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TORSION RADICALS OF LATTICE ORDERED GROUPS

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The notion of torsion radical of lattice ordered groups has been introduced by J. Martinez [12]. Several concrete types of torsion radicals have been investigated by P. Conrad [3]. To each torsion radical ϱ there corresponds a torsion class A_ϱ consisting of all lattice ordered groups G having the property that $\varrho(G) = G$; the torsion radical ϱ is uniquely determined by A_ϱ. Let $\mathscr{G}$ be the class of all lattice ordered groups and let $\mathscr{R}$ be the class of all torsion radicals. Let $\varrho_1, \varrho_2 \in \mathscr{R}$. We put $\varrho_1 \leqq \varrho_2$ if $\varrho_1(G) \subseteq \varrho_2(G)$ for each $G \in \mathscr{G}$. The relation $\leqq$ is a partial order on the class $\mathscr{R}$. In [12] it has been proven that the partially ordered class $(\mathscr{R}; \leqq)$ is a complete lattice.

The least and the greatest element of $\mathscr{R}$ will be denoted by $\bar{0}$ or $\bar{\varrho}$, respectively. For $\varrho \in \mathscr{R}$ we denote by $A(\varrho)$ the class of all elements of $\mathscr{R}$ covering ϱ; the torsion radicals belonging to $A(\varrho)$ are said to be atoms over ϱ. The class of all principal torsion radicals will be denoted by $\mathscr{P}$. The symbols Ab and Repr denote the torsion class consisting of all abelian or representable lattice ordered groups, respectively. The one-element torsion class (containing the zero group $\{0\}$ only) is said to be trivial.

The content of this paper is as follows. In § 1 there are given the basic definitions. Torsion classes generated by linearly ordered groups are investigated in § 2. Principal torsion classes are dealt with in § 3. Covering relations in the partially ordered class $(\mathscr{R}; \leqq)$ will be examined in § 4 and § 5.

Sample results: If A is a torsion class generated by linearly ordered groups, then A cannot be represented as a product BC of nontrivial torsion classes B, C. The class $\mathscr{P}$ is an ideal of $\mathscr{R}$. A torsion class A is principal if and only if there exists a cardinal α such that for each $G \in A$ and each $0 < x \in G$ we have card $[0, x] \leqq \alpha$. If ϱ is a principal torsion radical, then the class $A(\varrho)$ is infinite and each $\varrho' \in A(\varrho)$ is principal. If $A \in \{\text{Ab}, \text{Rep}\}$ and if ϱ is the torsion radical corresponding to A, then the class $A(\varrho)$ is infinite. For each $\varrho \in \mathscr{R}$ there exists $\varrho' \in \mathscr{R}$ with $\varrho \leqq \varrho'$ such that (i) $A(\varrho') = \emptyset$, and (ii) if $\varrho \leqq \varrho_1 \in \mathscr{R}$ and $A(\varrho_1) = \emptyset$, then $\varrho' \leqq \varrho_1$. There exists a torsion radical σ with $\sigma < \bar{\varrho}$ such that (i) $A(\sigma) = \emptyset$, and (ii) if $\varrho_1 \in \mathscr{R}$ and $A(\varrho_1) = \emptyset$, then $\sigma \leqq \varrho_1$. There exists no dual atom in $\mathscr{R}$.

1. PRELIMINARIES

The standard denotations for lattices and lattice ordered groups will be used (cf. Conrad [2] and Fuchs [4]). The symbols $\subseteq$ and $\subset$ will be applied for denoting the containment or proper containment of classes, respectively.

Let $\mathscr{G}$ be the class of all lattice ordered groups and let ϱ be a mapping of $\mathscr{G}$ into $\mathscr{G}$ such that the following conditions are fulfilled for each $G \in \mathscr{G}$:

(i) $\varrho(G)$ is a convex l-subgroup of G.

(ii) If G_1 is a convex l-subgroup of G, then $\varrho(G_1) = \varrho(G) \cap G_1$.

(iii) If φ is a homomorphism of G onto a lattice ordered group G_1, then $\varphi(\varrho(G)) = \varrho(\varphi(G))$.

Under these assumptions ϱ is said to be a torsion radical.

The system of all convex l-subgroups of a lattice ordered group G will be denoted by $c(G)$; this system is partially ordered by inclusion. It is well-known that $c(G)$ is a complete lattice; we denote the lattice operation in $c(G)$ by $\wedge$, $\vee$. If $G_1 \in C(G)$, $\{G_i\}_{i \in I} \subseteq c(G)$, then

$$G_1 \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (G_1 \wedge G_i)\,.$$

A nonempty class C of lattice ordered groups is called a torsion class if it has the following properties:

(a) If $G \in C$ and if $G_1 \in c(G)$, then $G_1 \in C$.

(b) If $G \in \mathscr{G}$ and if $\{G_i\}_{i \in I}$ is a system of convex l-subgroups of G such that $G_i \in C$ for each $i \in I$, then $\bigvee_{i \in I} G_i$ belongs to C.

(c) The class C is closed with respect to homomorphisms.

There is a one-to-one correspondence between torsion radicals and torsion classes. Namely, if ϱ is a torsion radical, then the class $C^0(\varrho)$ of all $G \in \mathscr{G}$ with $\varrho(G) = G$ is a torsion class. Let C be a torsion class. For each $G \in \mathscr{G}$ we denote by $\varrho^0(C)(G)$ the join of all convex l-subgroups of G belonging to C. Then $\varrho^0(C)$ is a torsion radical. For each torsion radical ϱ and each torsion class C we have

$$C^0(\varrho^0(C)) = C\,, \quad \varrho^0(C^0(\varrho)) = \varrho\,.$$

Let $\mathscr{R}$ be the class of all torsion radicals. We consider $\mathscr{R}$ with the partial order $\leqq$ defined in the introduction. Then for $\varrho_1, \varrho_2 \in \mathscr{R}$ we have $\varrho_1 \leqq \varrho_2$ if and only if $C^0(\varrho_1) \subseteq C^0(\varrho_2)$. Let $\mathscr{R}_1$ be a nonempty subclass of $\mathscr{R}$. For each $G \in \mathscr{G}$ we put

$$\varrho_1(G) = \bigvee_{\varrho \in \mathscr{R}_1} \varrho(G)\,, \quad \varrho_2(G) = \bigwedge_{\varrho \in \mathscr{R}_1} \varrho(G)\,.$$

Then ϱ_1 is the least upper bound of $\mathscr{R}_1$ in $\mathscr{R}$, and ϱ_2 is the greatest lower bound of $\mathscr{R}_1$

in $\mathcal{R}$. We denote $\varrho_1 = \bigvee_{\varrho \in \mathcal{R}_1} \varrho$, $\varrho_2 = \bigwedge_{\varrho \in \mathcal{R}_1} \varrho$. If $\varkappa \in \mathcal{R}$, then

$$\varkappa \wedge \left(\bigvee_{\varrho \in \mathcal{R}_1} \varrho\right) = \bigvee_{\varrho \in \mathcal{R}_1} (\varkappa \wedge \varrho)$$

(cf. [12]).

The notions of torsion class and torsion radical can be generalized in such a way that in the conditions (ii) and (c) above we replace homomorphisms by isomorphisms. The corresponding generalized notions are called radical class and radical mapping, respectively. The class $\bar{\mathcal{R}}$ of all radical mappings is partially ordered analogously to $\mathcal{R}$. The partially ordered class $\bar{\mathcal{R}}$ has been investigated in [9].

2. TORSION CLASSES GENERATED BY LINEARLY ORDERED GROUPS

Let A be a nonempty class of lattice ordered groups. Let us denote by

$S_c(A)$ – the class of all lattice ordered groups H' such that H' is a convex l-subgroup of a lattice ordered group $H \in A$;

$H(A)$ – the class of all lattice ordered groups H' such that H' is a homomorphic image of some $H \in A$;

$l(A)$ – the class of all lattice ordered groups H' that can be expressed as $H' = \bigcup_{i \in I} H_i$, where H_i are convex l-subgroups of H', $H_i \in A$ for each $i \in I$, and the system $\{H_i\}_{i \in I}$ (partially ordered by inclusion) is a chain;

$u(A)$ – the class of all lattice ordered groups H' that can be written as $H' = \bigvee_{i \in I} H_i$, where H_i are convex l-subgroups of H' and $H_1 \in A$ for each $i \in I$.

2.1. Lemma. *Let φ be a homomorphism of a lattice ordered group G onto a lattice ordered group H. Let H' be a convex l-subgroup of H. Then $\varphi^{-1}(H')$ is a convex l-subgroup of G.*

Proof. $\varphi^{-1}(H')$ is clearly an l-subgroup of G. Let $g_1 \in G$, $g_2 \in \varphi^{-1}(H')$, $0 \leqq g_1 \leqq g_2$. Then $0 \leqq \varphi(g_1) \leqq \varphi(g_2)$, hence $\varphi(g_1) \in H'$ and thus $g_1 \in \varphi^{-1}(H')$. Therefore $\varphi^{-1}(H')$ is a convex l-subgroup of G.

2.2. Lemma. *Let $A \neq \emptyset$ be a class of lattice ordered groups. Let $C = H(S_c(A))$. Then C fulfils the conditions* (a) *and* (c) *from* § 1.

Proof. The validity of (c) follows immediately from the definition of C. Let $H_0 \in C$ and let H' be a convex l-subgroup of H_0. There exist $G \in A$, a convex l-subgroup G_1 of G and a homomorphism φ of G_1 onto H_0. Put $H_1 = \varphi^{-1}(H')$. According to 2.1, H_1 is a convex l-subgroup of G_1, hence H_1 is a convex l-subgroup of G. Thus $H' \in C$ and therefore C fulfils (a).

2.3. Lemma. *Let $A \neq \emptyset$ be a class of lattice ordered groups and let $C' = u(H(S_c(A)))$. Then C' is a torsion class.*

Proof. We have to verify that C' fulfils the conditions (a), (b) and (c) from § 1. The validity of (b) follows immediately from the definition of C'. Let $G \in C'$ and let G_1 be a convex l-subgroup of G. Let C be as in 2.2. There exists a system $\{G_i\}_{i\in I} \subseteq C$ such that each G_i is a convex l-subgroup of G and $G = \bigvee_{i\in I} G_i$. Hence $G_1 = \bigvee_{i\in I} (G_1 \wedge G_i)$, and all $G_1 \wedge G_i$ are convex l-subgroups of G_1. Moreover, according to 2.2, each $G_1 \wedge G_i$ $(i \in I)$ belongs to C. Hence $G_1 \in C'$ and so C' fulfils (a). Let φ be a homomorphism of G onto a lattice ordered group H_0. Denote $G_i' = \varphi(G_i)$ for each $i \in I$. Then G_i' are convex l-subgroups of H_0 and $H_0 = \bigvee_{i\in I} G_i'$. From 2.2 we obtain $G_i' \in C$ for each $i \in I$. Hence $H_0 \in C'$ and thus C' fulfils (c) as well.

If A_1 is a torsion class with $A \subseteq A_1$, then clearly $C' \subseteq A_1$. From this and from 2.3 we obtain:

2.4. Corollary. *Let $A \neq \emptyset$ be a class of lattice ordered groups. Then $u(H(S_c(A)))$ is the least torsion class having A as a subclass.*

In view of 2.4, $u(H(S_c(A)))$ will be said to be the torsion class generated by A; it will be denoted by $T(A)$. If A is a one-element class, then $T(A)$ will be called a principal torsion class, and the corresponding torsion radical will be said to be a principal radical.

2.5. Lemma. (Cf. [10].) *Let G be a lattice ordered group and let $H_1, H_2 \in c(G)$. Assume that both H_1 and H_2 are linearly ordered and that $H_1 \cap H_2 \neq \{0\}$. Then we have either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.*

Let $\{G_i\}_{i\in I}$ be a system of lattice ordered groups. The direct product and the direct sum (= discrete direct product) will be denoted by $\prod_{i\in I} G_i$ or by $\sum_{i\in I} G_i$, respectively. Without loss of generality we can assume that if $G = \prod_{i\in I} G_i$ or $G = \sum_{i\in I} G_i$, then $G_i \in c(G)$ for each $i \in I$.

2.6. Theorem. *Let $A \neq \emptyset$ be a class of linearly ordered groups. Let $G \in \mathcal{G}$. Then the following conditions are equivalent:*

(α) $G \in T(A)$.

(β) *G can be expressed as $G = \sum_{j\in J} G_j$, where each G_j belongs to $l(H(S_c(A)))$.*

Proof. We obviously have $l(H(S_c(A))) \subseteq u(H(S_c(A)))$. If $G = \sum_{j\in J} G_j$, then $\bigvee_{j\in J} G_j = G$ and G_j are convex l-subgroups of G. From this and from 2.4 we infer that (β) ⇒ (α).

Assume that (α) is valid. According to 2.4 G can be expressed as $G = \bigvee_{m\in M} K_m$, where each K_m is a convex l-subgroup of G and belongs to $H(S_c(A))$. The case $G = \{0\}$ being trivial, we can assume without loss of generality that $K_m \neq \{0\}$ for each $m \in M$. Let m be an arbitrary but fixed element of M. We denote by A_m the set of all K_{m_1} $(m_1 \in M)$ with $K_m \cap K_{m_1} \neq \{0\}$. From 2.5 it follows that the system A_m (partially ordered by inclusion) is a chain. Hence $\bar{A}_m = \bigcup K_{m_1}$ $(K_{m_1} \in A_m)$ is a convex l-subgroup of G. Thus $\bar{A}_m \in l(H(S_c(A)))$ for each $m \in M$. If $m, m' \in M$, then either $\bar{A}_m = \bar{A}_{m'}$ or $\bar{A}_m \cap \bar{A}_{m'} = \{0\}$. Let us denote by $\{G_j\}_{j\in J}$ the set of all $\bar{A}_m$ $(m \in M)$. Clearly $G_j \in l(H(S_c(A)))$ for each $j \in J$.

From $G = \bigvee_{m\in M} K_m$ we obtain $G = \bigvee_{j\in J} G_j$. Since $G_j \cap G_{j_1} = \{0\}$ for each pair of distinct elements $j, j_1 \in J$, we have $G = \sum_{j\in J} G_j$, thus (β) holds.

For a nonempty class A of lattice ordered groups we denote by $T_0(A)$ the radical class generated by A. From 2.2, 2.6 and [10], Thm. 3.4 we obtain:

2.7. Corollary. *Let $A = \emptyset$ be a class of linearly ordered groups. Then $T(A) = T_0(H(S_c(A)))$.*

If A is a radical class, then $\varrho^0(A)(G)$ has an analogous meaning as in the case of torsion classes (i.e., $\varrho^0(A)(G)$ is the join of all convex l-subgroups of G belonging to A). For each radical class A and each $G \in \mathcal{G}$, $\varrho^0(A)(G)$ is an l-ideal of G. Let A, B be radical classes. We denote by AB the class of all $G \in \mathcal{G}$ having the property that $G/\varrho^0(A)(G)$ belongs to B. Then AB is a radical class; if, moreover, A and B are torsion classes, then AB is a torsion class as well. Products of torsion classes have been investigated in [12], [8]; for products of radical classes cf. [10]. A radical class A is called complete if $AA = A$.

The following results have been established in [10]:

(*) *Let $A \neq \emptyset$ be a class of linearly ordered groups. Then $T_0(A)$ cannot be represented as a product BC of radical classes B, C distinct from the zero class $\{0\}$.*

(**) *Let $A \neq \emptyset$ be a class of linearly ordered groups. Let $G \in T_0(A)$. Then G cannot be represented as a direct product of an infinite number of nonzero lattice ordered groups.*

From 2.7 and (*) we obtain:

2.8. Theorem. *Let $A \neq \emptyset$ be a class of linearly ordered groups. Then $T(A)$ cannot be represented as a product BC of nonzero torsion classes B, C.*

From (**) and 2.7 we infer:

2.9. Theorem. *Let $A \neq \emptyset$ be a class of linearly ordered groups. Let $G \in T(A)$. Then G cannot be represented as a direct product of an infinite number of nonzero lattice ordered groups.*

3. PRINCIPAL TORSION RADICALS

Let $A = \{G\}$ be a one-element class of lattice ordered groups. Then $T(A)$ will be said to be the principal torsion class generated by G; we write also $T(A) = T(G)$. The corresponding principal torsion radical will be denoted by ϱ_G. From the definition of ϱ_G it follows that ϱ_G is the least element of the class $\{\varrho \in \mathcal{R} : \varrho(G) = G\}$. Let $\mathcal{P}$ be the class of all principal torsion radicals.

The following assertion follows immediately from Corollary 2.4.

3.1. Lemma. ([13], Lemma 1.1.) *Let $G, H \in \mathcal{G}$. Then the following conditions are equivalent:*

(a) $H \in T(G)$.

(b) *There are sets* $\{H_i\}_{i\in I} \subseteq c(H)$, $\{G_i\}_{i\in I} \subseteq c(G)$, *such that, for each* $i \in I$, H_i *is isomorphic with a factor lattice ordered group of* G_i, *and* $H = \bigvee_{i\in I} H_i$.

3.2. Lemma. *Let* $G \in \mathcal{G}$, $\varrho \in \mathcal{R}$, $\varrho \leqq \varrho_G$. *Let* $S = \{(G_j, G_j')\}_{j\in J}$ *be the set of all pairs* (G_j, G_j') *such that* $G_j \in c(G)$, G_j' *is an l-ideal of* G_j, *and* $G_j/G_j' \in C^0(\varrho)$. *Let* $G' = \sum_{j\in J} G_j''$, *where* G_j'' *is isomorphic with* G_j/G_j' *for each* $j \in J$. *Then* $\varrho = \varrho_{G'}$.

Proof. We have to verify that $C^0(\varrho) = T(G')$ is valid. According to the definition of G', $G' = \bigvee_{j\in J} G_j''$ and $G_j'' \in C^0(\varrho)$ for each $j \in J$, hence $G' \in C^0(\varrho)$. Thus $T(G') \subseteq$ $\subseteq C^0(\varrho)$.

Let $H \in C^0(\varrho)$. From $\varrho \leqq \varrho_G$ we obtain $H \in C^0(\varrho_G) = T(G)$. Hence the condition (b) from 3.1 is valid. From $H_i \in c(H)$ it follows that $H_i \in C^0(\varrho)$. Now from the definition of G' and from 3.1 we infer that $H \in T(G')$. Therefore $C^0(\varrho) \subseteq T(G')$.

3.3. Corollary. *Let* $\varrho_1 \in \mathcal{P}$, $\varrho_2 \in \mathcal{R}$, $\varrho_2 \leqq \varrho_1$. *Then* $\varrho_2 \in \mathcal{P}$.

3.4. Lemma. *Let* $A = \{G_i\}_{i\in I}$ *be a set of lattice ordered groups. Then* $T(A) =$ $= T(G)$, *where* $G = \sum_{i\in I} G_i$.

Proof. Since 2.4 implies $G \in T(A)$, hence $T(G) \subseteq T(A)$. Let $H_0 \in T(A)$, then according to 2.4 there is a set $\{H_i\}_{i\in I} \subseteq c(H_0) \cap H(S_c(A))$ such that $H = \bigvee_{i\in I} H_i$. From this and from 3.1 we obtain $H \in T(G)$. Hence $T(G) = T(A)$.

3.5. Corollary. *Let* I *be a set and let* $\{\varrho_i\}_{i\in I} \subseteq \mathcal{P}$. *Then* $\bigvee_{i\in I} \varrho_i \in \mathcal{P}$.

By summarizing, we infer from 3.3 – 3.5:

3.6. Theorem. *The class* $\mathcal{P}$ *is an ideal of the lattice* $\mathcal{R}$. *Moreover,* $\mathcal{P}$ *is closed with respect to taking joins of sets of torsion radicals belonging to* $\mathcal{P}$.

There are torsion radicals that fail to be principal (this follows, e.g., from Thm. 3.14 below); hence $\mathcal{R} \neq \mathcal{P}$. Put $\bar{\varrho}(G) = G$ for each $G \in \mathcal{G}$. Then $\bar{\varrho}$ is the greatest torsion radical. From $\mathcal{R} \neq \mathcal{P}$ and from 3.6 it follows that $\bar{\varrho}$ cannot be principal. Clearly $\bar{\varrho} = \bigvee_{G\in\mathcal{G}} \varrho_G$. Thus $\mathcal{P}$ fails to be closed with respect to arbitrary joins.

Let $G \in \mathcal{G}$. A subset $\{a_i\}_{i\in I}$ of G^+ is said to be disjoint if $a_i \wedge a_j = 0$ whenever $i, j \in$ $\in I$ and a_i, a_j are distinct elements. Put

$b(G) = \sup\{\text{card}\{a_i\}_{i\in I} : \{a_i\}_{i\in I} \text{ is a bounded disjoint subset of } G\}$,

$b_0(G) = \sup\{b(G_1) : G_1 \in H(S_c(\{G\}))\}$,

$m(G) = \sup\{\text{card}[0, x] : 0 \leqq x \in G\}$.

Then $b(G)$, $b_0(G)$ and $m(G)$ are increasing cardinal properties on $\mathcal{G}$ (in the sense introduced in [7]).

Let $G \in \mathcal{G}$ and let $S(G)$ be the set of all subgroups of G; if $S(G)$ is partially ordered by inclusion, then $S(G)$ is a complete lattice.

3.7. Lemma. (Cf. [11].) *The lattice* $c(G)$ *is a closed sublattice of* $S(G)$.

3.8. Theorem. *Let* $\{0\} \neq G \in \mathscr{G}$, $H \in T(G)$. *Then* $b(H) \leqq \max\{b_0(G), \aleph_0\}$, *and this estimate is the best possible.*

Proof. Let $\{H_i\}_{i\in I}$ be as in 3.1 (b), and let $\{h_j\}_{j\in J}$ be a bounded disjoint subset of H. Suppose that $0 < h_j < h \in H$ is valid for each $j \in J$ and that $h_{j_1} \neq h_{j_2}$ whenever j_1, j_2 are distinct elements of J. From 3.7 it follows that there are indices $i_1, \ldots, i_n \in I$ and elements $g_1 \in H_{i_1}, \ldots, g_n \in H_{i_n}$ with $h = g_1 + \ldots + g_n$. Thus $h \leqq g_1^+ + \ldots + g_n^+ = h'$ and hence $h_j \leqq h'$ for each $j \in J$. Without loss of generality we can suppose that $0 < g_k^+$ for $k = 1, 2, \ldots, n$. Let $j \in J$ be fixed; if $g_k^+ \wedge h_j = 0$ for $k = 1, \ldots, n$, then we should have $h' \wedge h_j = 0$, which is a contradiction. Hence there is $k \in \{1, \ldots, n\}$ with $g_k^+ \wedge h_j > 0$. For $k \in \{1, \ldots, n\}$ put $J(k) = \{j \in J : g_k^+ \wedge h_j > 0\}$. Then $J = J(1) \cup \ldots \cup J(n)$ and for each $k \in \{1, \ldots, n\}$, $\{g_k^+ \wedge h_j\}_{j\in J(k)}$ is a disjoint subset of H_{i_k}. Thus according to the definition of $b_0(G)$ we have card $J(k) \leqq b_0(G)$ for each $k \in \{1, \ldots, n\}$. This implies card $J \leqq n\, b_0(G)$. Therefore $b(H) \leqq \max\{b_0(G), \aleph_0\}$.

Let $\{G_K\}_{k\in K}$ be an infinite set of lattice ordered groups such that

(α) $\{G_k\}_{k\in K} \subseteq H(S_c(\{G\}))$,

(β) for each $G_1 \in H(S_c\{G\}))$ there is $k \in K$ such that G_1 is isomorphic with G_k.

Put $G' = \sum_{k\in K} G_k$. Clearly $G' \in T(G)$ and it is a routine to verify that $b(G') = \max\{b_0(G), \aleph_0\}$.

3.9. Corollary. *Let* $\{0\} \neq G \in \mathscr{G}$. *Then there is a cardinal* α *with* $b(H) \leqq \alpha$ *for each* $H \in T(G)$.

3.10. Remark. If A is a torsion class and if there is a cardinal α such that $b(H) \leqq \alpha$ for each $H \in A$, then A need not be principal.

Example. Let $A_1 \neq \emptyset$ be a class of nonzero linearly ordered groups, $A = T(A_1)$. From 2.6 it follows that we have $b(H) \leqq \aleph_0$ for each $H \in A$. The class A need not be principal (this can be verified by using Thm. 3.14 below and the fact that for each cardinal α there exists a linearly ordered group G such that card $[0, x] \geqq \alpha$ for each $0 < x \in G$).

3.11. Lemma. *Let* $G \in \mathscr{G}$, $0 \leqq a \in G$, $0 \leqq b \in G$ *and let* α *be an infinite cardinal. If* card $[0, a] \leqq \alpha$, card $[0, b] \leqq \alpha$, *then* card $[0, a + b] \leqq \alpha$.

For proving this the proof of Lemma 3.1 of [9] can be applied.

3.12. Lemma. *Let* $\{0\} \neq G \in \mathscr{G}$, $H \in T(G)$. *Then* $m(H) \leqq m(G)$.

Proof. Let $\{H_i\}_{i\in I}$ be as in 3.1 (b) and let $0 < h \in H$. Further, let $g_1, \ldots, g_n, h'$ be as in the proof of 3.8. Since $H_i \in H(S_c(\{G\}))$, we have card $[0, g_k] \leqq m(G)$ for $k = 1, \ldots, n$ and hence according to 3.11, card $[0, h] \leqq$ card $[0, h'] \leqq m(G)$. Therefore $m(H) \leqq m(G)$.

Since $G \in T(G)$, the estimate given in 3.12 is the best possible. Let us further remark that if $\{0\} \neq G \in \mathscr{G}$, then $m(G) \geqq \aleph_0$.

Let $G \in \mathcal{G}$, $0 \leqq x \in G$. Then

$$G_x = \bigcup [-nx, nx] \quad (n = 1, 2, 3, \ldots)$$

is the least convex l-subgroup of G containing the element x. From 3.11 it follows that if α is an infinite cardinal with card $[0, x] \leqq \alpha$, then card $G_x \leqq \alpha$.

3.13. Lemma. *Let α be an infinite cardinal. Let A_α be the class of all lattice ordered groups G with $m(G) \leqq \alpha$. Then A_α is a principal torsion class.*

Proof. Consider the conditions (a), (b) and (c) from the definition of torsion class (cf. § 1). The class A_α obviously fulfils the conditions (a) and (c). Let $H \in \mathcal{G}$ and let $H_1 = \{x \in H : \text{card } [0, x] \leqq \alpha\}$. From 3.11 it follows that H_1 is a convex l-subgroup of H. Hence H_1 is the largest convex l-subgroup of H belonging to A_α. This implies that A_α fulfils the condition (b) as well; hence A_α is a torsion class. We have to verify that A_α is principal.

Let us denote by $\bar{A}_\alpha$ the class of all lattice ordered groups G_1 with card $G_1 \leqq \alpha$. There exists a set $\{G_i\}_{i \in I}$ such that

(a) $G_i \in \bar{A}_\alpha$ for each $i \in I$;

(b) for each $G_1 \in \bar{A}_\alpha$ there is $i \in I$ such that G_1 is isomorphic with G_i.

Let $G = \sum_{i \in I} G_i$. Then we have $m(G) \leqq \alpha$, hence $G \in A_\alpha$. Let $H \neq \{0\}$ be an arbitrary element of A_α. For each $0 < x \in H$ let H_x be the convex l-subgroup of H generated by x. Then card $H_x \leqq \alpha$, hence there is $i \in I$ such that H_x is isomorphic with G_i. This together with 3.1 implies $H \in T(G)$. Thus $A_\alpha = T(G)$.

From 3.12, 3.13 and 3.3 we obtain:

3.14. Theorem. *Let A be a torsion class. Then the following conditions are equivalent*:

(a) *A is principal.*

(b) *There is a cardinal α such that $m(G) \leqq \alpha$ for each $G \in A$.*

4. COVERING RELATIONS

Let $\varrho_1, \varrho_2 \in \mathcal{R}$, $\varrho_1 \leqq \varrho_2$. The interval $[\varrho_1, \varrho_2]$ is defined to be the class $\{\varrho \in \mathcal{R} : : \varrho_1 \leqq \varrho \leqq \varrho_2\}$. If card $[\varrho_1, \varrho_2] = 2$, then ϱ_2 is said to cover ϱ_1 and in such a case we write $\varrho_1 \prec \varrho_2$; we also say that $[\varrho_1, \varrho_2]$ is a prime interval or that ϱ_2 is an atom over ϱ_1. The class of all atoms over ϱ_1 will be denoted by $A(\varrho_1)$. The relation $\varrho_1 \prec \varrho_2$ is obviously equivalent with the fact that (α) $C^0(\varrho_1)$ is a proper subclass of $C^0(\varrho_2)$, and (β) if A is a torsion class with $A \subseteq C^0(\varrho_2)$ such that $C^0(\varrho_1)$ is a proper subclass of A, then $A = C^0(\varrho_2)$. The above situation will be denoted also by writing $C^0(\varrho_1) \prec \prec C^0(\varrho_2)$ (i.e., $C^0(\varrho_2)$ covers $C^0(\varrho_1)$).

Let Ab be the class of all abelian lattice ordered groups. Since each variety of lattice ordered groups is a torsion class (Holland [5]), Ab is a torsion class. Let R be the additive group of all reals with the natural linear order. Let R_0 be the set of all l-

subgroups of R having more than one element. The trivial torsion class $\{\{0\}\}$ will be denoted by $\bar{0}_C$.

4.1. Proposition. *Let A be a torsion class, $A \subseteq$ Ab. Then the following conditions are equivalent:*

(a) $\bar{0}_C \prec A$.

(b) *There is $G \in R_0$ such that $A = T(G)$.*

Proof. Let $\bar{0}_C \prec A$. Hence there exists $H \in A$ with $H \neq \{0\}$. Let $0 \neq x \in H$. From the Axiom of Choice it follows that there exists a convex l-subgroup G_1 of H such that (i) $x \notin G_1$, and (ii) if $G' \in c(H)$ and $G_1 \subset G'$, then $x \in G'$. Let G_2 be the convex l-subgroup of H generated by the element x. It is well-known that $G_2/G_1 \in R_0$. Since A is a torsion class, we have $G_2/G_1 \in A$, whence $T(G_2/G_1) \subseteq A$. Because $T(G_2/G_1) \neq \bar{0}_C$ and $\bar{0}_C \prec A$, we obtain $A = T(G_2/G_1)$; thus (b) holds.

Assume that (b) is valid. Let A_1 be a torsion class with $\bar{0}_C \neq A_1 \subseteq A$. Choose $\{0\} \neq H \in A_1$ and let x, G_1, G_2 be as above. Then $G_2/G_1 \in R_0$ and $G_2/G_1 \in A_1$, hence $G_2/G_1 \in A$. If $G_3 \neq \{0\}$ is a homomorphic image of a convex l-subgroup of G, then $G_3 = G$. From this and from 2.6 it follows that each lattice ordered group belonging to A and distinct from $\{0\}$ can be written as $\sum_{j \in J} G_j$, where each G_j is isomorphic with G. Therefore G_2/G_1 is isomorphic with G and thus $T(G) \subseteq A_1$. Hence $A_1 = A$, and so (a) holds.

From the above proof we also obtain the following corollary:

4.2. Corollary. *Let A be a torsion class, $A \neq \bar{0}_C$, $A \cap$ Ab $\neq \bar{0}_C$. Then there exists a torsion class A_1 such that $\bar{0}_C \prec A_1 \subseteq A$ and $A_1 \cap$ Ab $\neq \emptyset$.*

Let I be a linearly ordered set and for each $i \in I$ let G_i be a lattice ordered group such that G_i is linearly ordered whenever i is not the greatest element of I. We denote by $G = \Gamma_{i \in I} G_i$ the lexicographic product of lattice ordered groups G_i $(i \in I)$ (cf. e.g., [4], [9]). If $I = \{1, 2\}$, then we also write $G = G_1 \circ G_2$. (For some basic properties of lexicographic products cf. [9], p. 452–453.)

4.3. Proposition. *Let $G \in \mathscr{G}$. There exists a principal torsion class A such that $T(G) \prec A$.*

Proof. For each ordinal α and each linearly ordered group G_1 let $G_1(\alpha)$ be the lexicographic product $\Gamma_{\beta < \alpha} G_\beta$, where each G_β is isomorphic with G_1. Let $G_1 \in R_0$. From 3.14 it follows that there is an ordinal α such that $G_1(\alpha) \notin T(G)$; let α be the least ordinal having the mentioned property. Put $G' = G \times G_1(\alpha)$, $A = T(G')$. Then $G' \in T(G)$, $G \in A$, hence $T(G)$ is a proper subclass of A.

Let A_1 be a torsion class such that $T(G)$ is a proper subclass of A_1 and $A_1 \subseteq A$. Then according to 3.3 there is $G_3 \in \mathscr{G}$ with $A_1 = T(G_3)$. We have $G_3 \in A$. Hence in view of 2.4, G_3 can be expressed as $G_3 = \bigvee_{i \in I} H_i$, where $\{H_i\}_{i \in I} \subseteq c(G_3)$, and for each $i \in I$ there are $G_i, G_i' \in c(G')$ such that G_i is an l-ideal of G_i' and H_i is isomorphic to G_i'/G_i. We have $G_i' = (G_i' \cap G) \times (G_i' \cap G_1(\alpha))$ and an analogous relation holds for G_i. The factor lattice ordered group G'_i/G_i is isomorphic to

$$(G'_i \cap G)/(G_i \cap G) \times (G'_i \cap G_1(\alpha))/(G_i \cap G_1(\alpha)) .$$

For each $i \in I$ there is an ordinal $\alpha_i \leqq \alpha$ such that $G'_i \cap G_1(\alpha)/(G_i \cap G_1(\alpha))$ is isomorphic to $G_1(\alpha_i)$. If $i \in I$ and $\alpha_i < \alpha$, then $G_1(\alpha_i) \in T(G)$, hence $G'_i \in T(G)$. In the case $\alpha_i < \alpha$ for each $i \in I$ we would have $G_3 \in T(G)$, thus $A_1 \subseteq T(G)$, which is a contradiction. Thus there is $i \in I$ with $\alpha_i = \alpha$. Then $G_1(\alpha) \in A_1$, hence $G' \in A_1$ and we conclude that $A_1 = A$, completing the proof.

4.4. Proposition. *Let ϱ be a principal torsion radical. Then the class of all atoms over ϱ is infinite.*

Proof. There are infinitely many nonisomorphic types of lattice ordered groups belonging to R_0. Let G_1 and G_2 be elements of R_0 and suppose that G_1 and G_2 are not isomorphic. Let α and β be the corresponding ordinals constructed as in the proof of 4.3 and let $G'(G_1) = G \times G_1(\alpha)$, $G'(G_2) = G \times G_2(\beta)$. In view of 4.3 it suffices to verify that $T(G'(G_1)) \neq T(G'(G_2))$.

Assume that $T(G'(G_1)) = T(G'(G_2))$. Hence $G_2(\beta) \in T(G'(G_1))$. Thus according to 2.4 there are

$$\{H_i\}_{i\in I} \subseteq c(G_2(\beta)), \quad \{G_i\}_{i\in I}, \quad \{G'_i\}_{i\in I} \subseteq c(G'(G_1))$$

such that $G_2(\beta) = \bigvee_{i\in I} H_i$ and for each $i \in I$, G_i is an l-ideal of G'_i and G'_i/G_i is isomorphic to H_i. Each H_i is isomorphic to some $G_2(\beta_i)$, $\beta_i \leqq \beta$. Similarly as in the proof of 4.3, G'_i/G_i is isomorphic to

$$(G \cap G'_i)/(G \cap G_i) \times (G_1(\alpha) \cap G'_i)/(G_1(\alpha) \cap G_i) . \tag{1}$$

Thus $G_2(\beta_i)$ is isomorphic to (1). Since $G_2(\beta_i)$ is linearly ordered, it is directly indecomposable and hence either

(a) $G_2(\beta_i)$ is isomorphic to $(G_1(\alpha) \cap G'_1)/(G_1(\alpha) \cap G_i)$;

or

(b) $G_2(\beta_i)$ is isomorphic to $(G \cap G'_i)/(G \cap G_i)$.

There is an ordinal α_1 such that $(G_1(\alpha) \cap G'_1)/(G_1(\alpha) \cap G_i)$ is isomorphic to $G_1(\alpha_1)$; since G_1 and G_2 fail to be isomorphic, $G_2(\beta_i)$ cannot be isomorphic to $G_1(\alpha_1)$. Thus (a) cannot hold. Therefore (b) is valid and hence $H_i \in T(G)$ for each $i \in I$. Thus $G_2(\beta) \in T(G)$, which is a contradiction. We infer that $T(G'(G_1)) \neq T(G'(G_2))$.

The following proposition shows that all prime intervals in $\mathscr{R}$ can be constructed from prime intervals with principal endpoints.

4.5. Proposition. *Let $[\varrho, \varrho']$ be a prime interval in $\mathscr{R}$. Then there exists a prime interval $[\varrho_1, \varrho_2]$ in $\mathscr{R}$ such that* (i) *ϱ_1, ϱ_2 are principal, and* (ii) *$\varrho_1 \leqq \varrho$, $\varrho \vee \varrho_2 = \varrho'$.*

Proof. Put $A = C^0(\varrho)$, $A' = C^0(\varrho')$. From $\varrho < \varrho'$ it follows that A is a proper subclass of A', hence there exists $G \in A' \setminus A$. Put $\varrho_2 = T(G)$. Then $\varrho_2 \leqq \varrho'$ and $\varrho_3 \nleqq \varrho$.

This together with $\varrho < \varrho'$ implies $\varrho \vee \varrho_2 = \varrho'$. Put $\varrho_1 = \varrho \wedge \varrho_2$. Since $\mathscr{R}$ is distributive we infer that $\varrho_1 \prec \varrho_2$. Moreover, in view of 3.3, ϱ_1 is principal.

For any $\varrho \in \mathscr{R}$ let $(\varrho]$ be the principal ideal of $\mathscr{R}$ generated by ϱ, i.e., $(\varrho] = \{\varrho_1 \in \mathscr{R} : \varrho_1 \leqq \varrho\}$. For any class $X \neq \emptyset$ of torsion radicals we denote

$$a(X) = \bigcup_{x \in X} A(x) .$$

6.6. Corollary. *For each $\varrho \in \mathscr{R}$ we have*

$$A(\varrho) = \{\varrho \vee \varrho_1 : \varrho_1 \in a((\varrho] \cap \mathscr{P})\} \setminus \{\varrho\} .$$

4.7. Corollary. *Let $\varrho \in \mathscr{R}$. The following conditions are equivalent:*

(a) $A(\varrho) = \emptyset$.

(b) *If ϱ_1, ϱ_2 are principal torsion radicals wiht $\varrho_1 \prec \varrho_2$ and if $\varrho_1 \leqq \varrho$, then $\varrho_2 \leqq \varrho$.*

From 4.6 and 3.5 we obtain:

4.8. Corollary. *Let ϱ and ϱ' be torsion radicals such that (i) ϱ' covers ϱ, and (ii) ϱ is principal. Then ϱ' is principal as well.*

4.9. Proposition. *Let $\sigma \in \mathscr{R}$. There exists $\varrho' \in \mathscr{R}$ such that the following conditions are fulfilled:*

(i) *$\sigma \leqq \varrho'$ and $A(\varrho') = \emptyset$.*

(ii) *If $\varrho'' \in \mathscr{R}$ is such that $\sigma \leqq \varrho''$ and $A(\varrho'') = \emptyset$, then $\varrho' \leqq \varrho''$.*

Proof. Denote $L_1 = (\sigma] \cap \mathscr{P}$, $\varrho_1 = \sigma \vee \sup a(L_1)$. Let $\alpha > 1$ be an ordinal and suppose that we have defined the classes L_β and torsion radicals ϱ_β for each ordinal $\beta < \alpha$. We define L_α as follows. If α is non-limit, $\alpha = \beta_1 + 1$, then we put $L_{{}_1} = (\varrho_{\beta_1}] \cap \mathscr{P}$. In the case when α is a limit ordinal we denote $L_\alpha = \bigcup_{\beta < \alpha} (\varrho_\beta] \cap \mathscr{P}$. In both cases we put $\varrho_\alpha = \sigma \vee \sup a(L_\alpha)$. Let ϱ' be the join of the class consisting of all torsion radicals ϱ_α. From 4.7 it follows that (ii) is valid. Clearly $\sigma \leqq \varrho'$.

Let σ_1 be a principal torsion radical generated by a lattice ordered group G. Suppose that $\sigma_1 \leqq \varrho'$. Hence $\varrho'(G) = G$ and thus $G = \bigvee_\alpha \varrho_\alpha(G)$. Thus there is an ordinal α_0 such that $G = \bigvee_{\alpha \leqq \alpha_0} \varrho_\alpha(G)$. Because $\varrho_\alpha \leqq \varrho_{\alpha_0}$ holds for each $\alpha < \alpha_0$, we obtain $G = \varrho_{\alpha_0}(G)$, whence $\sigma_1 \leqq \varrho_{\alpha_0}$, and thus for each $\sigma_2 \in a(\sigma_1)$ we have $\sigma_2 \leqq \varrho_{\alpha_0 + 1} \leqq \varrho'$. This together with 4.7 implies $A(\varrho') = \emptyset$.

The torsion radical σ uniquely determines ϱ'; let us put $\varrho' = \varrho'(\sigma)$. If $\sigma_1, \sigma_2 \in \mathscr{R}$, $\sigma_1 \leqq \sigma_2$, then from the construction given in the proof of 4.9 it follows that for solving the question on the existence of $\varrho \in \mathscr{R}$ with $\varrho \neq \bar{\varrho}$ and $A(\varrho) = \emptyset$ it suffices to verify whether $\varrho'(\overline{0}) \neq \bar{\varrho}$.

4.9.1. Remark. It is an open question whether for each principal torsion radical ϱ the following condition (*) holds:

(*) There exists $\varrho' \in \mathscr{R}$ such that (i) $\varrho' \in A(\varrho)$, and (ii) if $\varrho_1 \in \mathscr{R}$, $\varrho_1 < \varrho'$, then $\varrho_1 < \varrho'$, then $\varrho_1 \leqq \varrho$.

4.10. Proposition. *Let ϱ be a principal torsion radical generated by a lattice ordered group G. Let $R_1 \in R_0$. Suppose that α is an ordinal such that*

(i) $R_1(\alpha) \circ G \notin T(G)$.

(ii) $R_1(\beta) \circ G \in T(G)$ *for each ordinal β with $\beta < \alpha$.*

(iii) $R_1(\alpha) \circ (G/G_1) \in T(G)$ *for each l-ideal G_1 of G with $G_1 \neq \{0\}$.*

Then $(*)$ is valid.

Proof. a) Let ϱ' be the principal torsion radical generated by $G' = R_1(\alpha) \circ G$. First we have to verify that the torsion class $T(G)$ is covered by $T(G')$. Since $G \in c(G')$, we have $T(G) \subseteq T(G')$; from (i) we obtain $T(G) \neq T(G')$. Let A be a torsion class such that $T(G) \subset A \subseteq T(G')$. According to 3.3, there exists $H \in \mathscr{G}$ such that $A = T(H)$. We have $H \in T(G')$, hence there are $\{H_i\}_{i \in I} \subseteq c(H)$, $\{G_i\}_{i \in I}$, $\{G'_i\}_{i \in I} \subseteq c(G')$ such that $H = \bigvee_{i \in I} H_i$, and for each $i \in I$, G_i is an l-ideal of G'_i having the property that G'_i/G_i is isomorphic with H_i. Let $i \in I$. If $G'_i \neq G'$ or $G_i \neq \{0\}$, then according to (ii) and (iii) we have $H_j \in T(G)$. Hence there is $i \in I$ with $G_i = \{0\}$, $G'_i = G'$. Thus $G' \in A$, implying $A = T(G')$. Therefore $T(G) \prec T(G')$.

b) Let A be any torsion class with $A \subset T(G')$. Then A is principal; let A be generated by a lattice ordered group H. Further let H_i, G_i, G'_i $(i \in I)$ be as in a). If there exists $i \in I$ with $G_i = \{0\}$ or $G'_i = G'$, then $A = T(G')$, which is a contradiction. Thus $G_i \neq \{0\}$ and $G'_i \neq G'$ for each $i \in I$ and hence $H_i \in T(G)$ for each $i \in I$. We infer that $H \in T(G)$ and thus $A \subseteq T(G)$. Hence $(*)$ is valid.

Let $R_1 \in R_0$. If we put $G = R_1$, $\alpha = 1$, we obtain:

4.11. Corollary. *Let $R_1 \in R_0$. Then $T(R_1)$ is the unique torsion class covered by $T(R_1 \circ R_1)$.*

Let us denote by N_0 the additive group of all integers with the natural linear order. Let $R_1 \in R_0$ and let n be a positive integer. Let $f(R_1, n)$ be the set of all $(n + 1)$-tuples $x = (x_1, \ldots, x_n, x_0)$ such that $x_i \in R_1$ for $i = 1, \ldots, n$, and $x_0 \in N_0$. Let $x, y \in f(R_1, n)$. We put $x + y = z$, where

$$z_0 = x_0 + y_0 \,, \quad z_i = x_i + y_{i-j} \,,$$

$j \in \{0, 1, \ldots, n - 1\}$, $j \equiv x_0 \pmod n$. Further we put $x \leqq y$ if either $x_0 < y_0$, or $x_0 = y_0$ and $x_i \leqq y_i$ $(i = 1, \ldots, n)$. Then $f(R_1, n)$ is a lattice ordered group.

Lattice ordered groups $f(N_0, n)$ have been used in [6]; $f(N_0, 2)$ is described in [1], p. 311, Example 9.

4.12. Lemma. *Let $R_1 \in R_0$ and let n be a positive integer, $n \geqq 2$. Then $T(f(R_1, n))$ covers $T(R_1 \times N_0)$.*

Proof. Clearly R_1, $N_0 \in T(f(R_1, n))$, hence $T(R_1 \times N_0) \subseteq T(f(R_1, n))$. Now 2.6 implies $f(R_1, n) \notin T(R_1 \times N_0) = T(\{R_1, N_0\})$, thus $T(R_1 \times N_0) \subset (f(R_1, n))$.

Let G_1 be the set of all $x \in f(R_1, n)$ with $x_0 = 0$. Then G_1 is an l-ideal of $f(R_1, n)$ isomorphic with R_1^n. Moreover, $f(R_1, n)$ is a lexico extension of G_1; i.e., if $y \in f(R_1, n) \setminus G_1$, then either $y > x$ for each $x \in G_1$ or $y < x$ for each $x \in G_1$. Let A

be a torsion class with $T(R_1 \times N_0) \subset A \subseteq T(f(R_1, n))$. Then A is a principal torsion class generated by a lattice ordered group H. There exist $\{H_i\}_{i\in I} \subseteq c(H)$, $\{G_i\}_{i\in I}$, $\{G_i'\}_{i\in I} \subseteq c(f(R_1, n))$ such that $H = \bigvee_{i\in I} H_i$ and for each $i \in I$, G_i is an l-ideal of G_i' and $H_i \neq \{0\}$ is isomorphic to G_i'/G_i. If $G_i' \subseteq G_1$ for each $i \in I$, then $H_1 \in T(R_1)$ for each $i \in I$, hence $H \in T(R_1 \times N_0)$ and thus $A \subseteq T(R_1 \times N_0)$, which is a contradiction. Hence there is a nonempty subset I_1 of I such that, for each $i \in I_1$, G_i' is not a subset of G_1. From this it follows that $G_i' = f(R_1, n)$ for each $i \in I_1$. Hence $G_i \in \{\{0\}, G_1\}$ is valid for each $i \in I_1$. If $G_i = G_1$ for each $i \in I_1$, then (because in this case G_i'/G_i is isomorphic to N_0 and hence $H_i \in T(R_1 \times N_0)$) we should have $H \in T(R_1 \times N_0)$, a contradiction. Thus there is $i \in I_1$ with $G_i = \{0\}$, and hence H_i is isomorphic with $f(R_1, n)$. Therefore $A = T(f(R_1, n))$, completing the proof.

By using similar consideration as in the proof of 4.12 we get:

4.13. Lemma. *Let $R_1, R_2 \in R_0$ and let n_1, n_2 be positive integers. If $f(R_1, n_1) = f(R_2, n_2)$, then $R_1 = R_2$ and $n_1 = n_2$.*

Let us denote by Repr the class of all representable lattice ordered groups. If n is a positive integer, $n \geqq 2$, then clearly for each $R_1 \in R_0$, $f(R_1, n)$ is non-abelian and non-representable. On the other hand, $R_1 \times N_0 \in \text{Ab}$. Put

$$\varrho_{\text{Ab}} = \varrho^0(\text{Ab}), \quad \varrho_{\text{Repr}} = \varrho^0(\text{Repr}).$$

From 4.12, 4.13 and 4.6 we obtain:

4.14. Proposition. *Let $\varrho \in \{\varrho_{\text{Ab}}, \varrho_{\text{Repr}}\}$. Then there are infinitely many torsion radicals covering ϱ.*

5. ON THE TORSION CLASSES X^G

Let G be a lattice ordered group. From the fact that $\mathscr{R}$ is a complete lattice and from the relations between torsion radicals and torsion classes it follows that there is a largest torsion class X^G such that $\varrho^0(X^G)(G) = \{0\}$ (i.e., $\varrho^0(X^G)$ is the join of all torsion radicals ϱ having the property that $\varrho(G) = \{0\}$). The lattice ordered group G is said to be homogeneous if for each $\varrho \in \mathscr{R}$ we have either $\varrho(G) = \{0\}$ or $\varrho(G) = G$ (cf. [13]).

Martinez [13] proved several results concerning the relations between homogeneity of G and properties of the torsion class X^G. Let us quote the following theorem:

5.1. Theorem. ([13], Thm. 4.1.) *Let G be an l-group.* (i) *If G is homogeneous, then X^G is a complete, meet irreducible torsion class.* (ii) *If X^G is meet irreducible, G has a non-trivial homogeneous l-ideal.* (iii) *If X is any complete, meet irreducible torsion class, there is a homogeneous l-group H so that $X = X^H$.*

5.1.1. Remark. In 5.1 (ii) we must also assume that $G \neq \{0\}$. (In fact, if $G = \{0\}$, then $X^G = \mathscr{G}$, $\mathscr{G}$ is meet irreducible and G has no non-trivial l-ideal.)

(Let us also remark that in [12] and [13] the denotations for a torsion class A and for the corresponding torsion radical $\varrho^0(A)$ are not distinguished, i.e., $\varrho^0(A)$ is denoted by A; the universe of all torsion classes is denoted by $\mathscr{T}$.)

Again, let G be any lattice ordered group. In [12], § 4, it is remarked that 'it would be convenient if X^G were meet-irreducible in $\mathscr{T}$, but in general it is not clear what happens with classes that contain X^G properly'.

The condition

(α) X^G is meet-irreducible in $\mathscr{T}$

can be expressed, in our terminology, by the equivalent condition

(α_1) $\varrho^0(X^G)$ is meet-irreducible in $\mathscr{R}$.

In the example 5.1.2 below it will be shown that there exist lattice ordered groups G such that the condition (α_1) fails to hold.

5.1.2. Example. Let $R_1, R_2 \in R_0$, $R_1 \neq R_2$, $G = R_1 \times R_2$. Then for $H \in \mathscr{G}$ we have $H \in X^G$ if and only if $\varrho_H(G) = \{0\}$. From

$$\varrho_{R_1} \wedge \varrho_{R_2} = \bar{0}\,, \quad \varrho_{R_i} \wedge \varrho^0(X^G) = \bar{0} \quad (i = 1, 2)$$

we obtain

$$\varrho_{R_i} \vee \varrho^0(X^G) \succ \varrho^0(X^G) \quad (i = 1, 2)\,,$$

$$(\varrho_{R_1} \vee \varrho^0(X^G)) \wedge (\varrho_{R_2} \vee \varrho^0(X^G)) = \varrho^0(X^G)\,,$$

hence $\varrho^0(X^G)$ is finitely meet-reducible.

From 5.1 (i) we obtain immediately:

5.2. Corollary. *Let G be a homogeneous lattice ordered group, $G \neq \{0\}$. Let $\varrho_1 = \varrho^0(X^G)$. Then* card $A(\varrho_1) \leqq 1$.

5.3. Lemma. *Let G be a lattice ordered group, $\varrho_1 = \varrho^0(X^G)$, $\varrho_2 \in A(\varrho_1)$, $H = \varrho_2(G)$. Then H is a nontrivial homogeneous l-ideal of G.*

Proof. H is an l-ideal of G. From $\varrho_2 > \varrho_1$ and from the definition of X^G it follows that $H \neq \{0\}$. Let $\varrho \in \mathscr{R}$ and suppose that $\varrho(H) = H_1 \neq \{0\}$. Denote $\varrho' = \varrho \wedge \varrho_2$. Then $\varrho_1(H) = \{0\}$, $\varrho'(H) = H_1$, hence $\varrho' \nleqq \varrho_1$ and so $\varrho_1 < \varrho' \vee \varrho_1$. On the other hand, $\varrho' \leqq \varrho_2$, thus $\varrho' \vee \varrho_1 \leqq \varrho_2$. Since $\varrho_1 \prec \varrho_2$, we obtain $\varrho' \vee \varrho_1 = \varrho_2$. Thus

$$H = \varrho_2(H) = (\varrho' \vee \varrho_1)(H) = \varrho'(H) \vee \varrho_1(H) = H_1\,.$$

Therefore H is homogeneous.

5.4. Lemma. *Let G and ϱ_1 be as in 5.3. Let $\varrho_2, \varrho_3 \in A(\varrho_1)$, $\varrho_2 \neq \varrho_3$. Then $\varrho_2(G) \cap \varrho_3(G) = \{0\}$.*

Proof. We have $\varrho_2 \wedge \varrho_3 = \varrho_1$, whence $\varrho_2(G) \cap \varrho_3(G) = (\varrho_2 \wedge \varrho_3)(G) = \varrho_1(G) = \{0\}$.

From 5.3 and 5.4 we obtain immediately:

5.5. Theorem. *Let G be a lattice ordered group, $\varrho_1 = \varrho^0(X^G)$. Then $A(\varrho_1)$ cannot be a proper class (i.e., $A(\varrho_1)$ is a set).*

5.6. Theorem. *There exists a torsion class $A \neq \mathcal{G}$ such that there are no atoms over $\varrho^0(A)$.*

Proof. There are linearly ordered groups $G_i \in R_0$ $(i \in I = \{1, 2, 3, \ldots\})$ such that G_i is not isomorphic to G_j whenever i and j are distinct positive integers. Let $G = \Gamma_{i \in I} G_i$ be the lexicographic product of the system $\{G_i\}_{i \in I}$, where I is linearly ordered in the natural way. Let $H \neq \{0\}$ be a convex l-subgroup of G, $H \neq G$. Then there is a positive integer $n > 1$ such that H is isomorphic with $\Gamma_{i \in I_1} G_i$, where $I_1 = \{i \in I : i \geqq n\}$. Choose $m \in I$, $m > n$ and let H_m be the set of all $g \in G$ with $g(i) = 0$ for each $i \in I$, $i < m$. Put $\varrho = \varrho_H$. From Thm. 2.6 it follows that $\varrho(H) = H_m$. Hence there does not exist any homogeneous l-ideal of G distinct from $\{0\}$. This together with 5.3 implies $A(\varrho_1) = \emptyset$, where $\varrho_1 = \varrho^0(X^G)$. Clearly $X^G \neq \mathcal{G}$.

From 5.6 and 4.9 we obtain:

5.6.1. Corollary. *There exists a torsion radical $\varrho < \bar{\varrho}$ such that* (i) *$A(\varrho) = \emptyset$;* (ii) *if $\varrho_1 \in \mathcal{R}$ and $A(\varrho_1) = \emptyset$, then $\varrho \leqq \varrho_1$.*

5.7. Proposition. *Let $\{G_i\}$ $(i \in I)$ be a nonempty set of linearly ordered groups. There exists a linearly ordered group G such that G is homogeneous and $\varrho_{G_i} \leqq \varrho_G$ is valid for each $i \in I$.*

Proof. In view of the Axiom of Choice we can suppose that I is well-ordered (any linear ordering of I would suffice for our purposes). Put $H = \Gamma_{i \in I} G_i$, $G = \Gamma_{n \in N} H_n$, where N is the set of all positive integers with the natural linear order and $H_n = H$ for each $n \in N$. Let $\varrho \in \mathcal{R}$, $\varrho(G) = K \neq \{0\}$. Hence $K \in C^0(\varrho)$. There exists $K_1 \in c(K)$ such that K_1 is isomorphic to G. Thus $G \in C^0(\varrho))$ and so $\varrho(G) = G$; therefore G is homogeneous. According to 2.4, $\varrho_{G_i} \leqq \varrho_G$ for each $i \in I$.

5.8. Corollary. *Let α be cardinal. There exists a homogeneous linearly ordered group G with* card $G \geqq \alpha$.

Let $\varrho \in \mathcal{R}$. We denote by ϱ^δ the join of all torsion radicals ϱ_1 with $\varrho_1 \wedge \varrho = \bar{0}$. Then we have also $\varrho \wedge \varrho^\delta = \bar{0}$.

5.9. Lemma. *Let $\varrho \in \mathcal{R}$. Then ϱ is principal if and only if the class $[\bar{0}, \varrho]$ is a set.*

Proof. Assume that ϱ is principal, $\varrho = \varrho^0(T(G))$, $G \in \mathcal{G}$. Then each $\varrho_1 \in [\bar{0}, \varrho]$ is principal as well, i.e., $\varrho_1 = \varrho^0(T(G_1))$ for some $G_1 \in \mathcal{G}$. Let $S = \{(G_j, G_j')\}_{j \in J}$ be the set of all pairs G_j, $G_j' \in c(G)$ such that G_j is an l-ideal of G_j'. According to 3.1 there exists a subset $I \subseteq J$ and a system $\{H_i\}_{i \in I} \subseteq c(G_1)$ such that $G_1 = \bigvee_{i \in I} H_i$ and H_i is isomorphic to G_i'/G_i for each $i \in I$. Consider the mapping $f : [\bar{0}, \varrho] \to S$ defined by $f(\varrho_1) = \{(G_i, G_i')\}_{i \in I}$. If $\varrho_1, \varrho_2 \in [\bar{0}, \varrho]$ and $f(\varrho_1) = f(\varrho_2)$, then $\varrho_1 = \varrho_2$. Hence $[\bar{0}, \varrho]$ is a set.

Conversely, assume that $[\bar{0}, \varrho]$ is a set. Put $[\bar{0}, \varrho]_1 = [\bar{0}, \varrho] \cap \mathscr{P}$. Then $[\bar{0}, \varrho]_1$ is a set as well and clearly $\varrho = \sup [\bar{0}, \varrho]_1$ holds in $\mathscr{R}$. From this and from 3.5 it follows that ϱ is principal.

5.10. Theorem. *Let ϱ be a principal torsion radical. Then $A(\varrho^\delta)$ is a set and* card $A(\varrho^\delta)$ = card $([\bar{0}, \varrho] \cap A(\bar{0}))$.

Proof. Put $A_1 = [\bar{0}, \varrho] \cap A(\bar{0})$. For each $\varrho_1 \in A_1$ we set $f(\varrho_1) = \varrho_1 \vee \varrho^\delta$. From the definition of ϱ^δ and from the distributivity of $\mathscr{R}$ it follows that f is a one-to-one mapping of A_1 onto $A(\varrho^\delta)$. Since ϱ is a principal torsion radical according to 5.9, A_1 must be a set. Hence $A(\varrho^\delta)$ is a set and card $A(\varrho^\delta)$ = card A_1.

We put $(\varrho^\delta)^\delta = \varrho^{\delta\delta}$. Clearly $\varrho \leqq \varrho^{\delta\delta}$ for each $\varrho \in \mathscr{R}$.

5.11. Proposition. *Let $G \neq \{0\}$ be an abelian lattice ordered group and let ϱ be the principal torsion radical corresponding to G. Then $\varrho < \varrho^{\delta\delta}$.*

Proof. Let A be the principal torsion class generated by G and let α be a cardinal with $\alpha >$ card G. Let I be a linearly ordered set with card $I = \alpha$. According to 4.2 there is $A_1 \in \mathscr{R}$ with $A_1 \cap \mathrm{Ab} \neq \emptyset$, $0_C \prec A_1 \subseteq A$. Let $\{0\} \neq G_1 \in A_1 \cap \mathrm{Ab}$. Put $H = \Gamma_{i \in I} G_i$, where $G_i = G_1$ for each $i \in I$, $G' = H \circ G$. Let ϱ' be the principal torsion radical corresponding to G'. Then $\varrho \leqq \varrho'$. We have card $G' \geqq \alpha$, hence in view of 3.12, G' cannot belong to A. Thus $\varrho < \varrho'$. From the definition of ϱ' we obtain $\varrho' \wedge \varrho^\delta = \bar{0}$. Hence $\varrho' \leqq \varrho^{\delta\delta}$. Therefore $\varrho < \varrho^{\delta\delta}$.

Remark. The question whether the assumption of the commutativity can be cancelled in 5.11 remains open.

5.12. Proposition. *There exists no any dual atom in $\mathscr{R}$.*

Proof. By way of contradiction, suppose that ϱ is a dual atom in $\mathscr{R}$. Then according to 4.5 there exist $\varrho_1, \varrho_2 \in \mathscr{P}$ such that $\varrho_1 \leqq \varrho$, $\varrho_1 \prec \varrho_2$ and $\varrho \vee \varrho_2 = \bar{\varrho}$. There is $G \in \mathscr{G}$ with $\varrho_2 = \varrho_G$. Let G' be as in the proof of 5.11. If $\varrho_1 = \varrho$, then in view of 4.8 we should have $\bar{\varrho} \in \mathscr{P}$, whence $\mathscr{R} = \mathscr{P}$, which is a contradiction. Thus $\varrho_1 < \varrho$, and thus $\varrho_2 \nleqq \varrho$. Therefore $\varrho(G) \subset G$. This implies $\varrho(G') \subset G'$. From 3.12 we obtain $\varrho_2(G') \subset G'$. Further, we have

$$G' = \bar{\varrho}(G') = (\varrho_2 \vee \varrho)(G') = \varrho_2(G') \vee \varrho(G').$$

If both $\varrho(G')$ and $\varrho_2(G')$ are subsets of G, then $G' = G$, which is impossible. Each l-ideal of G' is comparable with G; if either $\varrho_2(G') \supseteq G$ or $\varrho(G') \supseteq G$, then $\varrho_2(G')$ and $\varrho(G')$ are comparable, whence $\varrho(G') \subset G'$, a contradiction.

By analogous reasoning we can verify the validity of the following proposition:

5.13. Proposition. *Let $\varrho \in \{\varrho^0(\mathrm{Ab}), \varrho^0(\mathrm{Repr})\}$. Then no torsion radical is covered by ϱ.*

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