T^* is cyclic.

(4): Let T be transitive and cyclic and let $x, y, z, u \in G$, $(x, y, z) \in T^*$, $(x, z, u) \in T^*$. Then $(u, z, x) \in T$, $(z, y, x) \in T$ and by 1.6. (4) $(u, y, x) \in T$. Hence $(x, y, u) \in T^*$ and T^* is transitive.

1.9. Denotation. Let G be a set, let T be a ternary relation on G and let $x_0 \in G$. We denote by ϱ_{T,x_0} the binary relation on G defined as follows:

$$(x, y) \in \varrho_{T,x_0} \Leftrightarrow (x_0, x, y) \in T .$$

The following theorem shows why we call a ternary relation T "asymmetric" and "transitive".

1.10. Theorem. *Let G be a set, let T be a ternary relation on G. Then:*

(1) *ϱ_{T,x_0} is a transitive binary relation on G for each $x_0 \in G$ if and only if the ternary relation T is transitive.*

(2) *If T is cyclic then ϱ_{T,x_0} is an asymmetric binary relation on G for each $x_0 \in G$ if and only if T is asymmetric.*

Proof. (1) Let ϱ_{T,x_0} be a transitive binary relation for each $x_0 \in G$ and let $(x, y, z) \in T$, $(x, z, u) \in T$. Then $(y, z) \in \varrho_{T,x}$, $(z, u) \in \varrho_{T,x}$, thus $(y, u) \in \varrho_{T,x}$ and $(x, y, u) \in T$. Hence T is transitive. Conversely, let T be transitive and $x_0 \in G$, $(x, y) \in \varrho_{T,x_0}$, $(y, z) \in \varrho_{T,x_0}$. Then $(x_0, x, y) \in T$, $(x_0, y, z) \in T$, thus $(x_0, x, z) \in T$ and $(x, z) \in \varrho_{T,x_0}$. This implies that ϱ_{T,x_0} is transitive.

(2) Let T be cyclic and let ϱ_{T,x_0} be an asymmetric binary relation for each $x_0 \in G$. Let $(x, y, z) \in T$. Then $(z, x, y) \in T$, thus $(x, y) \in \varrho_{T,z}$ and $(y, x) \bar{\in} \varrho_{T,z}$. This implies $(z, y, x) \bar{\in} T$ and T is asymmetric. Conversely let T be cyclic and asymmetric and let $x_0 \in G$, $(x, y) \in \varrho_{T,x_0}$. Then $(x_0, x, y) \in T$, thus $(x, y, x_0) \in T$ and $(x_0, y, x) \bar{\in} T$. This means $(y, x) \bar{\in} \varrho_{T,x_0}$, and ϱ_{T,x_0} is asymmetric.

2. CYCLIC ORDER

2.1. Definition. Let G be a set, C a ternary relation on G which is asymmetric, transitive and cyclic. Then C is called a *cyclic order* on the set G and the pair (G, C) is called a *cyclically ordered set*. If, moreover, card $G \geqq 3$ and C is complete then C is called a *complete cyclic order* on G and (G, C) is called *complete cyclically ordered set* or a *cycle*. If $C = \emptyset$ then C is called a *discrete cyclic order* and (G, C) is called a *discrete cyclically ordered set*.

2.2. Remark. *If C is a cyclic order on a set G then C^* is also a cyclic order on G. If (G, C) is a cycle then (G, C^*) is a cycle.*

Proof follows from 1.8.

2.3. Remark. *Let C be a cyclic order on a set G, let $H \subseteq G$. Then the restriction $C_{|H} = C \cap H^3$ is a cyclic order on H.*

Proof is trivial.

With respect to this fact, any subset H of a cyclically ordered set (G, C) will be mentioned as a cyclically ordered set with the induced ternary relation $C_{|H} = C \cap \cap H^3$. Especially, if $C_{|H}$ is a complete cyclic order on H, then H is called a *cycle* in (G, C). For instance, if $x, y, z \in G$, $(x, y, z) \in C$, then $\{x, y, z\}$ is a cycle in (G, C).

2.4. Definition. Let (G, C) be a cyclically ordered set and let H be a cycle in (G, C). H is called a *maximal cycle* in (G, C), iff it is not contained in any cycle in (G, C) as a proper subset.

2.5. Theorem. *Let (G, C) be a cyclically ordered set. Then any cycle in (G, C) is contained in a maximal cycle in (G, C).*

Proof. Let H be a cycle in (G, C). Denote by $\mathcal{S}$ the set of all cycles in (G, C) containing H; $\mathcal{S}$ is ordered by the set inclusion. Let $\mathcal{S}_1 \subseteq \mathcal{S}$ be a chain in $\mathcal{S}$. Put $K = \bigcup \mathcal{S}_1$; we show that K is a cycle in (G, C). Let $x, y, z \in K$, $x \neq y \neq z \neq x$. Then $x \in H_i$, $y \in H_j$, $z \in H_k$ where $H_i, H_j, H_k \in \mathcal{S}_1$. As $\mathcal{S}_1$ is a chain with respect to set inclusion, one of the sets H_i, H_j, H_k contains both others, i.e. there exists $H_m \in \in \mathcal{S}_1$ such that $H_i \subseteq H_m$, $H_j \subseteq H_m$, $H_k \subseteq H_m$. Then $x, y, z \in H_m$ and as H_m is a cycle, either $(x, y, z) \in C$ or $(z, y, x) \in C$ holds. This implies that K is a cycle in (G, C). Thus each chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$ and $\mathcal{S}$ contains a maximal element according to Zorn's lemma.

2.6. Definition. Let (G, C), (H, D) be cyclically ordered sets, let $\varphi : G \to H$ be a bijection. φ is called an *isomorphism*, iff $x, y, z \in G$, $(x, y, z) \in C \Leftrightarrow (\varphi(x), \varphi(y), \varphi(z)) \in D$. φ is called an *antiisomorphism*, iff $x, y, z \in G$, $(x, y, z) \in C \Leftrightarrow (\varphi(z), \varphi(y), \varphi(x)) \in D$. Cyclically ordered sets (G, C), (H, D) are called *isomorphic* (*antiisomorphic*), iff there exists an isomorphism (antiisomorphism) $\varphi : (G, C) \to (H, D)$.

2.7. Definition. Let I be a set and let (G_i, C_i) be a cyclically ordered set for any $i \in I$. Let the sets G_i $(i \in I)$ be pairwise disjoint. Put $G = \bigcup_{i \in I} G_i$, $C = \bigcup_{i \in I} C_i$. Then (G, C) is called the *direct sum* of cyclically ordered sets (G_i, C_i) $(i \in I)$; we write $(G, C) = \sum_{i \in I} (G_i, C_i)$. If $I = \{1, \ldots, n\}$ we write $\sum_{i \in I} (G_i, C_i) = (G_1, C_1) + \ldots \ldots + (G_n, C_n)$.

2.8. Lemma. *Let I be a set and let (G_i, C_i) be a cyclically ordered set for any $i \in I$. Let the sets G_i $(i \in I)$ be pairwise disjoint. Then $\sum_{i \in I} (G_i, C_i)$ is a cyclically ordered set.*

Proof. Trivial.

2.9. Definition. Let (G, C) be a cyclically ordered set, let $x_0 \in G$. (G, C) is called

x_0 – *connected,* iff the following condition holds: $x \in G - \{x_0\} \Rightarrow$ there exists $y \in G$ such that $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$. (G, C) is called *strongly* x_0 – *connected,* iff the following condition holds: $x, y \in G - \{x_0\}$, $x \neq y \Rightarrow (x_0, x, y) \in C$ or $(x_0, y, x) \in C$.

The concept of the strong x_0 – connectedness is, however, superfluous, because of:

2.10. Theorem. *Let (G, C) be a cyclically ordered set such that* card $G \geqq 3$. *If (G, C) is strongly x_0 – connected for some $x_0 \in G$, then (G, C) is a cycle.*

Proof. Let $x, y, z \in G$, $x \neq y \neq z \neq x$. If $x_0 \in \{x, y, z\}$ then $(x, y, z) \in C$ or $(z, y, x) \in C$ by definition of strong x_0 – connectedness. Thus let $x_0 \bar{\in} \{x, y, z\}$. It is $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$, and $(x_0, x, z) \in C$ or $(x_0, z, x) \in C$, and $(x_0, y, z) \in$ $\in C$ or $(x_0, z, y) \in C$. Suppose first $(x_0, x, y) \in C$. If $(x_0, z, x) \in C$, then $(z, x, y) \in C$ by 1.6 (2). If $(x_0, x, z) \in C$, then in the case $(x_0, y, z) \in C$ we have $(x_0, y, z) \in C$, $(x_0, x, y) \in C \Rightarrow (x, y, z) \in C$ by 1.6 and in the case $(x_0, z, y) \in C$ we have $(x_0, z, y) \in$ $\in C$, $(x_0, x, z) \in C \Rightarrow (x, z, y) \in C$ by 1.6. Suppose now $(x_0, y, x) \in C$. If $(x_0, x, z) \in C$, then $(y, x, z) \in C$ by 1.6. If $(x_0, z, x) \in C$ then in the case $(x_0, y, z) \in C$ we have $(y, z, x) \in C$ and in the case $(x_0, z, y) \in C$ the assumption $(x_0, y, x) \in C$, $(x_0, z, y) \in C$ implies $(z, y, x) \in C$. Thus the relation C is complete and (G, C) is a cycle.

3. ORDER AND CYCLIC ORDER

3.1. Theorem. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. For any $x, y \in G$ put $x <_{C,x_0} y \Leftrightarrow (x_0, x, y) \in C$ or $x_0 = x \neq y$. Then $<_{C,x_0}$ is an order on G with the least element x_0.*

Proof. By 1.10 and 1.4 $<_{C,x_0}$ is an asymmetric and transitive binary relation on $G - \{x_0\}$, i.e. it is an order on $G - \{x_0\}$. But $x_0 <_{C,x_0} y$ for any $y \in G - \{x_0\}$ by definition of the relation $<_{C,x_0}$, i.e. x_0 is the least element in $(G, <_{C,x_0})$ and $<_{C,x_0}$ is an order on G.

3.2. Remark. Dually we can define $x <^{C,x_0} y \Leftrightarrow (x, y, x_0) \in C$ or $x \neq y = x_0$; then $<^{C,x_0}$ is an order on G with the greatest element x_0.

3.3. Lemma. *Let C, D be cyclic orders on the set G, let $x_0 \in G$. If $C \subseteq D$, then $<_{C,x_0} \subseteq <_{D,x_0}$.*

Proof. Trivial.

3.4. Lemma. *Let C be a complete cyclic order on the set G, let $x_0 \in G$. Then $<_{C,x_0}$ is a linear order on G.*

Proof. If $x, y \in G - \{x_0\}$, $x \neq y$, then $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$. In the first case we have $x <_{C,x_0} y$, in the second $y <_{C,x_0} x$. Further, $x_0 <_{C,x_0} y$ holds for any $y \in G - \{x_0\}$. Thus $<_{C,x_0}$ is a linear order on G.

3.5. Theorem. *Let G be a set, let $<$ be an order on G. Define a ternary relation $C_<$ on G by $(x, y, z) \in C_< \Leftrightarrow x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on G.*

Proof. Let $x, y, z \in G$, $(x, y, z) \in C_<$. Then $x < y < z$ or $y < z < x$ or $z < x < y$. Thus neither $z < y < x$ nor $y < x < z$ nor $x < z < y$ holds, i.e. $(z, y, x) \in C_<$ does not hold. The relation $C_<$ is thus asymmetric.

Let $x, y, z, u \in G$, $(x, y, z) \in C_<$, $(x, z, u) \in C_<$. Then either $x < y < z$ or $y < z < x$ or $z < x < y$, and either $x < z < u$ or $z < u < x$ or $u < x < z$ holds. It is easy to see that only the following possibilities do not lead to a contradiction:

$$\begin{array}{ll} x < y < z\,, & x < z < u\,, \\ x < y < z\,, & u < x < z\,, \\ y < z < x\,, & z < u < x\,, \\ z < x < y\,, & z < u < x\,. \end{array}$$

In the first case we have $x < y < u$, in the second $u < x < y$, in the third $y < u < x$ and in the fourth $u < x < y$. Thus in all cases $(x, y, u) \in C_<$ and $C_<$ is transitive. Let $x, y, z \in G$, $(x, y, z) \in C_<$. Then from the definition of the relation $C_<$ it follows that $(y, z, x) \in C$, and C is cyclic. Thus the relation $C_<$ on G is asymmetric, transitive and cyclic, i.e. it is a cyclic order on G.

3.6. Lemma. *Let G be a set, let $<_1, <_2$ be orders on G. If $<_1 \subseteq <_2$, then $C_{<_1} \subseteq C_{<_2}$.*

Proof. Trivial.

3.7. Lemma. *Let G be a set, let $<$ be a linear order on G. If* card $G \geqq 3$, *then $C_<$ is a complete cyclic order on G.*

Proof. Let $(G, <)$ be a linearly ordered set and let $x, y, z \in G$, $x \neq y \neq z \neq x$. Then just one of the following possibilities holds:

$$\begin{array}{lll} x < y < z\,, & y < z < x\,, & z < x < y\,, \\ z < y < x\,, & y < x < z\,, & x < z < y\,. \end{array}$$

In the first three cases we obtain $(x, y, z) \in C$, in the last three cases we have $(z, y, x) \in C_<$. Hence $C_<$ is a complete cyclic order on G.

3.8. Theorem. *Let $(G, <)$ be an ordered set with the least element x_0. Then $< \, = \, <_{C_<, x_0}$.*

Proof. Let $x, y \in G - \{x_0\}$. Then $x < y \Leftrightarrow x_0 < x < y \Leftrightarrow (x_0, x, y) \in C_< \Leftrightarrow \Leftrightarrow x <_{C_<, x_0} y$. Further, the element $x_0 \in G$ is the least element with respect to both $<$ and $<_{C_<, x_0}$. Thus $< \, = \, <_{C_<, x_0}$.

3.9. Theorem. *Let G be a set, $x_0 \in G$, C a cyclic order on G. Then $C_{<_{C,x_0}} \subseteq C$.*

Proof. Let $x, y, z \in G$, $(x, y, z) \in C_{<_{C,x_0}}$. Then $x <_{C,x_0} y <_{C,x_0} z$ or $y <_{C,x_0}$

$<_{C,x_0} z <_{C,x_0} x$ or $z <_{C,x_0} x <_{C,x_0} y$. Suppose first $x_0 \bar{\in} \{x, y, z\}$. Then in the first case we have $(x_0, x, y) \in C$, $(x_0, y, z) \in C$. By 1.6. (2), $(x, y, z) \in C$ holds. In the second case, we obtain $(x_0, y, z) \in C$, $(x_0, z, x) \in C$, thus, by 1.6, we have $(y, z, x) \in C$ and $(x, y, z) \in C$. In the third case, we obtain $(x_0, z, x) \in C$, $(x_0, x, y) \in C$, thus $(z, x, y) \in C$ and $(x, y, z) \in C$. Now let $x_0 \in \{x, y, z\}$. By an even permutation of the sequence (x, y, z) we can get that x_0 is in this sequence at the first place, i.e. it suffices to consider the case $(x_0, x, y) \in C_{<_{C,x_0}}$. Then $x_0 <_{C,x_0} x <_{C,x_0} y$ and hence $(x_0, x, y) \in$ $\in C$. Thus $C_{<_{C,x_0}} \subseteq C$.

3.10. Remark. In general the equality does not hold in 3.9.

Example. $G = \{x_0, y, z, u, v, w\}$, $C = \{(x_0, y, z), (y, z, x_0), (z, x_0, y), (u, v, w), (v, w, u), (w, u, v)\}$. Then

$$<_{C,x_0} = \{(x_0, y), (x_0, z), (x_0, u), (x_0, v), (x_0, w), (y, z)\},$$
$$C_{<_{C,x_0}} = \{(x_0, y, z), (y, z, x_0), (z, x_0, y)\}.$$

The fact that $C = C_{<_{C,x_0}}$ is not valid in general leads to the question under which assumptions this equality holds.

3.11. Lemma. *Let (G, C) be a complete cyclically ordered set, let $x_0 \in G$. Then $C = C_{<_{C,x_0}}$.*

Proof. It suffices to show that $C_{<_{C,x_0}}$ is a complete cyclic order on G. By 3.4, $<_{C,x_0}$ is a linear order on G and by 3.7. $C_{<_{C,x_0}}$ is a complete cyclic order on G.

3.12. Definition. Let (G, C) be a cyclically ordered set, let $x_0 \in G$. (G, C) is called x_0 – *stable* iff the following condition holds:

$x, y \in G - \{x_0\}$, $(z, x, y) \in C$ for some $z \in G \Rightarrow (x_0, x, y) \in C$ or $(x_0, y, x) \in C$.

3.13. Remark. Let (G, C) be a cyclically ordered set. If C is a complete cyclic order or a discrete cyclic order then (G, C) is x_0 – stable for each $x_0 \in G$.

3.14. Theorem. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. (G, C) is x_0 – stable if and only if each maximal cycle in (G, C) contains the element x_0.*

Proof. 1. Let (G, C) be x_0 – stable and let H be a maximal cycle in (G, C). Then card $H \geqq 3$ and assume $x_0 \bar{\in} H$. For any $x, y \in H$, $x \neq y$ there exists an element $z \in H$ such that $(z, x, y) \in C$ or $(z, y, x) \in C$. As (G, C) is x_0 – stable, the above mentioned implies $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$. This means that the cyclically ordered set $(H \cup \{x_0\}, C_{|H \cup \{x_0\}})$ is strongly x_0 – connected. By 2.10 $(H \cup \{x_0\}, C_{|H \cup \{x_0\}})$ is a cycle. But this contradicts the maximality of H.

2. Let each maximal cycle in (G, C) contain the element x_0 and let $x, y \in G - \{x_0\}$, $z \in G$ be such elements that $(z, x, y) \in C$. As $\{x, y, z\}$ is a cycle in (G, C), by 2.5 there exists a maximal cycle H in (G, C) such that $\{x, y, z\} \subseteq H$. According to the assumption $x_0 \in H$ holds. As H is a cycle, we have $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$. Thus (G, C) is x_0 – stable.

3.15. Theorem. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. Then the following statements are equivalent:*
(A) $C = C_{<_{C,x_0}}$,
(B) (G, C) *is* x_0 – *stable.*

Proof. 1. Let (A) hold. Suppose that (B) does not hold, i.e. there exist elements $x, y, z \in G - \{x_0\}$ such that $(z, x, y) \in C$, $(x_0, x, y) \bar{\in} C$, $(x_0, y, x) \bar{\in} C$. Then the elements x, y are incomparable in the order $<_{C,x_0}$ and thus $(z, x, y) \bar{\in} C_{<_{C,x_0}}$. Hence $C \neq C_{<_{C,x_0}}$, contradicting (A). Therefore (B) holds.

2. Let (B) hold and let $x, y, z \in G$, $(x, y, z) \in C$. Suppose first $x_0 \bar{\in} \{x, y, z\}$. Then $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$, and $(x_0, x, z) \in C$ or $(x_0, z, x) \in C$, and $(x_0, y, z) \in C$ or $(x_0, z, y) \in C$. This implies that $(\{x_0, x, y, z\}, C_{|\{x_0,x,y,z\}})$ is a strongly x_0 – connected cyclically ordered set, thus, by 2.10, $\{x_0, x, y, z\}$ is a cycle in (G, C). By 3.11, we have

$$C_{|\{x_0,x,y,z\}} = (C_{|\{x_0,x,y,z\}})_{<_{C_{|\{x_0,x,y,z\}},x_0}}$$

so that

$$(x, y, z) \in (C_{|\{x_0,x,y,z\}})_{<_{(C_{|\{x_0,x,y,z\}}),x_0}},$$

i.e. $(x, y, z) \in C_{<_{C,x_0}}$.

Now suppose $x_0 \in \{x, y, z\}$, for instance $x_0 = x$. Then $(x_0, y, z) \in C$, thus $y <_{C,x_0} z$. Hence $x_0 <_{C,x_0} y <_{C,x_0} z$, and $(x_0, y, z) \in C_{<_{C,x_0}}$. Analogously we consider the cases $x_0 = y$ and $x_0 = z$. Hence $C \subseteq C_{<_{C,x_0}}$ and by 3.9, $C = C_{<_{C,x_0}}$. This means that (A) holds.

By means of this theorem we prove further the following statement:

3.16. Lemma. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. Then*

$$C_{<_{C,x_0}} = (C_{<_{C,x_0}})_{<_{(C_{<_{C,x_0}}),x_0}}.$$

Proof. By 3.15 it suffices to show that $(G, C_{<_{C,x_0}})$ is x_0 – stable. Let $x, y \in G - \{x_0\}$, $z \in G$ and let $(z, x, y) \in C_{<_{C,x_0}}$. Suppose $(x_0, x, y) \bar{\in} C_{<_{C,x_0}}$, $(x_0, y, x) \bar{\in} C_{<_{C,x_0}}$. As x_0 is the least element in $(G, <_{C,x_0})$, this means that the elements x, y are incomparable with respect to $<_{C,x_0}$. But then $(z, x, y) \bar{\in} C_{<_{C,x_0}}$, a contradiction.

4. CONNECTEDNESS

4.1. Definition. Let (G, C) be a cyclically ordered set, let $A \subseteq G$, $A \neq \emptyset$. The subset A is called *connected*, iff the following condition holds: $x, y \in A, x \neq y \Rightarrow$ there exist a natural number n and elements $x_i, y_i, z_i \in A$ $(1 \leqq i \leqq n)$ such that $(x_i, y_i, z_i) \in C$ for all $i = 1, \ldots, n$, $x \in \{x_1, y_1, z_1\}$, $y \in \{x_n, y_n, z_n\}$ and $\{x_i, y_i, z_i\} \cap \{x_{i+1}, y_{i+1}, z_{i+1}\} \neq \emptyset$ for $i = 1, \ldots, n - 1$.

4.2. Remark. *Let (G, C) be a cyclically ordered set, let $A \subseteq G$, $A \neq \emptyset$. If $(A, C_{|A})$ is x_0 – connected for some $x_0 \in A$ then A is connected.*

Proof. Let $x, y \in A$, $x \neq y$. If $x \neq x_0 \neq y$, then there exist $z, u \in A$ such that $(x_0, x, z) \in C$ or $(x_0, z, x) \in C$, and $(x_0, y, u) \in C$ or $(x_0, u, y) \in C$. As $\{x_0, x, z\} \cap \cap \{x_0, y, u\} \supseteq \{x_0\}$, the condition in 4.1 is satisfied by $n = 2$. If $x = x_0$ then there exists $z \in A$ such that $(x_0, y, z) \in C$ or $(x_0, z, y) \in C$ so that the condition of 4.1 is satisfied by $n = 1$.

4.3. Remark. *Let (G, C), (H, D) be cyclically ordered sets, let $\varphi : G \to H$ be an isomorphism (antiisomorphism). If A is a connected subset of (G, C) then $\varphi(A)$ is a connected subset of (H, D).*

Proof. Trivial.

4.4. Theorem. *Let (G, C) be a cyclically ordered set and let $\{A_i; i \in I\}$ be a system of connected subsets of (G, C). If $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then $\bigcup_{i \in I} A_i$ is connected.*

Proof. Let $x, y \in \bigcup_{i \in I} A_i$, $x \neq y$. Then there exist $i, j \in I$ such that $x \in A_i$, $y \in A_j$. According to the assumption $A_i \cap A_j \neq \emptyset$; choose an element $x_0 \in A_i \cap A_j$. Suppose first $x \neq x_0 \neq y$. As $x, x_0 \in A_i$ and A_i is connected, there exist a natural number n and elements $x_k, y_k, z_k \in A_i$ $(1 \leqq k \leqq n)$ such that $(x_k, y_k, z_k) \in C$ for each $k = 1, \ldots, n$, $x \in \{x_1, y_1, z_1\}$, $x_0 \in \{x_n, y_n, z_n\}$ and $\{x_k, y_k, z_k\} \cap \{x_{k+1}, y_{k+1}, z_{k+1}\} \neq \emptyset$ for $k = 1, \ldots, n - 1$. As $x_0, y \in A_j$ and A_j is connected, there exist a natural number m and elements $u_l, v_l, w_l \in A_j$ $(1 \leqq l \leqq m)$ such that $(u_l, v_l, w_l) \in C$ for each $l = 1, \ldots, m$, $x_0 \in \{u_l, v_l, w_l\}$, $y \in \{u_m, v_m, w_m\}$, and $\{u_l, v_l, w_l\} \cap \{u_{l+1}, v_{l+1}, w_{l+1}\} \neq \emptyset$ for $l = 1, \ldots, m - 1$. Put $x_{n+l} = u_l$, $y_{n+l} = v_l$, $z_{n+l} = w_l$ for $l = 1, \ldots, m$. Then $x_k, y_k, z_k \in \bigcup_{i \in I} A_i$ $(1 \leqq k \leqq n + m)$, $(x_k, y_k, z_k) \in C$ for $k = 1, \ldots, n + m$, $x \in \{x_1, y_1, z_1\}$, $y \in \{x_{n+m}, y_{n+m}, z_{n+m}\}$ and $\{x_k, y_k, z_k\} \cap \{x_{k+1}, y_{k+1}, z_{k+1}\} \neq \emptyset$ for $k = 1, \ldots, n + m - 1$. If, for instance, $x = x_0 \neq y$, then $x, y \in A_j$ and thus the condition of 4.1 is satisfied, A_j being connected.

4.5. Corollary. *Let (G, C) be a cyclically ordered set, let $\{A_i; i \in I\}$ be a system of connected subsets of (G, C). If $\bigcap_{i \in I} A_i \neq \emptyset$ then $\bigcup_{i \in I} A_i$ is connected.*

4.6. Corollary. *Let (G, C) be a cyclically ordered set and let $\{A_i; i \in I\}$ be a monotone system of connected subsets of (G, C). Then $\bigcup_{i \in I} A_i$ is connected.*

4.7. Lemma. *Let (G, C) be a cyclically ordered set and let (A_n) be a finite or countable (of type ω) sequence of connected subsets of (G, C). If $A_n \cap A_{n+1} \neq \emptyset$ for all n for which A_{n+1} is defined, then $\bigcup A_n$ is connected.*

Proof. A_1 is connected according to the assumption. If $A_1 \cup \ldots \cup A_n$ is connected and if A_{n+1} is defined then $(A_1 \cup \ldots \cup A_n) \cap A_{n+1} \supseteq A_n \cap A_{n+1} \neq \emptyset$, and $A_1 \cup \ldots \cup A_n \cup A_{n+1}$ is connected by 4.4. Thus the assertion is proved by induction in the case when the sequence (A_n) is finite. If (A_n) is a sequence of type ω then put

$B_n = A_1 \cup \ldots \cup A_n$ for all natural n. Then (B_n) is a monotone sequence, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ and, according to the first part of the proof, each B_n is connected. Hence 4.6 implies that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ is connected.

4.8. Theorem. *Let (G, C) be a cyclically ordered set. Then, for each $x \in G$, there exists a maximal connected subset A of (G, C) containing x.*

Proof follows from 4.6 and from Zorn's lemma.

4.9. Definition. A maximal connected subset of a cyclically ordered set (G, C) is called a *component* of (G, C).

4.10. Remark. Each element of a cyclically ordered set (G, C) is contained in a component of (G, C).

4.11. Lemma. *Let (G, C) be a cyclically ordered set, let $\{A_i;\ i \in I\}$ be the set of all its components. Then $\{A_i;\ i \in I\}$ is a decomposition of the set G.*

Proof. From 4.10 it follows that $\bigcup_{i \in I} A_i = G$. Assume $A_i \cap A_j \neq \emptyset$ for some $i, j \in I$, $i \neq j$. Then $A_i \cup A_j$ is connected according to 4.4 and this contradicts the maximality of components. Hence $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\{A_i;\ i \in I\}$ is a decomposition of G.

5. CHARACTERIZATIONS OF COMPONENTS

5.1. Theorem. *A subset A of a cyclically ordered set (G, C) is a component of (G, C) if and only if:*
(i) *A is connected.*
(ii) *For any $x \in A$, $y \in G$, $z \in G - A$ the conditions $(x, y, z) \bar{\in} C$, $(z, y, x) \bar{\in} C$ hold.*

Proof. 1. Let A be a component of (G, C). Then A is connected, i.e. (i) holds. Assume that there exist elements $x \in A$, $y \in G$, $z \in G - A$ such that, for instance, $(x, y, z) \in C$. As A is connected, $\{x, y, z\}$ is connected and $A \cap \{x, y, z\} \supseteq \{x\}$, 4.4 implies that $A \cup \{x, y, z\}$ is connected. But $A \cup \{x, y, z\} \supseteq A$, $A \cup \{x, y, z\} \neq A$ and this contradicts the maximality of A. Hence (ii) holds.

2. Let A be a subset of (G, C) satisfying (i), (ii). Then A is connected; if A is not a component of (G, C) then there exists a connected subset B of (G, C) such that $A \subseteq B$, $A \neq B$. Choose elements $x \in A$, $z \in B - A$. As $x, z \in B$, $x \neq z$ and B is connected, there exist a natural number n and elements $x_i, y_i, z_i \in B$ $(1 \leqq i \leqq n)$ such that $(x_i, y_i, z_i) \in C$ for all $i = 1, \ldots, n$, $x \in \{x_1, y_1, z_1\}$, $z \in \{x_n, y_n, z_n\}$ and $\{x_i, y_i, z_i\} \cap \{x_{i+1}, y_{i+1}, z_{i+1}\} \neq \emptyset$ for $i = 1, \ldots, n - 1$. Assume that there does not exist any $i \in \{1, \ldots, n\}$ such that $\{x_i, y_i, z_i\} \cap A \neq \emptyset \neq \{x_i, y_i, z_i\} \cap (B - A)$. Then $\{x_i, y_i, z_i\} \subseteq A$ or $\{x_i, y_i, z_i\} \subseteq B - A$ for any $i = 1, \ldots, n$. As $x \in \{x_1, y_1, z_1\}$, $z \in \{x_n, y_n, z_n\}$, we have $\{x_1, y_1, z_1\} \subseteq A$, $\{x_n, y_n, z_n\} \subseteq B - A$. Let $m \in \{1, \ldots, n\}$ be

the greatest number such that $\{x_m, y_m, z_m\} \subseteq A$. Thus $1 \leqq m < n$ and $\{x_{m+1}, y_{m+1}, z_{m+1}\} \subseteq B - A$. But this implies $\{x_m, y_m, z_m\} \cap \{x_{m+1}, y_{m+1}, z_{m+1}\} = \emptyset$ which is a contradiction. Thus there exists a number $i \in \{1, \ldots, n\}$ such that $\{x_i, y_i, z_i\} \cap \cap A \neq \emptyset \neq \{x_i, y_i, z_i\} \cap (B - A)$. If, for instance, $x_i \in A$, $z_i \in B - A$ then $x_i \in A$, $y_i \in G$, $z_i \in G - A$ and $(x_i, y_i, z_i) \in C$ which contradicts (ii). Hence A is a component of (G, C).

5.2. Theorem. *A nonempty subset A of a cyclically ordered set (G, C) is a component of (G, C) if and only if A is a minimal subset of G with the property* (ii) *from* 5.1.

Proof. 1. Let A be a component of (G, C). Then A has the property (ii) according to 5.1. Let $B \subseteq A$, $B \neq A$ be any nonempty subset of A. Choose $x \in B$, $z \in A - B$. Then $x, z \in A$, $x \neq z$ and as A is connected, there exist a natural number n and elements $x_i, y_i, z_i \in A$ $(1 \leqq i \leqq n)$ with the properties from 4.1. When repeating the considerations of the second part of the proof of 5.1 we find an index $i \in \{1, \ldots, n\}$ such that $\{x_i, y_i, z_i\} \cap B \neq \emptyset \neq \{x_i, y_i, z_i\} \cap (A - B)$. Thus there exist elements $u \in B$, $v \in G$, $w \in G - B$ such that $(u, v, w) \in C$ or $(w, v, u) \in C$, and the set B has not the property (ii). Hence A is a minimal subset of G with the property (ii).

2. Let A be a minimal nonempty subset of G with the property (ii). Assume that A is not connected and choose any element $x \in A$. By 4.10 there exists a component B of the cyclically ordered set $(A, C_{|A})$ containing x. As A is not connected, $B \subseteq A$, $B \neq A$ hold. We prove that B has the property (ii). Suppose that there exist elements $x \in B$, $y \in G$, $z \in G - B$ such that, for instance, $(x, y, z) \in C$. If $y \bar{\in} A$ then $x \in A$, $z \in G$, $y \in G - A$ and $(x, y, z) \in C$ which contradicts the property (ii). Analogously, if $z \bar{\in} A$ then $x \in A$, $y \in G$, $z \in G - A$ and $(x, y, z) \in C$ which is a contradiction. Thus $y, z \in A$, i.e. $x \in B$, $y \in A$, $z \in A - B$ and $(x, y, z) \in C$, thus also $(x, y, z) \in C_{|A}$. As B is a connected subset of $(A, C_{|A})$, $\{x, y, z\}$ is a connected subset of $(A, C_{|A})$ and $B \cap \{x, y, z\} \supseteq \{x\}$, 4.4 implies that $B \cup \{x, y, z\}$ is a connected subset of $(A, C_{|A})$. But $B \cup \{x, y, z\} \supseteq B$, $B \cup \{x, y, z\} \neq B$ and this contradicts the maximality of the component B. Hence there exist no elements $x \in B$, $y \in G$, $z \in G - B$ such that $(x, y, z) \in C$ or $(z, y, x) \in C$, and B has the property (ii). But this contradicts the minimality of the set A. We have proved that A is a connected subset of (G, C) and by 5.1 it is a component of (G, C).

6. CANONICAL REPRESENTATION OF A CYCLICALLY ORDERED SET

6.1. Theorem. *Let (G, C) be a cyclically ordered set, let $\{A_i;\ i \in I\}$ be the set of all its components. Then* $(G, C) = \sum_{i \in I} (A_i, C_{|A_i})$.

Proof. In 4.11 we have proved $G = \bigcup_{i \in I} A_i$ where the sets A_i $(i \in I)$ are pairwise disjoint. This implies that also $C_{|A_i}$ $(i \in I)$ are pairwise disjoint and $\bigcup_{i \in I} C_{|A_i} \subseteq C$.

Let $(x, y, z) \in C$. Then there exists an element $i \in I$ such that $x \in A_i$. Assume $z \in G - A_i$; then $x \in A_i$, $y \in G$, $z \in G - A_i$, $(x, y, z) \in C$ which contradicts 5.1. Assume $y \in G - A_i$; then $y \in A_j$ where $j \in I$, $j \neq i$ so that $y \in A_j$, $z \in G$, $x \in G - A_j$, $(y, z, x) \in C$. This contradicts also 5.1. Thus $y \in A_i$, $z \in A_i$ and $(x, y, z) \in C_{|A_i}$. Hence $C \subseteq \bigcup_{i \in I} C_{|A_i}$ and therefore $C = \bigcup_{i \in I} C_{|A_i}$. Hence $(G, C) = \sum_{i \in I} (A_i, C_{|A_i})$.

6.2. Definition. Let (G, C) be a cyclically ordered set, $\{G_i;\ i \in I\}$ the set of all its components. Put $C_i = C_{|G_i}$ for $i \in I$. Then the expression $(G, C) = \sum_{i \in I} (G_i, C_i)$ will be called the *canonical representation* of (G, C).

6.3. Remark. *Let (G, C) be a cyclically ordered set, let $(G, C) = \sum_{i \in I} (G_i, C_i)$ where (G_i, C_i) are cyclically ordered sets. If all sets G_i $(i \in I)$ are connected then they are components of (G, C), i.e. $(G, C) = \sum_{i \in I} (G_i, C_i)$ is the canonical representation of (G, C).*

Proof. Let $i \in I$ and $x \in G_i$, $y \in G$, $z \in G - G_i$. Then $(x, y, z) \bar{\in} C_i$, $(z, y, x) \bar{\in} C_i$, thus also $(x, y, z) \bar{\in} C$, $(z, y, x) \bar{\in} C$. The set G_i satisfies the conditions (i), (ii) in 5.1 and hence it is a component of (G, C).

6.4. Theorem. *Let (G, C), (H, D) be cyclically ordered sets, let $(G, C) = \sum_{i \in I} (G_i, C_i)$, $(H, D) = \sum_{i \in I} (H_j, D_j)$ be their canonical representations. Then $(G, C) \cong (H, D)$ if and only if there exists a bijection $\psi : I \to J$ such that $(G_i, C_i) \cong (H_{\psi(i)}, D_{\psi(i)})$ for all $i \in I$.*

Proof. 1. Let there exist a bijection $\psi : I \to J$ such that $(G_i, C_i) \cong (H_{\psi(i)}, D_{\psi(i)})$ for all $i \in I$. Let $\varphi_i : G_i \to H_{\psi(i)}$ be an isomorphism. Define a mapping $\varphi : G \to H$ by $x \in G$, $x \in G_i \Rightarrow \varphi(x) = \varphi_i(x)$. Obviously φ is an isomorphism of (G, C) onto (H, D).

2. Let $(G, C) \cong (H, D)$, let $\varphi : G \to H$ be an isomorphism. Let $i \in I$ be an element and choose $x \in G_i$. Then there exists $j \in J$ such that $\varphi(x) \in H_j$. We show $\varphi(G_i) \subseteq H_j$. By 4.3, $\varphi(G_i)$ is a connected subset of (H, D). Assume $\varphi(G_i) \nsubseteq H_j$; then $\varphi(G_i) \cup H_j$ is a connected subset of (H, D) by 4.4 and $\varphi(G_i) \cup H_j \neq H_j$. This contradicts the maximality of H_j. Thus $\varphi(G_i) \subseteq H_j$ and as $\varphi^{-1} : H \to G$ is an isomorphism, for the same reason we have $\varphi^{-1}(H_j) \subseteq G_i$. Therefore $\varphi(G_i) = H_j$. Define a mapping $\psi : I \to J$ by $\psi(i) = j$ where $\varphi(G_i) = H_j$. ψ is obviously a bijection and from its definition it follows that $G_i \cong H_{\psi(i)}$.

6.5. Lemma. *A cyclically ordered set (G, C) is connected if and only if the equality $(G, C) = (G_1, C_1) + (G_2, G_2)$ implies $G_1 = \emptyset$ or $G_2 = \emptyset$.*

Proof. If $(G, C) = (G_1, C_1) + (G_2, C_2)$ where $G_1 \neq \emptyset$ and $G_2 \neq \emptyset$, then (G, C) is obviously not connected, for if $x \in G_1$, $y \in G_2$, then there do not exist natural number n and elements $x_i, y_i, z_i \in G$ $(1 \leqq i \leqq n)$ with the properties from 4.1. Assume that (G, C) is not connected and let $(G, C) = \sum_{i \in I} (G_i, C_i)$ be its canonical representation. Then $\operatorname{card} I \geqq 2$; put $I = I_1 \cup I_2$ where $I_1 \neq \emptyset \neq I_2$, $I_1 \cap I_2 = \emptyset$, $(G_1, C_1) = \sum_{i \in I_1} (G_i, C_i)$, $(G_2, C_2) = \sum_{i \in I_2} (G_i, C_i)$. Obviously $(G, C) = (G_1, C_1) + (G_2, C_2)$ and $G_1 \neq \emptyset$, $G_2 \neq \emptyset$.

Now we give a certain characterization of x_0 – stable cyclically ordered sets.

6.6. Theorem. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. (G, C) is x_0 – stable if and only if $(G, C) = (G_0, C_0) + (H, D)$ where (G_0, C_0) is x_0 – connected and x_0 – stable and (H, D) is discrete.*

Proof. 1. Let $(G, C) = (G_0, C_0) + (H, D)$ where (G_0, C_0) is x_0 – connected and x_0 – stable and (H, D) is discrete. Let $x, y \in G - \{x_0\}$, $(z, x, y) \in C$. Then $(z, x, y) \in C_0$ and thus $x, y, z \in G_0$. As (G_0, C_0) is x_0 – stable, either $(x_0, x, y) \in C_0$ or $(x_0, y, x) \in C_0$ holds, thus also $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$. Hence (G, C) is x_0 – stable.

2. Let (G, C) be x_0 – stable. Let $(G, C) = \sum_{i \in I} (G_i, C_i)$ be the canonical representation of (G, C). There exists $i_0 \in I$ such that $x_0 \in G_{i_0}$. Assume $C_i \neq \emptyset$ for some $i \in I$, $i \neq i_0$; then there exist $z, x, y \in G_i$ such that $(z, x, y) \in C_i \subseteq C$ and obviously $(x_0, x, y) \notin C$, $(x_0, y, x) \notin C$. This contradicts the x_0 – stability of (G, C). Hence $C_i = \emptyset$ for all $i \in I$, $i \neq i_0$ so that if we put $(G_0, C_0) = (G_{i_0}, C_{i_0})$, $H = \bigcup_{i \in I - \{i_0\}} G_i$, $D = \emptyset$, we have $(G, C) = (G_0, C_0) + (H, D)$ and (H, D) is discrete. Let $x \in G_0 - \{x_0\}$ be any element. As (G_0, C_0) is connected and $x \neq x_0$, there exist a natural number n and elements $x_i, y_i, z_i \in G_0$ $(1 \leqq i \leqq n)$ such that $(x_i, y_i, z_i) \in C_0$ for all $i = 1, \ldots, n$, $x \in \{x_1, y_1, z_1\}$, $x_0 \in \{x_n, y_n, z_n\}$ and $\{x_i, y_i, z_i\} \cap \{x_{i+1}, y_{i+1}, z_{i+1}\} \neq \emptyset$ for $i = 1, \ldots, n - 1$. Particularly $(x_1, y_1, z_1) \in C_0 = C$ and $x \in \{x_1, y_1, z_1\}$, for instance $x = x_1$. Then $(z_1, x, y_1) \in C$. If $y_1 = x_0$ then $(z_1, x, x_0) \in C = C_0$ and thus $(x_0, z_1, x) \in C_0$. If $y_1 \neq x_0$ then the x_0 – stability of (G, C) implies $(x_0, x, y_1) \in C = C_0$ or $(x_0, y_1, x) \in C = C_0$. Thus (G_0, C_0) is x_0 – connected and obviously it is x_0 – stable for $C_0 = C$.

From this theorem we derive further the following assertion (see 3.13):

6.7. Theorem. *Let (G, C) be a cyclically ordered set. Then the following statements are equivalent:*
(A) (G, C) is x – stable for any $x \in G$.
(B) C is a complete cyclic order or a discrete cyclic order.

Proof. (B) $\Rightarrow$ (A) holds by 3.13.

(A) $\Rightarrow$ (B): Let (A) hold. If C is discrete then (B) holds. Suppose therefore that C is not discrete. Choose any $x_0 \in G$; by 6.6 there is $(G, C) = (G_0, C_0) + (H, D)$ where (G_0, C_0) is x_0 – connected and x_0 – stable and (H, D) is discrete. Then $H = \emptyset$ for if there exists $x_1 \in H$ then for any $z, x, y \in G_0$ with $(z, x, y) \in C_0 = C$ we have $(x_1, x, y) \notin C$, $(x_1, y, x) \notin C$ and this contradicts the x_1 – stability of (G, C).

Thus $(G, C) = (G_0, C_0)$, i.e. (G, C) is x_0 – connected. We show that (G, C) is strongly x_0 – connected. Let $x, y \in G - \{x_0\}$, $x \neq y$ and assume $(x_0, x, y) \bar{\in} C$, $(x_0, y, x) \bar{\in} C$. As $y \in G - \{x_0\}$ and (G, C) is x_0 – connected, there exists $z \in G$ such that $(x_0, y, z) \in C$ or $(x_0, z, y) \in C$. Thus $(z, x_0, y) \in C$ or $(z, y, x_0) \in C$, but $(x, x_0, y) \bar{\in} C$, $(x, y, x_0) \bar{\in} C$. This is a contradiction because (G, C) is x – stable. Thus it must be $(x_0, x, y) \in C$ or $(x_0, y, x) \in C$ for any $x, y \in G - \{x_0\}$, $x \neq y$ and (G, C) is strongly x_0 – connected. Then it follows from 2.10 that (G, C) is a cycle.

6.8. Corollary. *Let (G, C) be a cyclically ordered set. Then the following statements are equivalent:*
(A) $C = C_{<_{C,x}}$ *for any* $x \in G$.
(B) *C is a complete cyclic order or a discrete cyclic order.*

Proof follows from 6.7 and 3.15.

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