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THE STRUCTURE OF THE MINIMUM GROUP KERNEL OF A REGULAR SEMIGROUP

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1. Introduction. A general question in the theory of semigroups has been a characterization of the minimum group congruence on a semigroup, where such a congruence exists. Stoll, in [11], developed a basic theory in terms of neat, reflexive, and left unitary subsets of the semigroup. For the class of regular semigroups, however, an explicit determination was sought beginning with inverse, orthodox and conventional semigroups.

After Section 2, which gives necessary preliminaries and definitions, Section 3 develops the property of reflexiveness and gives a result which characterizes, in terms of necessary and sufficient conditions, the relationship of the minimum reflexive subsemigroup to the set of idempotents and to the minimum group kernel of a regular semigroup. In particular, in a completely simple semigroup, the set idempotents is shown to be the minimum reflexive subset of the semigroup.

Section 4 then ties together the work of Section 3 and that of R. R. Feigenbaum in [4]. The main result characterizes the minimum group kernel of a regular semigroup in terms of R, the reflexive subsemigroup generated by the set of idempotents, and in terms of U, the minimum full self-conjugate subsemigroup. The relationship of R and U to each other and to the minimum group kernel is also determined.

2. Preliminaries. The notation of Clifford and Preston, [1], [2], will be used throughout the paper. If S is regular and there is no danger of ambiguity, E will be used to denote E_S, the set of idempotents of S. The set of inverses of an element $c \in S$ will be denoted by $V(c)$.

If $cEc' \subseteq E$ for all $c \in S$ and $c' \in V(c)$, S is called *conventional*; if E is also a subsemigroup, S is *orthodox*; if E is commutative, S is called an *inverse* semigroup.

When a congruence ϱ is such that S/ϱ is the maximal homomorphic image of S of type C, as in [2; p. 275] and [2; Theorem 11.25 (A), p. 276], then ϱ will be called the *minimum congruence* on S of type C and S/ϱ will be called the *maximum homomorphic image* of S of type C. In other words, S/ϱ is the maximum C-image if and only if ϱ is of type C and $\varrho \subseteq \sigma$ for each congruence σ which is of type C.

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If σ is a group congruence on S, then H, the group identity of S/σ, is a subsemigroup of S and can be used to describe σ. In particular, if we define the relation σ_H on S by

$$\sigma_H = \{(a, b) \in S \times S\colon ax, bx \in H \text{ for some } x \text{ in } S\}, \tag{2.1}$$

then $\sigma = \sigma_H$. The subsets which generate group images of S, relative to the relation defined in equation (2.1), R. Stoll [11; p. 478] calls, after Dubriel, the normal and unitary subsemigroups of S. They are also characterized by [11; Theorem 3, p. 477]:

(2.2) (i) H is a subsemigroup of S;
(ii) if $a, b \in S$ and $ab \in H$ then $ba \in H$;
(iii) if $a, b \in S$ and a and $ab \in H$ then $b \in H$;
(iv) H is a neat subset of S.

Recall from [2; p. 16] that a subset H is called *neat* if it is a right neat subset of S and a left neat subset of S; i.e., if for each $s \in S$, there exists an $x \in S$ and $y \in S$ such that $sx \in H$ and $ys \in H$.

Clifford and Preston [2; pp. 55, 56] describe a subset H of S as being *reflexive* if it satisfies (ii) of (2.2) and *left unitary* if it satisfies (iii) of (2.2). Henceforth, a subsemigroup of S satisfying (ii)–(iii) of (2.2) will be called a normal subsemigroup of S. If S is a regular semigroup, then E is inherently neat. Note that the concept of a neat normal subsemigroup coincides with what Levi [8; p. 142] called a complete normal subsemigroup.

Since the kernel of a congruence on a regular semigroup is the union of all the congruence classes containing an idempotent, the subsemigroup H of (2.1) must necessarily contain E. In the following, the kernel of the maximum [minimum] group homomorph [congruence] will be denoted by K.

For reference, we state the following theorem which gave a partial description of the minimum neat normal subsemigroup generated by a neat subset; that is, the description contained in the following result was complete except for a precise description of the reflexive subsemigroup generated by the neat subset being used.

Lemma 2.3. [9; Theorem 4.2, p. 400]. *Let S be a semigroup containing a neat subset D. Denote by R the reflexive subsemigroup of S generated by D. The subset $H = \{s \in S : bs \in R \text{ for some } b \in R\}$ is the minimum neat normal subsemigroup of S containing D. Thus σ_H of equation (2.1) is the minimum group congruence on S generated by D. If S regular and $D = E$, then $H = K$.*

3. The Relationship of E, R, and K. In this section for a regular semigroup S we investigate the properties of R, the minimum reflexive subsemigroup of S generated by E, and its relationship to K, the kernel of the minimum group congruence on S. Since E is a neat subset of a regular semigroup, then Lemma 2.3, with $D = E$, implies at least that $E \subseteq R \subseteq K$.

Some natural questions arise; viz, when is E reflexive or equal to K? The next result shows that the minimum reflexive subset containing E need not be a subsemigroup of S, and, in fact, may equal E.

Theorem 3.1. *If S is a completely simple semigroup, then E is a reflexive subset of S.*

Proof. If $x, y \in S$ with $xy \in E$, then xy is a left identity for x. Thus, $(yx)(yx) = y((xy)x) = y(x)$, and E is reflexive.

The converse of the preceding theorem is false. In particular, if S is any semigroup in which E is a rectangular band, then by [9; Corollary 4.5, p. 401] $E = R$. Thus while E is reflexive, S need not be completely simple. In fact, it is straight forward to show that for such S, we actually have $E = R = K$.

It is also natural to ask whether or not R must equal one of E or K. The answer is provided by the following:

Remark 3.2. There exists semigroups in which $E \subsetneqq R \subsetneqq K$. For example, let S_1 be the symmetric inverse semigroup on $\{1, 2\}$, and let S_2 be the full transformation semigroup on $\{1, 2, 3\}$. Then $E_1 = R_1 \subsetneqq K_1$, $E_2 \subsetneqq R_2 = K_2$, and the semigroup $S = S_1 \times S_2$ satisfies the remark with $E = E_1 \times E_2$, $R = R_1 \times R_2$, and $K = K_1 \times K_2$.

The next theorem now categorizes the possible relationships between E, R and K, and gives necessary and sufficient conditions on E and R in order that these relationships hold.

Theorem 3.3. *Let S be a regular semigroup. Denote the idempotents of S by E, the minimum reflexive subsemigroup generated by E by R, and the minimum group kernel (neat normal subsemigroup) by K.*

(i) $E = R$ *if and only if E is a reflexive subsemigroup of S.*

(ii) $R = K$ *if and only if R is a left unitary subsemigroup of S.*

(iii) $E = R = K$ *if and only if E is a left unitary subset of S.*

Proof. Parts (i) and (ii) are immediate from the definitions of R and K respectively.

For part (iii), $E = K$ clearly implies that E is left unitary. Conversely, if E is left unitary and $ab = e \in E$, then $aba = ea$ and $eaba = ea$. It follows then that for $(ea)' \in V(ea)$, $(ea)'\, ea(ba) = (ea)'\, eaba = (ea)'\, ea$. Since $(ea)'\, ea \in E$, then E being left unitary implies that $ba \in E$; i.e., E is reflexive. That E is a subsemigroup follows from [7; Lemma 2.1, p. 149]. Thus by (i), $E = R = K$.

Some additional characteristics of R are contained in the following result. We will denote the set of inverses of a set A by $V(A)$; that is $V(A) = \bigcup_{x \in A} V(x)$.

Theorem 3.4. *Let S be a regular semigroup with E, R, and K as in Theorem 3.3. Then:*

(i) $V(E) \subseteq R$, $V(R) \subseteq K$, $V(K) = K$.

(ii) *For all $c \in S$ and $c' \in V(c)$, $cRc' \subseteq R$.*

Proof. If $x \in V(e)$, for $e \in E$, then $ex, xe \in E \subseteq R$ implies that $x = xex = xeex \in \in R$. The second and third parts of (i) follow from the left unitary property of K.

For (ii), let $c \in S$, $c' \in V(c)$ and $b \in R$. Since $c'cb \in R$, then $cRc' \subseteq R$.

Note that part (ii) of Theorem 3.4 is comparable to that for E and K; viz, if S is conventional, [9; p. 396], then $cEc' \subseteq E$ for all $c \in S$ and $c' \in V(c)$, and if S is regular, then $cKc' \subseteq K$ for all $c' \in V(c)$.

Remark 3.5. If S is an inverse [orthodox, conventional, regular] semigroup, then K is an inverse [orthodox, conventional, regular] subsemigroup of S. The proofs are all immediate from the definitions.

The next result is presented for the case of an inverse semigroup, and it motivates the discussion to follow. Recall that for the natural partial order ω on an inverse semigroup S, $E\omega = \cup\{e\omega : e \in E\}$ where $e\omega = \{a \in S : ea = e\}$. For a subset A of S, we define $A\omega$ as $\{x \in S : ax \in A$ for some $a \in A\}$; A will be called closed if $A\omega = A$.

Theorem 3.6. *Let S be an inverse semigroup. Denote the minimum group kernel of S by K, and the natural partial order on S by ω. Then,*

(i) $K = \{a \in S : ea = ae = e$ *for some* $e \in E\}$.

(ii) $E\omega = K$.

(iii) *K is a closed inverse subsemigroup of S.*

Proof. For (i), $K = \{a \in S : ea = e$ for some $e \in E\}$ by [2; p. 193]. Let $a \in S$, $e \in E$ and $ea = e$. Then we have $ae(e)\,ae = ae$, $e(ae)\,e = e$, and $ae \in V(e)$. Thus, since S is inverse, $ae = e$.

To show (ii), let $x \in E\omega$, say $x \in e\omega$ for some $e \in E$. Since $ex = e$, it follows as in the proof of (i) that $ex = xe = e$; i.e., $x \in K$. Conversely, if $x \in K$, then there exists $e \in E$ such that $ex = xe = e$ and hence $x \in e\omega$.

Lastly, by [2; Lemma 7.9, p. 43] part (iii) is true.

Theorem 3.6 suggests that the notion of left unitary closure may give rise to more general considerations in a regular semigroup. That such is the case was shown in Feigenbaum [4; Chapter IV]. From Feigenbaum [4; pp. 5, 6, and 21] we have the following definitions: a subset H of a regular semigroup S will be called *full* if $E \subseteq H$, and *self-conjugate* if $cHc' \subseteq H$ for all $c \in S$ and $c' \in V(c)$; the *closure* of H will be $H\omega = \{s \in S : hs \in H$ for some $h \in H\}$; H will be called *closed* if $H = H\omega$.

We have immediately that any full subset is neat, and that R, by Theorem 3.4 (ii), is a self-conjugate subsemigroup of S. Moreover, closure as defined above coincides with the concept of left unitary as used in Lemma 2.3. Thus we see that $R\omega = K$.

From [4; p. 19], we define $\mathscr{C} = \{C \subseteq S : C$ is a full, selfconjugate subsemigroup of $S\}$. Since $S \in \mathscr{C}$, $\mathscr{C} \neq \emptyset$. If $U = \cap\mathscr{C}$; i.e., U is the smallest member of $\mathscr{C}$, then by [4; Theorem 4.2, p. 20] the minimum group congruence on a regular semigroup S

can be characterized by:

(3.7) $$\beta_U = \{(a, b) \in S \times S : ab' \in U\omega \text{ for some } b' \in V(b)\}\,.$$

We are thus lead to determining the relationships between the sets U, R, and K, and the congruences σ_K of (2.1) and β_U of (3.7).

4. The Relationship of E, U, R, **and** K. We begin with the following result.

Lemma 4.1. *Let U be defined as in section 3. Then $U\omega$ is closed and self-conjugate.*

Proof. Since $U\omega \subseteq (U\omega)\,\omega$, suppose $x \in (U\omega)\,\omega$. There exists $y \in U\omega$ such that $yx \in U\omega$, and therefore there exists $u, v \in U$ such that uy, $vyx \in U$. Since $(uy)\,u(y'vy) \in U^3 \subseteq U$, then $(uyuy'vy)\,x = u(yuy')\,vyx \in U^3 \subseteq U$ implies that $x \in U\omega$.

Next, let $c \in S$, $c' \in V(c)$, and $x \in U\omega$. There exists $u \in U$ such that $ux \in U$. Since $c(u\,.\,xux')\,c' \in U$ and $c'c \in E \subseteq U$, then $(c(u\,.\,xux')\,c')\,cxc' = c((ux)\,u(x'c'cx))\,c' \in cU^3c' \subseteq cUc' \subseteq U$; i.e., $cxc' \in U\omega$.

Theorem 4.2. *Let S be a regular semigroup. Denote the set of idempotents of S by E, the minimum full self-conjugate subsemigroup of S by U, the minimum reflexive subsemigroup of S by R, and the minimum group kernel by K. Let β_U and σ_K be defined as in equations (3.7) and (2.1) respectively. Then*:

(i) $E \subseteq U \subseteq R \subseteq K$.

(ii) $U\omega = R\omega = K$.

(iii) $\beta_U = \sigma_K$.

Proof. To establish (i), we need only show that $U \subseteq R$. By Theorem 3.4, R is a full self-conjugate subsemigroup of S. Thus by the definition of U, $U \subseteq R$.

For (ii), let $ab \in U\omega$ where $a, b \in S$. There exists $u \in U$ such that $uab \in U$, and therefore, $a'(uab)\,a = (a'ua)\,ba \in U$. Thus $ba \in U\omega$ and $U\omega$ is reflexive.

By the definition of R, $R \subseteq U\omega$. By Lemma 4.1, $(U\omega)\,\omega = U\omega$, and therefore $K = R\omega \subseteq (U\omega)\,\omega = U\omega \subseteq R\omega = K$.

Part (iii) follows from (ii) since $a\,\beta_U b$ iff $\exists b' \in V(b)$ such that $ab' \in U\omega$ iff $\exists b' \in V(b)$ such that $ab' \in K$ iff $a\sigma_K b$.

In Remark 4.3 below, we show that $U \neq R$ in general. Thus Theorem 4.2 says that beginning with E, the minimum group kernel for a regular semigroup can be found through either reflexiveness or self-conjugacy, with left unitary closure completing the process; i.e., both Stoll's and Feigenbaum's theory yield the same result. This is amplied in Theorem 4.4 where a description of K is given.

Remark 4.3. Let $S = A \cup B$ where $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ for $i, j \in \{1, 2\}$. Define multiplication in S by

$$a_{ij}a_{kl} = \begin{cases} a_{il} & \text{if } j = k\,, \\ b_{il} & \text{if } j \neq k\,; \end{cases}$$

$$b_{ij}b_{kl} = a_{ij}b_{kl} = b_{ij}a_{kl} = b_{il}\,.$$

Then S is an orthodox semigroup with $E = U$ and $R = K = S$.

Theorem 4.4. *If S is a regular semigroup, then the minimum group congruence on S may be written as either*

$$\{(a, b) \in S \times S : eae = eu'u_1be \text{ for some } e \in E,\ u_1 \text{ and } u \in U, \text{ and } u' \in V(u)\}$$

or as

$$\{(a, b) \in S \times S : gag = gr'r_1bg \text{ for some } g \in E,\ r_1 \text{ and } r \in R, \text{ and } r' \in V(r)\}\,.$$

Proof. By Theorem 4.2, $U\omega = R\omega = K$ implies that if $(a, b) \in \sigma_K = \beta_U$, then there exists r_1, $r \in R$ and there exists u_1, $u \in U$ such that $rab' = r_1$ and $uab' = u_1$. By Howie and Lallement [8; Lemma 1.1, p. 146] there exists $g, e \in E$ such that $gr'r = b'bg = g$ and $eu'u = b'be = e$. Thus we have $eae = e(u'u_1)be$ and $gag = g(r'r_1)\,bg$. The converse is evident.

Note that if $V(R) \subseteq R$, or $V(U) \subseteq U$, then Theorem 4.4 could be simplified further. It is an open question as to whether or not $V(R) \subseteq R$ or $V(U) \subseteq U$ in general.

With the relationship between E, U, R, and K determined, the results of R. R. Stoll and R. Feigenbaum have been pulled together. While a construction for R has appeared in [10], a construction for U is yet unknown.

We conclude the paper with an overview table. Recall that R is the minimum reflexive subsemigroup of S generated by E, and U is the minimum full self-conjugate subsemigroup of S.

S	Kernel of S	
Inverse:	$\{x \in S\colon exe = e, \text{ some } e \in E\}$	Munn
Orthodox:	$\{x \in S\colon ex = f, \text{ some } e, f \in E\}$	Meakin
Conventional:	$\{x \in S\colon \mathrm{ex} = f, \text{ some } e, f \in E\}$	Masat
Regular:	$\{x \in S\colon er'sx = f, \text{ some } e, f \in E \text{ and } r, s \in R,\ r' \in V(v)\}$	Masat
	or	
	$\{x \in S\colon eu'vx = f, \text{ some } e, f \in E \text{ and } u, v \in U,\ u' \in V(u)\}$	Feigenbaum

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