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SOME CURVATURE IDENTITIES FOR COMMUTATIVE SPACES

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1. INTRODUCTION

A commutative space has been defined by Roberts and Ursell, [RU], as a Riemannian manifold in which "every two random steps commute". More explicitly, let us consider a compact Riemannian manifold (M, g) and a point $P \in M$, the position of which is not fixed but given only by a probability distribution $p(P)$ (with respect to the Riemannian measure). Consider a random step starting from the point P such that the new position P' is situated on the sphere $S(P, r)$ with center P and (small) radius r, and all positions on this sphere are equally probable when measured by the solid angle with center P. (M, g) is called a *commutative space* if the probability distribution of the position of P after two (small) random steps of lengths r, s does not depend on the order of performing these steps.

We can express this property by means of certain mean-value operators on (M, g), and we can also modify it in such a way that it will make sense for non-compact Riemannian manifolds, too.

Let (M, g) be an arbitrary n-dimensional Riemannian manifold, $m \in M$ a point, and $r > 0$ a small number such that the geodesic sphere $S_m(r)$ lies in a normal neighborhood of m. For any continuous function f defined on the sphere $S_m(r)$, put

$$\mathscr{L}_m(r, f) = \int_{S^{n-1}(1)} (f \circ \exp_m)(ru)\, \mathrm{d}u \Big/ \int_{S^{n-1}(1)} \mathrm{d}u\,,$$

where S^{n-1} is the unit sphere with center 0 in the tangent space M_m, $\exp_m : M_m \to M$ is the exponential map at m, and $\mathrm{d}u$ denotes the volume element of the (Euclidean) sphere $S^{n-1}(1)$ (cf. [K]).

One can easily see that, for each point $m \in M$, each continuous function f near m, and each sufficiently small number $r > 0$, the function

$$\mathscr{L}(r, f)(x) = \mathscr{L}_x(r, f)$$

is defined and continuous for all $x \in B_m(s)$, where $B_m(s)$ is a small geodesic ball with center m and radius s. In accordance with [RU], we can now state:

Definition. A Riemannian manifold (M, g) is called a *commutative space* if the following is satisfied: for any point $m \in M$, any two sufficiently small numbers $r, s > > 0$, and for each continuous function f defined on the geodesic ball $B_m(r + s)$ we have

$$\mathscr{L}_m(r, \mathscr{L}(s, f)) = \mathscr{L}_m(s, \mathscr{L}(r, f)) .$$

It is well-known that *each harmonic space and also each locally symmetric space are commutative spaces in our sense.*

In the following, we shall suppose that all Riemannian manifolds under consideration are *analytic*. Then one can express the commutativity property by means of certain linear differential operators, called *Euclidean Laplacians* (of higher order).

Let U_m be a normal neighborhood of $m \in (M, g)$, and define a linear differential operator $\tilde{\Delta}_m$ on U_m by the formula

$$\tilde{\Delta}_m f = \bar{\Delta}_m(f \circ \exp_m) \circ \exp_m^{-1} ,$$

where $\bar{\Delta}_m$ denotes the (ordinary) Laplacian defined in the whole tangent space (M_m, g_m) and f is a smooth function on U_m. In any system $(x_1, \ldots, x_n)$ of adapted normal coordinates in U_m, we can write

$$\tilde{\Delta}_m f = \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x_i)^2}.$$

We shall define a sequence of global linear differential operators $\tilde{\Delta}^{(k)}$ of order $2k$, $k = 1, 2, \ldots$, by the formula

$$(\tilde{\Delta}^{(k)} f)(m) = (\tilde{\Delta}_m^k f)(m) , \quad m \in M .$$

Here $\tilde{\Delta}^{(1)} = \Delta$ is the ordinary Laplacian of (M, g). The explicit formulas for $\tilde{\Delta}^{(2)}$ and $\tilde{\Delta}^{(3)}$ (which are not iterations of $\tilde{\Delta}^{(1)}$, in general) have been calculated by Gray and Willmore, [GW], and we shall present them in the next section.

Let $r > 0$ be given, and let f be an analytic function on (M, g). In each part of the manifold (M, g) where the mean-value operator $\mathscr{L}(r, f)$ is defined, we have the "Pizzetti-like formula" (cf. [RU], [K]):

$$\mathscr{L}(r, f) = \sum_{k=0}^{\infty} \left[\frac{\tilde{\Delta}^{(k)} f}{2^k . k! \, n(n + 2) \ldots (n + 2k - 2)} \right] r^{2k} .$$

Hence we obtain immediately:

Theorem. *An analytic Riemannian manifold (M, g) is a commutative space if and only if all differential operators $\tilde{\Delta}^{(1)}, \tilde{\Delta}^{(2)}, \ldots, \tilde{\Delta}^{(k)}, \ldots$ commute.*

It is well-known that each analytic commutative space satisfies an infinite sequence of curvature identities, the so called *Ledger's conditions of odd order*, which are known from the theory of harmonic spaces (see [RU], [RWW]).

The purpose of this article is to show that we can obtain a number of other curvature identities just dealing with the mutual commutativity of the operators $\tilde{\Delta}^{(1)}, \tilde{\Delta}^{(2)}$ and $\tilde{\Delta}^{(3)}$. (Yet, it remains an open problem whether or not all curvature identities for a commutative space can be derived from the odd Ledger's conditions.)

2. THE SUMMARY OF USEFUL CONVENTIONS AND FORMULAS

We make two *major conventions*:

a) For expressing a tensor field T on (M, g) in a coordinate form we shall always use local fields of *orthonormal* frames, if not otherwise stated. Thus, in our coordinate expressions we can always use *lower* indices only.

b) We shall use an *Einstein convention* which is modified to our situation: every occurrence of the same lower index in two places will indicate the summation with respect to these indices.

Further *conventions*:

c) The coordinate components of a covariantly derived tensor field $\nabla^k T$ will be written in the form $\nabla^k_{i_1 \dots i_k} T_{j_1 \dots j_l}$.

d) The symbol $\sigma(T_{i_1 \dots i_k})$ indicates the sum $\sum_\sigma T_{i_{\sigma(1)} \dots i_{\sigma(k)}}$ running over all cyclic permutations σ of the set $\{1, \dots, k\}$.

e) The scalar product $\langle T, U \rangle$ of two tensor fields T, U of the same degree k is defined by the formula $\langle T, U \rangle = T_{i_1 \dots i_k} U_{i_1 \dots i_k}$. In particular, we put $\|T\|^2 = \langle T, T \rangle$.

. . . . .

We shall denote by R_{ijkl} the components of the *curvature tensor* on (M, g), by $\varrho_{ij} = R_{ikjk} = \varrho_{ji}$ the components of the *Ricci tensor*, and by $\tau = \varrho_{ii}$ the *scalar curvature*.

Proposition A. *The 4-th order differential operator $\tilde{\Delta}^{(2)}$ is given by the formula*

(2.1)
$$\tilde{\Delta}^{(2)} f = \Delta^2 f + \tfrac{2}{3}\langle \nabla^2 f, \varrho \rangle + \tfrac{1}{3}\langle \nabla f, \nabla \tau \rangle = $$
$$= \Delta^2 f + \tfrac{2}{3}\varrho_{ij}(\nabla^2_{ij} f) + \tfrac{1}{3}(\nabla_i \tau)(\nabla_i f).$$

Proposition B. *The 6-th order differential operator $\tilde{\Delta}^{(3)}$ is given by the formula*

(2.2)
$$\begin{aligned}\tilde{\Delta}^{(3)} f = {} & \Delta^3 f + 2\varrho_{ij}(\nabla^4_{ijkk} f) + 2(\nabla_i \varrho_{jk})(\nabla^3_{ijk} f) + \\ & + \tfrac{2}{5}(\nabla^2_{kk}\varrho_{ij})(\nabla^2_{ij} f) - \tfrac{12}{5} R_{ikjl}\varrho_{kl}(\nabla^2_{ij} f) + \\ & + \tfrac{52}{15}\varrho_{ik}\varrho_{jk}(\nabla^2_{ij} f) + \tfrac{4}{15} R_{abci} R_{abcj}(\nabla^2_{ij} f) + \\ & + \tfrac{8}{3}(\nabla_i \varrho_{jk})\varrho_{ij}(\nabla_k f) - \tfrac{4}{5}(\nabla_k \varrho_{ij})\varrho_{ij}(\nabla_k f) + \\ & + \tfrac{4}{3} R_{kijl}(\nabla_j \varrho_{il})(\nabla_k f) + \tfrac{2}{15} R_{jilh}(\nabla_k R_{jilh})(\nabla_k f).\end{aligned}$$

For the proof of both formulas, see [GW].

The following special identities are very useful in calculations:

For functions we have

$$\Delta f = \nabla^2_{ii} f, \quad \nabla^2_{ij} f = \nabla^2_{ji} f. \tag{2.3}$$

For general tensor fields we have

$$\nabla^2_{ij} - \nabla^2_{ji} = -R_{ij} \tag{2.4}$$

where R_{ij} denotes the derivation of the tensor algebra determined by the corresponding curvature transformation. Further,

$$\nabla^3_{uui} f - \nabla^3_{iuu} f = \varrho_{iu}(\nabla_u f), \tag{2.5}$$

$$\nabla^4_{uuij} f - \nabla^4_{ijuu} f = (\nabla_i \varrho_{ju} + \nabla_j \varrho_{iu} - \nabla_u \varrho_{ij})(\nabla_u f) + \tag{2.6}$$
$$+ \varrho_{ju} \nabla^2_{iu} f + \varrho_{iu} \nabla^2_{ju} f + 2R_{aiju} \nabla^2_{au} f.$$

Let $(x_1, \ldots, x_n)$ be a *system of adapted normal coordinates* in the neighborhood of $m \in M$. (Thus, the corresponding tangent frame at m is orthonormal.) Then we have the following equalities which are valid *at* m:

$$\nabla^2_{ij} f = \frac{\partial^2 f}{\partial x_i \, \partial x_j}, \tag{2.7}$$

$$\nabla^3_{ijk} f = \frac{\partial^3 f}{\partial x_i \, \partial x_j \, \partial x_k} - \frac{1}{3} R_{jikl} \frac{\partial f}{\partial x_l} - \frac{1}{3} R_{kijl} \frac{\partial f}{\partial x_l}. \tag{2.8}$$

Moreover, for *any* system of local coordinates we have identically

$$\nabla_i f = \frac{\partial f}{\partial x_i}. \tag{2.9}$$

All the relations above are to be found in [GW] or they can be derived immediately. We conclude with the following

Lemma. *The "1st odd Ledger's condition"* $\sigma(\nabla_i \varrho_{jk}) = 0$ *always implies*

$$\nabla_i \tau = 0, \quad \nabla_j \varrho_{ij} = 0. \tag{2.10}$$

Proof. From the usual Bianchi identities we obtain $\nabla_j \tau = 2\nabla_k \varrho_{jk}$ (see [GV], p. 161). Putting $i = k$ in Ledger's condition, we get $\nabla_j \tau = -2\nabla_k \varrho_{jk}$. Hence the result follows.

3. THE NEW CURVATURE IDENTITIES

Theorem 1. *The differential operators* $\tilde{\Delta}^{(1)} = \Delta$ *and* $\tilde{\Delta}^{(2)}$ *commute on* (M, g) *if and only if the following conditions hold*:

(3.1) *Ledger's condition* $\sigma(\nabla_i \varrho_{jk}) = 0$,

(3.2) $\nabla_l \|\varrho\|^2 = \frac{4}{3} R_{ijkl}(\nabla_i \varrho_{jk})$, $(l = 1, 2, \ldots, n)$.

Proof. According to (2.1) and (2.3) we have

$$Lf = (\Delta\tilde{\Delta}^{(2)} - \tilde{\Delta}^{(2)}\Delta)f =$$
$$= \tfrac{2}{3}\Delta\langle\nabla^2 f, \varrho\rangle + \tfrac{1}{3}\Delta\langle\nabla f, \nabla\tau\rangle - \tfrac{2}{3}\langle\nabla^2(\Delta f), \varrho\rangle - \tfrac{1}{3}\langle\nabla(\Delta f), \nabla\tau\rangle =$$
$$= \tfrac{2}{3}(\nabla^2_{uu}\varrho_{ij})(\nabla^2_{ij}f) + \tfrac{2}{3}\varrho_{ij}(\nabla^4_{uuij}f) + \tfrac{4}{3}(\nabla_u\varrho_{ij})(\nabla^3_{uij}f) + \tfrac{1}{3}(\nabla^3_{uui}\tau)(\nabla_i f) +$$
$$+ \tfrac{2}{3}(\nabla^2_{ui}\tau)(\nabla^2_{ui}f) + \tfrac{1}{3}(\nabla_i\tau)(\nabla^3_{uui}f) - \tfrac{2}{3}\varrho_{ij}(\nabla^2_{ij}(\Delta f)) - \tfrac{1}{3}\nabla_i(\Delta f)(\nabla_i\tau) =$$
$$= \tfrac{2}{3}\varrho_{ij}(\nabla^4_{uuij}f - \nabla^4_{ijuu}f) + \tfrac{1}{3}(\nabla_i\tau)(\nabla^3_{uui}f - \nabla^3_{iuu}f) + \tfrac{4}{3}(\nabla_u\varrho_{ij})(\nabla^3_{uij}f) +$$
$$+ \tfrac{2}{3}(\nabla^2_{uu}\varrho_{ij})(\nabla^2_{ij}f) + \tfrac{2}{3}(\nabla^2_{ui}\tau)(\nabla^2_{ui}f) + \tfrac{1}{3}(\nabla^3_{uui}\tau)(\nabla_i f).$$

Choose a fixed point $m \in M$ and a system of adapted normal coordinates $(x_1, \ldots, x_n)$ in the neighborhood of m. Then the above tensorial identity still holds *at the point m* if we replace the coordinate components with respect to a moving orthonormal frame by the coordinate components with respect to the normal coordinates $(x_1, \ldots \ldots, x_n)$. Using (2.5)–(2.9) we can see that the third order part of $(Lf)_m$ is the expression $\tfrac{4}{3}(\nabla_u\varrho_{ij})(\partial^3 f/\partial x_u\, \partial x_i\, \partial x_j)_m$. Hence L is a second order differential operator at m if and only if $\sigma(\nabla_u\varrho_{ij}) = 0$ at m. If this is the case, we obtain $(\nabla_i\tau)_m = 0$ according to (2.10), and hence

$$(Lf)_m = \tfrac{2}{3}\varrho_{ij}(\nabla^4_{uuij}f - \nabla^4_{ijuu}f)_m + \tfrac{2}{3}(\nabla^2_{uu}\varrho_{ij})(\nabla^2_{ij}f)_m + \tfrac{4}{3}(\nabla_i\varrho_{jk})(\nabla^3_{ijk}f)_m\,.$$

We can now prove that the 2nd order part of $(Lf)_m$ vanishes. In fact, using (2.6), (2.7), (2.8), (3.1) and routine calculations, we can see that the second order part $(L^{(2)}f)_m$ equals

$$(L^{(2)}f)_m = (\tfrac{4}{3}\varrho_{il}\varrho_{lj} + \tfrac{4}{3}\varrho_{ab}R_{iabj} + \tfrac{2}{3}\nabla^2_{uu}\varrho_{ij})(\partial^2 f/\partial x_i\, \partial x_j)_m\,.$$

Further, using (3.1), (2.4) and (2.10) at m, we get

$$(\nabla^2_{uu}\varrho_{ij})_m = -\nabla^2_{ui}\varrho_{ju} - \nabla^2_{uj}\varrho_{iu} = R_{ui}(\varrho_{ju}) + R_{uj}(\varrho_{iu}) - \nabla^2_{iu}\varrho_{ju} - \nabla^2_{ju}\varrho_{iu} =$$
$$= -R_{uijl}\varrho_{lu} - R_{uiul}\varrho_{jl} - R_{ujil}\varrho_{lu} - R_{ujul}\varrho_{il}\,,$$

i.e.,

(3.3) $$(\nabla^2_{uu}\varrho_{ij})_m = -2\varrho_{il}\varrho_{jl} - 2R_{uijl}\varrho_{lu}\,.$$

Hence the result follows.

Thus the equation $(Lf)_m = 0$ is reduced to vanishing of the first order part. According to (2.5) (2.6) (2.8) and (3.1) this can be written as

$$\tfrac{2}{3}\varrho_{ij}(-2\nabla_l\varrho_{ij})(\partial f/\partial x_l) - \tfrac{4}{9}(\nabla_i\varrho_{jk})(R_{jikl} + R_{kijl})(\partial f/\partial x_l) = 0\,,$$

and with respect to the first Bianchi identity, this is equivalent to

$$\tfrac{2}{3}\nabla_l\Big(\sum_{i,j=1}^{n}(\varrho_{ij})^2\Big) = \tfrac{8}{9}(\nabla_i\varrho_{jk})(R_{ijkl}),\quad l = 1, \ldots, n\,,$$

q.e.d.

Theorem 2. *Let* Δ *and* $\tilde{\Delta}^{(2)}$ *commute on* (M, g). *Then* Δ *and* $\tilde{\Delta}^{(3)}$ *commute if and only if the following conditions hold*:

$$\sigma(\nabla_i A_{jk}) = 0\,, \quad (i, j, k = 1, \ldots, n)\,, \tag{3.4}$$

$$\nabla^2_{uu} A_{ij} + 2A_{il}\varrho_{jl} + 2A_{au}R_{aiju} + 2\nabla_i B_j = 0\,, \quad (i, j = 1, \ldots, n)\,, \tag{3.5}$$

$$\tfrac{4}{3}(\nabla_u A_{ij})\,R_{uijk} - 2A_{ij}(\nabla_k \varrho_{ij}) + \nabla^2_{uu}B_k + B_i\varrho_{ik} = 0\,, \quad (k = 1, \ldots, n)\,, \tag{3.6}$$

where

$$A_{ij} = \tfrac{8}{5}R_{ikjl}\varrho_{kl} + \tfrac{8}{3}\varrho_{ik}\varrho_{jk} + \tfrac{4}{15}R_{abci}R_{abcj}\,, \tag{3.7}$$

$$B_k = \tfrac{1}{15}\nabla_k[\|R\|^2 - 16\|\varrho\|^2]\,. \tag{3.8}$$

Proof. Ledger's condition (3.1) and Formula (2.8) yield a tensorial identity

$$(\nabla_i\varrho_{jk})(\nabla^3_{ijk}f) = -\tfrac{2}{3}(\nabla_j\varrho_{il})(R_{kijl})(\nabla_k f)\,. \tag{3.9}$$

This means that, on the right-hand side of (2.2), the third term and the last but one term cancel each other. Hence we can write

$$\tilde{\Delta}^{(3)}f = \Delta^3 f + 2\varrho_{ij}(\nabla^4_{ijkk}f) + \bar{A}_{ij}(\nabla^2_{ij}f) + \bar{B}_k(\nabla_k f)\,, \tag{3.10}$$

where

$$\bar{A}_{ij} = \tfrac{2}{5}\nabla^2_{kk}\varrho_{ij} - \tfrac{12}{5}R_{ikjl}\varrho_{kl} + \tfrac{52}{15}\varrho_{ik}\varrho_{jk} + \tfrac{4}{15}R_{abci}R_{abcj}\,,$$

$$\bar{B}_k = \tfrac{8}{3}(\nabla_i\varrho_{jk})\,\varrho_{ij} - \tfrac{4}{5}(\nabla_k\varrho_{ij})\,\varrho_{ij} + \tfrac{2}{15}R_{jilh}(\nabla_k R_{jilh})\,.$$

Using Formula (3.3) we obtain that $\bar{A}_{ij} = A_{ij}$, and using Ledger's condition we get $\bar{B}_k = B_k$.

Now, we can write

$$\begin{aligned} Nf = (\Delta\tilde{\Delta}^{(3)} - \tilde{\Delta}^{(3)}\Delta)f &= 2(\nabla^2_{uu}\varrho_{ij})(\nabla^4_{ijkk}f) + \\ &+ 4(\nabla_u\varrho_{ij})(\nabla^5_{uijkk}f) + 2\varrho_{ij}[\nabla^6_{uuijkk}f - \nabla^6_{ijkkuu}f] + \\ &+ (\nabla^2_{uu}A_{ij})(\nabla^2_{ij}f) + 2(\nabla_u A_{ij})(\nabla^3_{uij}f) + A_{ij}(\nabla^4_{uuij}f - \nabla^4_{ijuu}f) + \\ &+ (\nabla^2_{uu}B_k)(\nabla_k f) + 2(\nabla_u B_k)(\nabla^2_{uk}f) + B_k(\nabla^3_{uuk}f - \nabla^3_{kuu}f)\,. \end{aligned}$$

We shall prove that the sum of the first, second and third term on the right-hand side vanishes. In fact, we have

$$\begin{aligned} 2\varrho_{ij}(\nabla^6_{uuijkk}f - \nabla^6_{ijkkuu}f) &= 2\varrho_{ij}(\nabla^4_{uuij} - \nabla^4_{ijuu})(\nabla^2_{kk}f) = \\ &= 2\varrho_{ij}[-2(\nabla_u\varrho_{ij})(\nabla^3_{ukk}f) + 2\varrho_{iu}\nabla^4_{jukk}f + 2R_{aiju}(\nabla^4_{aukk}f)]\,. \end{aligned} \tag{3.11}$$

Here we have used (2.6) and Ledger's condition. From (3.9) we obtain easily (using the symmetry $\varrho_{jk} = \varrho_{kj}$)

$$4(\nabla_i\varrho_{jk})(\nabla^5_{ijkuu}f) = \tfrac{8}{3}(\nabla_i\varrho_{jk})\,R_{ijkl}(\nabla^3_{luu}f)\,. \tag{3.12}$$

From (3.11) we get

$$\begin{aligned} &2(\nabla^2_{uu}\varrho_{ij})(\nabla^4_{ijkk}f) + 2\varrho_{ij}(\nabla^6_{uuijkk}f - \nabla^6_{ijkkuu}f) = \\ &= 2[\nabla^2_{uu}\varrho_{ij} + 2\varrho_{li}\varrho_{lj} + 2R_{aiju}\varrho_{au}]\,\nabla^4_{ijkk}f - 4\varrho_{ij}(\nabla_u\varrho_{ij})(\nabla^3_{ukk}f)\,. \end{aligned}$$

Here the first term is zero due to (3.3). The second term equals $-2\nabla_l\|\varrho\|^2(\nabla^3_{luu}f)$. According to Formula (3.2), this cancels the right-hand side of (3.12) and we are finished.

Because A_{ij} is obviously a symmetric tensor, we get now

$$\begin{aligned} Nf = {} & 2(\nabla_u A_{ij})(\nabla^3_{uij}f) + (\nabla^2_{uu}A_{ij})(\nabla^2_{ij}f) + \\ & + A_{ij}[-2(\nabla_u \varrho_{ij})(\nabla_u f) + 2\varrho_{iu}(\nabla^2_{ju}f) + 2R_{aiju}(\nabla^2_{au}f)] + \\ & + (\nabla^2_{uu}B_k)(\nabla_k f) + 2(\nabla_u B_k)(\nabla^2_{uk}f) + B_k\varrho_{ku}(\nabla_u f)\,, \end{aligned}$$

which is a 3[rd]-order differential operator.

Obviously, Nf becomes a 2[nd]-order differential operator if and only if $\sigma(\nabla_u A_{ij}) = 0$. If this is the case, then using normal coordinates and (2.8) we get

$$2(\nabla_u A_{ij})(\nabla^3_{uij}f) = \tfrac{4}{3}(\nabla_u A_{ij})(R_{uikl})(\nabla_l f)\,.$$

Let us write Nf in the form $C_{ij}(\nabla^2_{ij}f) + D_k(\nabla_k f)$. Using normal coordinates once again, we see that $Nf = 0$ identically if and only if $C_{ij} + C_{ji} = 0$, $D_k = 0$ on M, which is exactly (3.5) and (3.6) respectively. Q.E.D.

Corollary 1. *On a commutative space, the function* $3\|R\|^2 - 8\|\varrho\|^2$ *is harmonic.*

Proof. We can see easily from (3.7) and (3.8) that $A_{ii} = \frac{16}{15}\|\varrho\|^2 + \frac{4}{15}\|R\|^2$, $\nabla_i B_i = \frac{1}{15}\Delta[\|R\|^2 - 16\|\varrho\|^2]$. Now, putting $i = j$ in (3.5) and summing up, we obtain $\Delta(A_{ii}) + 2\nabla_i B_i = 0$, which was to be proved.

Corollary 2. *On a compact commutative space, the volume of a geodesic ball* $B_m(r)$ *can be expressed in the form*

$$\operatorname{vol}(B_m(r)) = \operatorname{vol}(B_0(r))(1 + c_1 r^2 + c_2 r^4 + O(r^6))\,, \quad r \to 0\,,$$

where c_1, c_2 *are constants independent of* $m \in M$, *and* $B_0(r)$ *is the volume of a Euclidean ball of the same dimension and radius.*

Proof. According to [GV] we have

$$\operatorname{vol}(B_m(r)) = \operatorname{vol}(B_0(r))(1 + A(m)\,r^2 + B(m)\,r^4 + O(r^6))\,, \quad r \to 0\,,$$

where

$$A(m) = -\frac{\tau}{6(n+2)}\,, \quad B(m) = \frac{-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\Delta\tau}{360(n+2)(n+4)}\,.$$

Here $\tau = \text{const.}$ according to (2.11), and $B(m)$ is constant because it is a harmonic function.

The meaning of Corollary 2 is the following: It is known that, for a locally symmetric or harmonic space, the volume $\operatorname{vol}(B_m(r))$ depends only on the radius r and not on the point $m \in M$. A natural question arises whether this is still true for the commutative spaces.

Remark. It was communicated to me in a letter by L. Vanhecke that Corollary 1 is a (non-trivial) consequence of the first and second odd Ledger's condition.

Theorem 3. *If Δ commutes with $\tilde{\Delta}^{(2)}$ and $\tilde{\Delta}^{(3)}$, then $\tilde{\Delta}^{(2)}$ and $\tilde{\Delta}^{(3)}$ commute if and only if the following identities are satisfied:*

$$(3.13) \qquad \sigma[\varrho_{li}(\nabla_l A_{jk}) - A_{li}(\nabla_l \varrho_{jk})] = 0 \quad (i, j, k = 1, \ldots, n),$$

$$(3.14) \qquad (\varrho_{ai}A_{jl} + \varrho_{ij}A_{la}) R_{iljb} + (\varrho_{bi}A_{jl} + \varrho_{ij}A_{lb}) R_{ilja} + \\ + \varrho_{ij}(\nabla^2_{ij}A_{ab}) - A_{ij}(\nabla^2_{ij}\varrho_{ab}) + 2\varrho_{al}(\nabla_l B_b) - B_k(\nabla_k \varrho_{ab}) = 0 \quad (a, b = 1, \ldots, n),$$

$$(3.15) \qquad \varrho_{ij}A_{ab}(\nabla_i R_{jabl} + \nabla_a R_{ibjl}) + \varrho_{ij}(\nabla^2_{ij}B_l - B_k R_{kijl}) + \\ + \tfrac{4}{3}R_{ajbl}[-\varrho_{ij}(\nabla_i A_{ab}) + A_{ij}(\nabla_i \varrho_{ab})] = 0 \quad (l = 1, \ldots, n).$$

Proof. The calculations are the same routine as for Theorem 1 and Theorem 2, only combined with numerous applications of (2.4). Let us also remark that, in our situation, it is sufficient to check the commutativity of the operators $\varrho_{ij}(\nabla^2_{ij}f)$ and $2\varrho_{ij}(\nabla^2_{ijkk}f) + A_{ij}(\nabla^2_{ij}f) + B_k(\nabla_k f)$ (cf. (2.1) and (3.10)).

Corollary 3. *On a commutative space we always have*

$$(\varrho_{ij}\nabla^2_{ij})(3\|R\|^2 - 8\|\varrho\|^2) = 0.$$

Proof. Putting $a = b$ in (3.14) and summing up we obtain the desired relation.

Remark. We can see easily that, in the Einstein case, the relations (3.13), (3.14) and (3.15) are reduced to (3.4), (3.5) and (3.6), respectively.

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