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ON THE TRANSFORMATION THEORY OF ORDINARY SECOND-ORDER LINEAR SYMMETRIC DIFFERENTIAL EXPRESSIONS

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This paper is dedicated to OTAKAR BORŮVKA
on the occasion of his eightieth birthday

1. INTRODUCTION

The purpose of this paper is to describe certain isometric (unitary) transformations of second-order linear symmetric (formally self-adjoint) differential expressions and associated differential equations. Also to discuss certain invariant properties of the differential equations under these transformations.

We are concerned here only with scalar second-order ordinary differential expressions and equations, defined on some interval of the real line. No attempt is made to discuss extensions to higher-order equations, nor to systems of differential equations; however, some general remarks in this respect are made at the end of the paper.

The classical transformation of second-order differential equations is named after J. Liouville (1837). In the notation adopted by Birkhoff and Rota [5, chapter X, sections 1 and 9] the Sturm-Liouville equation

$$(1.1)\qquad (p(x)\,u'(x))' + (\lambda\,\varrho(x) - q(x))\,u(x) = 0 \quad (x \in [a, b])\,,$$

where $'$ denotes differentiation with respect to $x \in [a, b]$, is transformed into the equation (the so-called Liouville normal form)

$$(1.2)\qquad \ddot{w}(t) + (\lambda - \hat{q}(t))\,w(t) = 0 \quad (t \in [0, c])$$

where $\cdot$ denotes differentiation with respect to $t \in [0, c]$, by means of the transformation

$$(1.3)\qquad w(t) = u(x)\,\{p(x)\,\varrho(x)\}^{1/4}, \quad t = \int_a^x \{\varrho(s)/p(s)\}^{1/2}\,\mathrm{d}s\,.$$

The application of this transformation requires the coefficient q to be continuous, and the coefficients p and ϱ to be positive and twice continuously differentiable on the closed interval $[a, b]$. The coefficient $\hat{q}$ is then continuous on the interval $[0, c]$, where $c = \int_a^b \{\varrho/p\}^{1/2}$; for details see [5, chapter X, theorem 6].

With a different notation this transformation is also considered by Ince [21, section 11.4].

It is shown in [5, chapter X, section 9] that the transformation (1.3) above has certain invariant properties between the two differential equations (1.1) and (1.2). In particular let the functions u and v on $[a, b]$ be transformed into functions f and g, respectively, on the interval $[0, c]$ by means of the Liouville transformation (1.3); then

$$\int_a^b \varrho(x)\, u(x)\, \bar{v}(x)\, \mathrm{d}x = \int_0^c f(t)\, \bar{g}(t)\, \mathrm{d}t\,. \tag{1.4}$$

This last result may be written in the form, using a standard inner-product notation,

$$(u, v)_\varrho = (f, g) \tag{1.5}$$

where ϱ now indicates a weighted integrable-square function space (recall ϱ is positive on $[a, b]$). In this way we may claim the Liouville transformation as an isometric (unitary) transformation, see Akhiezer and Glazman [2, section 36], and, in a sense to be made definite in a later section, the differential equations are unitarily equivalent.

It is clear that the application of the Liouville transformation requires considerable restriction on the coefficients p, q and ϱ of the original equation (1.1). One of the purposes of this paper is to consider the transformation theory of symmetric differential expressions and equations under minimal conditions on the coefficients.

The Liouville transformation has a history which extends back to at least 1837; see the remarks in Neuman [24, section 1]. We make no attempt to trace the origins of other transformations of this type but quote recent references as appropriate.

Transformations of the Liouville type have a number of important applications in the theory of differential equations of the form (1.1); see Ince [21, sections 11.4 and 11.5], Titchmarsh [29, sections 1.14 and 5.8], Eastham [8, sections 3.9 and 3.10] and, in the calculus of variations, Weinstock [30, section 8.2].

The contents of the paper are as follows: to complete this section we give some notations, followed by acknowledgements to those mathematicians who have contributed to discussions on transformation theory; section 2 is concerned with properties of symmetric differential expressions and equations; section 3 is devoted to definitions and properties associated with the so-called right-definite and left-definite cases of the second-order differential equation, and their classification theory; in section 4 the transformation theory of the right-definite case is considered, including a discussion of the Liouville transformation under minimal conditions; similarly in section 5 there is a discussion of transformation theory of the left-definite case; finally, some general remarks are made in section 6. The list of references is specific to this paper and should not be thought of as comprehensive for the subject as a whole.

Notations. R and C denote the real and complex number fields; I and J denote arbitrary, not degenerate intervals of R; the end-points of I are denoted by a and b with $-\infty \leqq a < b \leqq \infty$; closed and open end-points of I are denoted by $[,]$ and $(,)$ respectively; the use of $[$ for an end-point requires that $a > -\infty$, and similarly at b; the interval I is compact when $I = [a, b]$; all functions considered take values in R or C and are Lebesgue measurable; $f(\cdot)$ is used when it is needed to emphasize that f is a function; $C^{(r)}(I)$ denotes the class of all functions on I with r continuous derivatives; L denotes Lebesgue integration; $L(I)$, $L^2(I)$ denote the usual integration spaces; if $w \geqq 0$ on I then $L^2_w(I)$ denotes the w-weighted integrable-square function space; $L(a, b)$ sometimes denotes $L(I)$; AC represents absolute continuity; $L_{\text{loc}}(I)$ and $AC_{\text{loc}}(I)$ denote sets of functions which are L or AC on all compact sub-intervals of I (this use of 'loc' requires a distinction to be made between open and closed end-points of I); $\in$ and $\Rightarrow$ are logical symbols representing 'belongs to' and 'implies'; '$(x \in K)$' is to be read as 'the set of all x belonging to the set K'; i is the complex number (0,1); $\bar{z}$ denotes the conjugate complex number of $z \in C$; re$[\ldots]$ and im$[\ldots]$ denote the real and imaginary parts of a complex number; the symbol $a(b)$ is to be read as a respectively b.

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2. DIFFERENTIAL EXPRESSIONS AND EQUATIONS

Generalized or quasi-differential expressions were introduced in the nineteenth century; for some historical references see Bennewitz and Everitt [4, section 2] and Everitt and Zettl [17, section 1].

The general symmetric (*i.e.* formally self-adjoint) quasi-differential scalar expression of the second-order is considered in [17, section 4] and [4, section 2], as follows:

(i) let I be an arbitrary, non-degenerate interval of the real line

(ii) let the coefficients p and $q : I \to R$ and satisfy

$$p(x) \neq 0 \text{ (almost all } x \in I) \quad \text{and} \quad p^{-1}, \quad i.e. \quad 1/p, \in L_{\text{loc}}(I) \tag{2.1}$$

$$q \in L_{\text{loc}}(I) \tag{2.2}$$

(iii) let the coefficient $r : I \to C$ and satisfy

$$r \in L_{\text{loc}}(I) \tag{2.3}$$

(iv) if $f : I \to C$, define the quasi-derivatives $f^{[r]}$ $(r = 0, 1, 2)$ by, here $'$ denotes classical differentiation,

$$f^{[0]} = f, \quad f^{[1]} = p(f' - rf)$$

and

$$f^{[2]} = (p(f' - rf))' + \bar{r}p(f' - rf) - qf \quad \text{on} \quad I$$

(v) define the domain $D(\equiv D(p, q, r)) \subset AC_{\text{loc}}(I)$ on the differential expression M by

$$D = \{f : I \to C : f^{[r]} \in AC_{\text{loc}}(I) \text{ for } r = 0, 1\}$$

(vi) define the linear homogeneous quasi-differential expression $M : D \to L_{\text{loc}}(I)$ by

$$M[f] = \mathrm{i}^2 f^{[2]} = -f^{[2]} \quad (f \in D). \tag{2.4}$$

It may be shown that, under the given conditions on p, q and r, the domain D is dense in $L_{\text{loc}}(I)$; see [17, section 6].

The differential expression M is symmetric in the sense that Green's formula

$$\int_\alpha^\beta \{\bar{g}\, M[f] - f\, \overline{M}[g]\} = [fg]\,(\beta) - [fg]\,(\alpha) \quad (f, g \in D) \tag{2.5}$$

holds for any compact interval $[\alpha, \beta] \subset I$; here $[fg]$ is the skew-symmetric sesquilinear form determined by, for all $f, g \in D$,

$$[fg]\,(x) = f^{[0]}(x)\, \overline{g^{[1]}}(x) - f^{[1]}(x)\, \overline{g^{[0]}}(x) \quad (x \in I). \tag{2.6}$$

We consider also a general first-order symmetric differential expression on I, as follows:

(i) let $w : I \to R$ and satisfy

$$w \in L_{\text{loc}}(I) \tag{2.7}$$

(ii) let $\varrho : I \to R$ and satisfy

$$\varrho \in AC_{\text{loc}}(I) \tag{2.8}$$

(iii) define the domain D_0 by

$$D_0 = \{f : I \to C : f \in AC_{\text{loc}}(I)\}$$

(iv) define the differential expression $S : D_0 \to L_{\text{loc}}(I)$

$$S[f] = \mathrm{i}(\varrho f)' + \mathrm{i}\varrho f' + wf \quad (f \in D_0) \tag{2.9}$$
$$(= 2\mathrm{i}\varrho f' + \mathrm{i}\varrho' f + wf).$$

Clearly D_0 is dense in $L_{\text{loc}}(I)$ and integration by parts shows that S is symmetric with Green's formula

$$\int_\alpha^\beta \{\bar{g}\, S[f] - f\, \bar{S}[g]\} = [2\mathrm{i}\varrho f\bar{g}]_\alpha^\beta \quad (f, g \in D_0) \tag{2.10}$$

for all compact $[\alpha, \beta] \subseteq I$; again the right-hand side is a skew-symmetric sesquilinear form on domain D_0.

It should be noted that all first-order symmetric differential expressions have essentially complex-valued coefficients. The seemingly more general symmetric expression $\mathrm{i}(\varrho y)' + \mathrm{i}\bar{\varrho} y'$, where $\varrho : I \to C$ and $\varrho \in AC_{\mathrm{loc}}(I)$, can be written as $\mathrm{i}(\mathrm{re}[\varrho]\, y)' + \mathrm{i}\,\mathrm{re}[\varrho]\, y' - \mathrm{im}[\varrho]\, y$, and the latter term can be absorbed into the term wy.

We note that the differential expressions are sympathetic to each other in that the domains of M and S satisfy $D \subset D_0$.

With the differential expressions M and S we associate the second-order scalar differential equation on the interval I

$$M[y] = \lambda S[y] \quad \text{on} \quad I, \tag{2.11}$$

where $\lambda \in C$ is a parameter. This is a quasi-differential equation; the existence of solutions may be seen by writing (2.11) in the following equivalent system form

$$Y' = AY \quad \text{on} \quad I,$$

where $Y = [y_1 y_2]^T$ and A is the 2×2 function matrix defined by

$$A = \begin{bmatrix} r - \mathrm{i}\lambda\varrho p^{-1} & p^{-1} \\ q - R & -\bar{r} - \mathrm{i}\lambda\varrho p^{-1} \end{bmatrix} \tag{2.12}$$

and $R = \lambda w + \mathrm{i}\lambda\varrho(r - \bar{r}) + \lambda^2\varrho^2 p^{-1}$. We note that, from the given conditions on the coefficients p, q, r, w and ϱ, it follows that $R \in L_{\mathrm{loc}}(I)$, and from this that the matrix $A \in L_{\mathrm{loc}}(I)$. Standard existence theorems for linear differential systems, see Naimark [23, sections 15 and 16], Zettl [31, section 2] and [17, sections 3 and 5] then show that the quasi-derivatives associated with the differential equation (2.11) are to be defined as

$$y_\lambda^{[0]} = y, \quad y_\lambda^{[1]} = p(y' - ry) + \mathrm{i}\lambda\varrho y \quad \text{on} \quad I, \tag{2.13}$$

where the notation now indicates a dependence, in general, on the parameter λ.

These results imply that given any point $c \in I$ and any two complex numbers α and β, there exists a unique solution $y : I \times C \to C$ of (2.11) such that

$$\begin{gathered} y_\lambda^{[r]}(\cdot, \lambda) \in AC_{\mathrm{loc}}(I) \quad (r = 0, 1;\ \lambda \in C), \\ y_\lambda^{[0]}(c, \lambda) = \alpha, \quad y_\lambda^{[1]}(c, \lambda) = \beta \quad (\lambda \in C), \end{gathered} \tag{2.14}$$

$y_\lambda^{[r]}(x, \cdot)$ is holomorphic on C $(r = 0, 1;\ x \in I)$. These results imply, in turn, that for such a solution y we have y and $p(y' - ry) \in AC_{\mathrm{loc}}(I)$; it should however be noted that $p(y' - ry)$ is not in general a natural quasi-derivative for the differential equation (2.11) except in the special case when $\varrho = 0$ on I.

In (2.14) we may replace α and β by two functions $\alpha(\lambda)$ and $\beta(\lambda)$ $(\lambda \in C)$, which are holomorphic on C, and the same properties continue to hold for the solution y of the differential equation.

The conditions (2. 1, 2, 3, 7 and 8) on the coefficients of (2.11) ensure that any point $c \in I$ is a regular point of the differential equation, *i.e.* solutions determined by the initial conditions (2.14) at c exist on I with the properties stated. The end-point $a(b)$ of I is a regular end-point of the equation if $a > -\infty$ $(b < \infty)$ and the conditions (2. 1, 2, 3, 7 and 8) hold with $I = [a, b)$ $(I = (a, b])$; if this is not the case then the end-point $a \geqq -\infty$ $(b \leqq \infty)$ is said to be a singular end-point of the differential equation.

It should be noted that the $L_{\mathrm{loc}}(I)$ conditions required for the application of the standard existence theorems to the differential equation (2.11), are both necessary and sufficient for existence; see Everitt and Race [16].

The differential equation (2.11) may be classified as right-definite or left-definite if certain additional restrictions are placed on the coefficients; properties of solutions of the equation are then considered in an appropriate Hilbert function space H. It is this framework which introduces the possibility of considering certain invariant properties of the equation under transformations. Details of these two cases are:

(i) *right-definite case*

(2.15) $$\varrho(x) = 0 \quad (x \in I), \quad w(x) \geqq 0 \quad (\text{almost all } x \in I),$$
$$H : L^2_w(I) \quad i.e. \quad \{f \text{ measurable on } I \text{ and } \textstyle\int_I w|f|^2 < \infty\}$$

(ii) *left-definite case*

(2.16) $$p(x) \geqq 0 \quad \text{and} \quad q(x) \geqq 0 \quad (\text{almost all } x \in I),$$
$$H : H^2_{p,q,r}(I) \quad i.e. \quad \{f \in AC_{\mathrm{loc}}(I) \text{ and } \textstyle\int_I \{p\,|f' - rf|^2 + q|f|^2\} < \infty\}.$$

We consider these two cases separately in sections 4 and 5 below.

The case

(2.17) $$w(x) = 0 \quad (\text{almost all } x \in I)$$

is not excluded from (2.15) although it does, in effect, remove the parameter λ from the differential equation, and certain properties of the equation become technically trivial. (In the case of the Liouville transformation (1.3) this would correspond to $\varrho(x) = 0$ $(x \in [a, b])$ which entirely negates use of this transformation.) For consideration of certain properties of the right-definite case the condition (2.16) has to strengthened to require

(2.18) $$w(x) > 0 \quad (\text{almost all } x \in I).$$

For a general discussion of certain theoretical aspects of the differential equation (2.11), in respect of the right- and left-definite cases, see the account of Pleijel [27].

It is possible to consider certain generalizations of the right-definite case; for a detailed account see the work of Atkinson [3, chapter 8, in particular section 8.1].

We consider now certain classifications of the right- and left-definite cases at singular end-points of I of the general differential equation (2.11), *i.e.* $M[y] = \lambda\, S[y]$ on I.

It is convenient to introduce a solution $\phi(\cdot, \lambda)$, for all $\lambda \in C$, of this differential equation determined by the following initial conditions at an arbitrary point $c \in I$:

$$\phi_\lambda^{[0]}(c, \lambda) = 0\,, \quad \phi_\lambda^{[1]}(c, \lambda) = 1 \quad (\lambda \in C)\,, \tag{3.1}$$

i.e. from (2.13 and 2.14), for all $\lambda \in C$,

$$\phi(c, \lambda) = 0\,, \quad p(\phi' - r\phi)(c, \lambda) + \mathrm{i}\lambda\, \varrho(c)\, \phi(c, \lambda) = 1\,. \tag{3.2}$$

3.1. Classification theory: right-definite case. The conditions (2.15) on the coefficients are now taken to hold so that the differential equation is

$$M[y] = \lambda w y \quad \text{on} \quad I \tag{3.3}$$

with quasi-derivatives $y^{[0]} = y$ and $y^{[1]} = p(y' - ry)$. A boundary condition, on the solution y, at the point c is a relationship between the quasi-derivatives of the form

$$y^{[0]}(c, \lambda) \cos \gamma - y^{[1]}(c, \lambda) \sin \gamma = 0 \tag{3.4}$$

satisfied for some $\gamma \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The special cases of (3.4)

$$y^{[0]}(c, \lambda) = 0\,, \quad \textit{i.e.} \quad \gamma = 0\,, \quad \text{and} \quad y^{[1]}(c, \lambda) = 0\,, \quad \textit{i.e.} \quad \gamma = \tfrac{1}{2}\pi\,,$$

are called the Dirichlet and Neumann boundary conditions, at the point c, respectively.

If a (respectively b) is a singular end-point of (3.3) then the equation is said to be *limit-point* (LP) at a (b) if for some value of $\lambda \in C$ we have, with ϕ defined by (3.1),

$$\phi(\cdot, \lambda) \notin L_w^2(a, c) \quad (\phi(\cdot, \lambda) \notin L_w^2(c, b))\,; \tag{3.5}$$

it then follows from the general theory of (3.3) that (3.5) holds for all $\lambda \in C$ with $\operatorname{im}[\lambda] \neq 0$. Otherwise the equation (3.3) is *limit-circle* (LC) at a (b) and for all $\lambda \in C$, real or complex,

$$\phi(\cdot, \lambda) \in L_w^2(a, c) \quad (\phi(\cdot, \lambda) \in L_w^2(c, b))\,; \tag{3.6}$$

in this case the general theory shows that all solutions of (3.3), for all $\lambda \in C$, lie in $L_w^2(a, c)$ ($L_w^2(c, b)$). For these results see [23, section 17.5] or [29, sections 2.1 and 2.19].

In the case when the coefficient w satisfies the stronger condition (2.18) we have the following alternative criterion for the LP classification; let the domain $\Delta \subset L_w^2(I)$ be defined by

$$\Delta = \{f : I \to C \mid \text{(i) } f^{[0]} \text{ and } f^{[1]} \in AC_{\mathrm{loc}}(I) \quad \text{(ii) } f \text{ and } w^{-1} M[f] \in L_w^2(I)\}\,; \tag{3.7}$$

then the equation (3.3) is LP at a (b) if and only if (see (2.6) for $[f, g]$)

$$\lim_{a+(b-)} [f, g] = \lim_{a+(b-)} (f^{[0]}\overline{g^{[1]}} - f^{[1]}\overline{g^{[0]}}) = 0 \quad (f, g \in \Delta) . \tag{3.8}$$

For some details of this result see [23, section 18.3].

Additionally, again with (2.18) holding, we make the following definitions:

(i) the equation (3.3) is *strong limit-point* (SLP) at a (b) if

$$\lim_{a+(b-)} f^{[0]}\overline{g^{[1]}} = 0 \quad (f, g \in \Delta) \tag{3.9}$$

(ii) the equation is Dirichlet (D) at a (b) if

$$|p|^{1/2} (f' - rf) \quad \text{and} \quad |q|^{1/2} f \in L^2(a, c)\, (L^2(c, b)) \quad (f \in \Delta) ; \tag{3.10}$$

is conditional Dirichlet (CD) at a (b) if

$$|p|^{1/2} (f' - rf) \in L^2(a, c) \quad (\in L^2(c, b)) \quad (f \in \Delta) \tag{3.11}$$

and

$$\lim_{x \to a+} \int_x^c q f \bar{g} \left(\lim_{x \to b-} \int_c^x q f \bar{g} \right)$$

exists and is finite for all $f, g \in \Delta$; and is weak Dirichlet (WD) at a (b) if

$$\lim_{x \to a+} \int_x^c \{p(f' - rf)\,\overline{(g' - rg)} + q f \bar{g}\} , \tag{3.12}$$

$$\left(\lim_{x \to b-} \int_c^x \{p(f' - rf)\,\overline{(g' - rg)} + q f \bar{g}\} \right)$$

exists and is finite for all $f, g \in \Delta$.

For these definitions see Everitt [12, section 1], Everitt [13, section 2 and references], Everitt, Giertz and McLeod [14], and Kalf [22].

For the motivation for, and connection between, these definitions we have the Dirichlet formula for the differential equation (3.3)

$$\int_\alpha^\beta \{p(f' - rf)\,\overline{(g' - rg)} + q f \bar{g}\} = f^{[1]}\overline{g^{[0]}} \Big|_\alpha^\beta + \int_\alpha^\beta w(w^{-1} M[f])\, \bar{g} \tag{3.13}$$

valid for all $[\alpha, \beta] \subseteq I$ and all $f, g \in \Delta$.

From these definitions and results it is possible to see that we have the following relationships which hold at either end-point (but not between end-points)

$$LP \Leftarrow SLP \Rightarrow WD \Leftarrow CD \Leftarrow D . \tag{3.14}$$

It is known that all these implication signs are strict, *i.e.* do not hold in the reverse direction, except for $SLP \Rightarrow WD$; it does not seem to be known at present if $WD \Rightarrow$ $\Rightarrow SLP$ although this looks unlikely.

We shall see below that for certain important transformations of the differential equation (3.3), many of these properties of the right-definite case remain invariant.

There is an extensive literature, not quoted here, to obtaining additional conditions on the coefficients p, q, w and r which place the differential equation in the LC, LP, SLP or D condition at a singular end-point; however, see the references in [12], [13] and Hinton [20].

We remark here, since it will be of importance later, that all these considerations of the right-definite case are given without any sign restrictions on the real-valued coefficient p.

3.2. Classification theory; left-definite case. The conditions (2.16) on the coefficients of the general differential equation (2.11), are now taken to hold. Whilst p and q are required to be non-negative on I it should be noted that, in general, ϱ is not null on I and that there is no sign restriction on w.

There is a limit-point (LP) and limit-circle (LC) classification theory of the differential equation (2.11) in the left-definite case; solutions are now considered in the Hilbert (Sobolev) function space $H^2_{p,q,r}(I)$ as defined in (2.16). For this purpose it is necessary to introduce the half-planes of $C : C_+ = \{\lambda \in C : \text{im}[\lambda] > 0\}$ and $C_- = \{\lambda \in C : \text{im}[\lambda] < 0\}$; in any statement where terms such as $C_\pm$ are used the upper sign or the lower sign should be read throughout.

For some aspects of the general theory see the account by Pleijel [27] and the results of Ong [26].

If a (b) is a singular end-point of the differential equation (2.11) then the equation is said to be $LP_\pm$ at a (b) in $C_\pm$ if for some (and then for all) $\lambda \in C_\pm$ the solution ϕ, see (3.1), satisfies

$$\phi(\cdot, \lambda) \notin H^2_{p,q,r}(a, c) \quad (\phi(\cdot, \lambda) \notin H^2_{p,q,r}(c, b)) ; \tag{3.15}$$

otherwise the equation is $LC_\pm$ at a (b) and then all solutions, for all $\lambda \in C_\pm$, lie in $H^2_{p,q,r}(a, c)$ $(H^2_{p,q,r}(c, b))$.

The distinction between C_+ and C_- is essential here; it is possible for the differential equation (2.11) to be in LP_+ and yet in LC_-; consider the example $-y'' = i\lambda y'$ on $[0, \infty)$. For this and other reasons, care has to be exercised in the consideration, in this context, of real values of λ; see the examples in Ong [26, pages 251 and 253].

We do not consider in this paper any other aspects of the classification problem for the left-definite case. In particular we do not consider the definition and properties of differential operators in the $H^2_{p,q,r}$ spaces associated with the differential equation (2.11), but see the work of Pleijel [27].

4. TRANSFORMATION THEORY: RIGHT-DEFINITE CASE

We consider again in this section the general differential equation (2.11) in the right-definite case, *i.e.* under the conditions (2.15); the equation then reduces to

$$M[y] = \lambda wy \quad \text{on} \quad I . \tag{3.3}$$

We recall that of the coefficients of M we have p and q real-valued but with no sign restrictions at this stage; r is complex-valued in general; w is non-negative on I and, where necessary, will be taken to satisfy the positive condition (2.18).

In the application of any transformation we adopt the notation used in Everitt [10]. In the original equation lower case letters x, y, p, q, r, w are used for variables and coefficients, a prime on lower case letters, *e.g.* y', denotes differentiation with respect to x, and the interval I has end-points a and b; in the transformed equation capital letters X, Y, P, Q, W represent new variables and transformed coefficients, Y' denotes differentiation with respect to X, and the interval I is transformed to J with end-points A and B; the differential expression M is transformed to an expression N.

The general theory of isometric transformations and unitary equivalence in Hilbert space is taken from Akhiezer and Glazman [2, sections 35 and 36].

4.1. Transformation to real coefficients. It is somewhat remarkable that the general symmetric differential expression M, given by (2.4), can always be isometrically transformed to a real symmetric differential expression, and this without loss of generality. The transformation to effect this result is due to C. D. Ahlbrandt (personal communication to the author) and is suggested by certain results in Ahlbrandt [1].

In the case of the Ahlbrandt transformation the new independent variable X is defined simply as $X(x) = x$ $(x \in I)$; the new interval $J = I$ so that $A = a$ and $B = b$; however, for consistency, we keep ot the notation given in section 4 above.

We commence with M defined in (2.4) with coefficients p, q and r. Let c be any point of the interval I and define the function $\mu : I \to C$ by

$$\mu(x) = \exp\left[\int_c^x r(t)\,\mathrm{d}t\right] \quad (x \in I)\,; \tag{4.1}$$

then $\mu(x) \neq 0$ $(x \in I)$ and $\mu \in AC_{\mathrm{loc}}(I)$.

Define the transformed coefficients $P, Q, W : J \to R$ by, for all $X \in J$ (equivalently for all $x \in I$)

$$P(X) = |\mu(x)|^2\, p(x)\,, \quad Q(X) = |\mu(x)|^2\, q(x)\,, \quad W(X) = |\mu(x)|^2\, w(x)\,; \tag{4.2}$$

we note that $P^{-1}, Q, W \in L_{\mathrm{loc}}(J)$.

Let the mapping $U : C(I) \to C(J)$ be defined by

$$F(X) = (Uf)(X) = \{\mu(x)\}^{-1} f(x) \quad (x \in I;\ f \in C(I))\,; \tag{4.3}$$

the inverse mapping is then $f = U^{-1}F = \mu F$.

Let the domain D of the differential expression M, as defined in section 2, be now denoted by $D(M)$; introduce a second differential expression N (compare with the definition of M) by

(i) define the quasi-derivatives on J

$$F^{[0]} = F\,, \quad F^{[1]} = PF'\,, \quad F^{[2]} = (PF')' - QF \tag{4.4}$$

(ii) define $D(N) \subset AC_{\mathrm{loc}}(J)$ by

$$D(N) = \{F : J \to C \mid F^{[r]} \in AC_{\mathrm{loc}}(J) \text{ for } r = 0, 1\}$$

(iii) define $N : D(N) \to L_{\mathrm{loc}}(J)$ by

$$N[F] = \mathrm{i}^2 F^{[2]} = -(PF')' + QF \quad (F \in D(N)). \tag{4.5}$$

It is clear that N is symmetric in the sense of section 2.

Now let $f \in D(M)$ and take $F = Uf = \mu^{-1}f$; clearly $F \in AC_{\mathrm{loc}}(J)$ and

$$f' = r\mu F + \mu F' \quad \textit{i.e.} \quad f' - rf = \mu F'. \tag{4.6}$$

This yields, on multiplication by the coefficient p and using (4.2),

$$f^{[1]} = p(f' - rf) = p\mu F' = \bar{\mu}^{-1}PF' = \bar{\mu}^{-1}F^{[1]} \tag{4.7}$$

and so $F^{[1]} \in AC_{\mathrm{loc}}(J)$; hence $F \in D(N)$. If given $F \in D(N)$ we take $f = U^{-1}F = \mu F$ then the argument may be reversed to give $f \in D(M)$. Thus

$$F = Uf \in D(N) \quad \textit{if and only if} \quad f \in D(M). \tag{4.8}$$

From (4.7) we obtain, if also $G = Ug$,

$$f^{[0]}\,\overline{g^{[1]}} = \mu F\,\overline{\bar{\mu}^{-1}}\,\overline{PG'} = F^{[0]}\,G^{[1]} \quad (f, g \in D(M)). \tag{4.9}$$

Again if $f \in D(M)$ then. from (4.6),

$$(p(f' - rf))' = (p\mu F')' = (\bar{\mu}^{-1}\, PF')'$$

and both factors in the last expression are in $AC_{\mathrm{loc}}(J)$; hence

$$(p(f' - rf))' = \bar{\mu}^{-1}(PF')' - \bar{\mu}^{-1}\bar{r}PF'. \tag{4.10}$$

Thus for $f \in D(M)$, and using the definition (4.2)

$$M[f] = -(p(f' - rf))' - \bar{r}p(f' - rf) + qf =$$

$$= -\bar{\mu}^{-1}(PF')' + \bar{\mu}^{-1}\bar{r}PF' - \bar{r}p\mu F' + q\mu F = \bar{\mu}^{-1}\{-(PF')' + QF\},$$

i.e.

$$N[F] = \bar{\mu}\, M[f] \quad (f \in D(M)) \tag{4.11}$$

and, equivalently,

$$M[f] = \bar{\mu}^{-1}\, N[F] \quad (F \in D(N)). \tag{4.12}$$

We note also the result

$$\int_\alpha^\beta M[f]\,\bar{g} = \int_\alpha^\beta N[F]\,\bar{G} \quad (f, g \in D(M)) \tag{4.12a}$$

valid for all $[\alpha, \beta] \subseteq I$; this together with (4.9) connects the Green's formula (2.5 and 6) for the differential expression M with the corresponding formula for N.

We may now transform the differential equation (3.3); let y be a solution of (3.3) on I and make the transformation (recall the notation given in section 4) to new independent and dependent variables

$$(4.13) \qquad X(x) = x\,, \quad Y(X) = \{\mu(x)\}^{-1}\, y(x) \quad (x \in I)\,.$$

We then have $y \in D(M)$, $Y \in D(N)$ from (4.8), and from (4.12)

$$N[Y] = \bar{\mu}\, M[y] = \bar{\mu}\lambda w y = \lambda W Y \quad \text{on} \quad J$$

so that Y satisfies the following symmetric differential equation with real coefficients P, Q and W

$$(4.14) \qquad -(PY')' + QY = \lambda W Y \quad \text{on} \quad J\,.$$

Note again that the transformed coefficients satisfy the basic requirements for the existence of solutions on J, *i.e.* P^{-1}, Q and $W \in L_{\mathrm{loc}}(J)$.

If the original solution y satisfies given initial conditions at $c \in I$ then these are transformed to

$$(4.15) \qquad \begin{aligned} Y(c) &= Y^{[0]}(c) = \mu(c)^{-1}\, y^{[0]}(c) = \mu(c)^{-1}\, y(c)\,,\\ (PY')(c) &= Y^{[1]}(c) = \bar{\mu}(c)\, y^{[1]}(c) = \bar{\mu}(c)\, p(y' - ry)(c)\,. \end{aligned}$$

A boundary condition at c of the form (3.4) is transformed to

$$(4.16) \qquad Y^{[0]}(c)\, |\mu(c)|^2 \cos\gamma - Y^{[1]}(c) \sin\gamma = 0$$

which may be rewritten as

$$Y^{[0]}(c) \cos\Gamma - Y^{[1]}(c) \sin\Gamma = 0$$

for some $\Gamma \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Note, however, that a Dirichlet or Neumann boundary condition is always transformed to a condition of the same kind.

If the end-point a (respectively b) of the differential equation (3.3) is singular then the end-point A (B) is singular for (4.14), and *vice versa*; this follows from the definition of regular and singular end-points given in section 2, and the definitions (4.2).

We now show that this transformation of the equation (3.3) to the equation (4.14) can be justifiably termed an isometric transformation, and that the classification properties described in section 3.1 are ivariant under the transformation.

We note that the non-negative condition (2.15) on w, *i.e.* $w \geqq 0$ on I, transforms under (4.2) to the corresponding condition $W \geqq 0$ on J. The definition (4.3) of the mapping U yields:

(4.17) (i) U maps $L^2_w(I)$ *onto* $L^2_W(J)$

(ii) $(Uf, Ug)_W = (f, g)_w \quad (f, g \in L^2_w(I))$

and this, in the terminology of [2, section 36] implies that U is an isometric mapping of $L^2_w(I)$ onto $L^2_W(J)$; similarly for U^{-1}.

Let $\Phi : J \times C \to C$ be the solution of the transformed equation (4.14) which satisfies the initial conditions, for $c \in J$,

$$\Phi^{[0]}(c, \lambda) = \Phi(c, \lambda) = 0\,, \quad \Phi^{[1]}(c, \lambda) = (P\Phi')(c, \lambda) = 1 \tag{4.18}$$

for all $\lambda \in C$. Then the following relation holds between Φ and the solution ϕ of (3.3) satisfying (3.1), on using (4.15),

$$\phi(x, \lambda) = \bar{\mu}(c)\,\mu(X)\,\Phi(X, \lambda) \quad (x \in I;\ \lambda \in C)\,. \tag{4.19}$$

Hence for any $[\alpha, \beta] \subseteq I$

$$\int_\alpha^\beta w(x)\,|\phi(x, \lambda)|^2\,dx = |\mu(c)|^2 \int_\alpha^\beta W(X)\,|\Phi(X, \lambda)|^2\,dX\,.$$

From the definition in section 3.1 it now follows that the transformed equation (4.14) is $LP\,(LC)$ at end-point $A\,(B)$ if and only if the original equation (3.3) is $LP\,(LC)$ at end-point $a\,(b)$.

To proceed further with the invariant properties of the map U we note that, from (4.2),

$$W(X) > 0 \quad (\text{almost all } X \in J) \tag{4.20}$$

if and only if the corresponding result (2.18) holds for w. Suppose now that (2.18) holds so that the domain $\Delta \subset L^2_w(I)$ is well-defined, see (3.7); we write $\Delta(T) = \Delta$ and define the linear differential operator $T : \Delta(T) \to L^2_w(I)$ by domain of T is $\Delta(T)$ and

$$Tf = w^{-1}\,M[f] \quad (f \in \Delta(T))\,. \tag{4.21}$$

From the differential expression N, see (4.5), and the positive weight W, define an operator

$$S : \Delta(S) \to L^2_W(J) \quad \text{with} \quad \Delta(S) = \{F : J \to C \mid \text{(i) } F^{[0]} \text{ and } F^{[1]} \in AC_{\mathrm{loc}}(J) \quad \text{(ii) } F \text{ and } W^{-1}N[F] \in L^2_W(J)\}$$

$$SF = W^{-1}\,N[F] \quad (F \in \Delta(S))\,. \tag{4.22}$$

From the results (4.8) and (4.11) we see that $f \in D(T)$ if and only $F = Uf \in D(S)$, *i.e.* $D(S) = UD(T)$; also that

$$\begin{aligned}(U^{-1}SU)\,f &= U^{-1}SF \\ &= U^{-1}(W^{-1}\,N[F]) \\ &= U^{-1}(W^{-1}\bar{\mu}\,M[f]) \\ &= U^{-1}(w\mu^{-1}\,M[f]) \\ &= w^{-1}\,M[f] \\ &= Tf \quad (f \in D(T))\,.\end{aligned}$$

Thus, see [2, section 36], the operators T and S are unitarily equivalent under the isometric map U.

The operators T and S are called the maximal operators for M in $L^2_w(I)$ and N in $L^2_W(J)$ respectively; these operators are not, in general, self-adjoint or even symmetric. However, $T(S)$ is self-adjoint in $L^2_w(I)\,(L^2_W(J))$ if and only if the differential equation (3.3) ((4.14)) is LP at both end-points a and b (A and B); for some details see [23, sections 18 and 19].

From the definition (3.9) for SLP and from the identity (4.9) it follows at once that the differential equation (3.3) is SLP at a (b) if and only if the transformed equation (4.14) is SLP at A (B).

The individual terms of the Dirichlet formula (3.13) for the equation (3.3) also have an invariant form (we omit the calculations)

$$\int_\alpha^\beta \{p(f' - rf)\overline{(g' - rg)} + qf\bar{g}\} = \int_\alpha^\beta \{PF'\bar{G}' + QF\bar{G}\}$$

$$\int_\alpha^\beta w(w^{-1} M[f])\,\bar{g} = \int_\alpha^\beta W(W^{-1} N[F])\,\bar{G} \tag{4.23}$$

both valid for all $[\alpha, \beta] \subseteq I$ and all $f, g \in D(M)$.

These last results, together with (4.9), show that the D, CD and WD classifications of the differential equation (3.3) at a (b) hold if and only if the transformed equation (4.14) is similarly classified at A (B).

We may summarize these results; every general right-definite symmetric differential equation of the form (3.3) may be isometrically transformed by the Ahlbrandt transformation (4.3) into a symmetric equation of real form (4.14) in such a way that all the associated classification properties, as detailed in section 3.1, remain invariant under the transformation. Separated, Dirichlet and Neumann boundary conditions of the form (3.4) remain invariant in form; certain differential operators associated with the differential equations are unitarily equivalent under the transformation.

4.2. Transformation to leading coefficient unity. In this section we suppose, if necessary, that the transformation in the previous section has been applied and rewrite the original right-definite equation in the generalized Sturm-Liouville form

$$M[y] = -(py')' + qy = \lambda wy \quad \text{on} \quad I \tag{4.24}$$

where $p, q, w : I \to R$ and $p^{-1}, q, w \in L_{\text{loc}}(I)$; also we have w non-negative on I, *i.e.* from (2.15)

$$w(x) \geqq 0 \quad (\text{almost all } x \in I)\,. \tag{4.25}$$

From section 2 above we recall that the quasi-derivatives for the equation (4.24) are defined to be, on I,

$$y^{[0]} = y\,, \quad y^{[1]} = py'\,, \quad y^{[2]} = (py')' - qy\,. \tag{4.26}$$

The differential expression M is defined by

$$(4.27)\qquad D(M) = \{f : I \to C \mid f^{[0]} \text{ and } f^{[1]} \in AC_{\mathrm{loc}}(I)\}\,,$$
$$M[f] = \mathrm{i}^2 f^{[2]} = -(pf')' + qf \quad (f \in D(M))\,.$$

In general there is no way to transform the symmetric equation (4.24) to a simpler symmetric form, *e.g.* to an equation with a reduced number of coefficients, unless additional conditions are placed on the coefficients p or w or both.

This situation is due essentially to the possibility that p may change sign on I and that w may have zeros (or even vanish in a set of positive measure) in I without violating the basic condition that p^{-1} and $w \in L_{\mathrm{loc}}(I)$, and the non-negative condition $w \geqq 0$ on I. If we take I to be the half-line $[0, \infty)$ then the following examples of p and w satisfy all these conditions but prevent any form of global isometric transformation on $[0, \infty)$ being applied to a differential equation incorporating such coefficients:

$$p(x) = \{\cos(x^{-1} + x)\}^{-1} \quad (x \in (0, \infty))\,,$$
$$w(x) = x^{\alpha} \quad (x \in (0, \infty)) \text{ with } \alpha > -1,\ \alpha \neq 0)\,,$$
$$w(x) = (x-1)^4 \quad (x \in [0, \infty))\,.$$

The coefficient p satisfies $p^{-1} \in C(0, \infty) \cap L_{\mathrm{loc}}[0, \infty)$ but oscillates infinitely often in sign in the neighbourhood of 0 and ∞. The first w gives difficulty at the end-point 0, and the second w at the interior point 1 when, for example, the Liouville transformation of section 1 is applied; these examples of w are mentioned again in the next section. (It is worth remarking that the 'infinities' of p defined above, *e.g.* the points $x \in (0, \infty)$ where $\cos(x^{-1} + x) = 0$, are not singularities of the differential equation involved; it is the zeros of p which may give rise to singular points, but for this coefficient $p(x) \neq 0$ $(x \in (0, \infty))$).

In this section we consider one additional constraint on the coefficients of the differential equation (4.24), that of restricting p to be of one sign on I. Without loss of generality we can take p to be non-negative on I, *i.e.*

$$(4.28)\qquad p(x) \geqq 0 \quad (\text{almost all } x \in I)\,.$$

It is then possible to transform the equation isometrically to an equation of the same form but with the leading coefficient p replaced by unity on the interval concerned. The transformation required is discussed by Eastham [9, section 3.4, (3.4.10)] and by F. V. Atkinson (personal communication to the author); see also the account in Everitt and Halvorsen [15, section 2].

We introduce a new independent variable in the differential equation (4.24) as follows: let $k \in I$, choose $K \in R$ and define the mapping $t(\cdot) : I \to R$ by

$$(4.29)\qquad X = t(x) = K + \int_k^x \{p(t)\}^{-1}\,\mathrm{d}t \quad (x \in I)$$

and

$$A = K - \int_a^k p^{-1}, \quad B = K + \int_k^b p^{-1}.$$

Then $t(k) = K$ and $-\infty \leqq A < B \leqq \infty$; define J to be the interval of R with end-points A and B, noting that the condition $p^{-1} \in L_{\text{loc}}(I)$ implies that if $a\,(b) \in I$ then $A\,(B) \in J$.

Clearly $t(\cdot) : I$ onto J, $t(\cdot) \in AC_{\text{loc}}(I)$; from (4.28) it follows that $t(\cdot)$ is strictly increasing on I and so has a strictly increasing, continuous inverse function, say, $T(\cdot) : J$ onto I; in fact $T(\cdot) \in AC_{\text{loc}}(J)$.

Define two new coefficients $Q, W : J \to R$ by

$$Q(X) = p(x)\,q(x) = p(T(X))\,q(T(X)) \quad (X \in J),$$
$$W(X) = p(x)\,w(x) = p(T(X))\,w(T(X)) \quad (X \in J)$$

from which it follows that, since $q, w \in L_{\text{loc}}(I)$,

(4.30) $$Q, W \in L_{\text{loc}}(J)$$

and

$$W(X) \geqq 0 \quad (X \in J).$$

Let $F : J \to C$ and $U : C(I) \to C(I)$
be defined by

(4.31) $$F(X) = (Uf)(X) = f(T(X)) \quad (X \in J)$$

so that

$$(U^{-1}F)(x) = F(t(x)) = f(x) \quad (x \in I).$$

Now introduce a second differential expression N by

(i) define the quasi-derivatives of F on J

$$F^{[0]} = F, \quad F^{[1]} = F', \quad F^{[2]} = F'' - QF$$

(recall the notation of section 4 : F' denotes differentiation of F with respect to X)

(ii) define the domain $D(N) \subset AC_{\text{loc}}(J)$

$$D(N) = \{F : J \to C \mid F^{[r]} \in AC_{\text{loc}}(J) \text{ for } r = 0, 1\}$$

(iii) define $N : D(N) \to L_{\text{loc}}(J)$

(4.32) $$N[F] = \mathrm{i}^2 F^{[2]} = -F'' + QF \quad (F \in D(N));$$

it is clear that N is symmetric in the sense of section 2 above.

We note that if functions g and ϱ are given such that (i) $g : I \to C$, and $g \in AC_{\text{loc}}(I)$, (ii) $\varrho : J \to I$, ϱ is strictly monotone increasing on J, and $\varrho \in AC_{\text{loc}}(J)$, then $g(\varrho(\cdot)) \in AC_{\text{loc}}(J)$. If we apply this result to $F = Uf$ with $f \in D(M)$ it follows that, since for all $X \in J$

$$F^{[0]}(X) = f(T(X)), \quad F^{[1]}(X) = (pf')(T(X)),$$

both $F^{[0]}$ and $F^{[1]} \in AC_{\mathrm{loc}}(J)$ and hence $F \in D(N)$.

The argument may be reversed to give

$$F = Uf \in D(N) \quad \textit{if and only if} \quad f \in D(M).$$

If now $f, g \in D(M)$ and, also, $G = Ug$ then

$$f^{[0]}\overline{g^{[1]}} = F^{[0]}\overline{G^{[1]}}. \tag{4.32a}$$

Also a calculation shows that, for all $f \in D(M)$,

$$N[F](X) = p(x)\, M[f](x) \quad \text{(almost all } x \in I)$$

and

$$M[f](x) = \{p(T(X))\}^{-1}\, N[F](X) \quad \text{(almost all } X \in J).$$

The differential equation (4.24) is transformed by

$$X = t(x) = K + \int_k^x p^{-1} \quad (x \in I), \quad Y(X) = y(T(X)) \quad (X \in J)$$

to the symmetric differential equation

$$-Y'' + QY = \lambda WY \quad \text{on} \quad J \tag{4.33}$$

noting that Q, W are real-valued and locally integrable on J.

Initial conditions are transformed by

$$Y(K) = Y^{[0]}(K) = y^{[0]}(k) = y(k),$$
$$Y'(K) = Y^{[1]}(K) = y^{[1]}(k) = (py')(k)$$

and boundary conditions of the form (3.4) are transformed to conditions of the same form; also Dirichlet and Neumann boundary conditions are left invariant.

The end-point a (b) of (4.24) is singular if and only if the end-point A (B) of (4.33) is singular.

As in section 4.1 the map U of (4.31) is an isometric map of $L^2_w(I)$ onto $L^2_W(J)$; similarly for U^{-1}.

If $\phi : J \times C \to C$ is the solution of the transformed equation (4.33) which satisfies the initial conditions

$$\Phi^{[0]}(K, \lambda) = 0, \quad \Phi^{[1]}(K, \lambda) = 1 \quad (\lambda \in C)$$

and ϕ the solution of (4.24) satisfying (3.1) then

$$\phi(x, \lambda) = \Phi(t(x), \lambda) \quad (x \in I;\ \lambda \in C)$$

and, for all $[\alpha, \beta] \subseteq I$,

$$\int_\alpha^\beta w(x)\, |\phi(x, \lambda)|^2\, dx = \int_{t(\alpha)}^{t(\beta)} W(X)\, |\Phi(X, \lambda)|^2\, dx \quad (\lambda \in C).$$

All the remaining results given in section 4.1 concerning invariance of *LP*, *LC*, *SLP*, *D*, *CD*, *WD* and other properties, now hold *mutatis mutandis* for the transformations of this section. In particular, when w and W satisfy the positivity condition (4.20), the corresponding maximal operators T and S are unitarily equivalent.

We summarize the results of this section; every generalized Sturm-Liouville equation of the form (4.24) *with a non-negative leading coefficient* p may be isometrically transformed into an equation of the same form but with leading coefficient unity. The transformation leaves invariant classification properties and boundary conditions; certain differential operators are unitarily equivalent under the transformation.

4.3. The Liouville transformation. This transformation is applicable to the generalized Sturm-Liouville equation

$$-(py')' + qy = \lambda wy \quad \text{on} \quad I \tag{4.34}$$

where the coefficients p, q and w are real-valued on I and satisfy, throughout this section, the following conditions

$$\begin{aligned}
&\text{(i) } p \text{ and } p' \in AC_{\text{loc}}(I), \text{ and } p(x) > 0 \ (x \in I),\\
&\text{(ii) } w \text{ and } w' \in AC_{\text{loc}}(I), \text{ and } w(x) > 0 \ (x \in I),\\
&\text{(iii) } q \in L_{\text{loc}}(I).
\end{aligned} \tag{4.35}$$

These are the minimal conditions for the application of the Liouville transformation, and so the transformation is only applicable to a restricted class of the equations of the form (4.34). However, the Liouville transformation is sometimes useful in considering specific examples when the coefficients p, q and w may be analytic on I and so certainly satisfy the differentiability conditions required by (4.35).

The Liouville transformation was mentioned in section 1 above but there in the notation of Birkhoff and Rota [5, chapter X, sections 1 and 5], and other references are given in that section. Additionally, see Eastham [9, section 4.1], Hille [19] and Reid [28, chapter V, section 1, example 3].

A reference is also made again to the paper by Neuman [24] which relates the Liouville transformation to the theory of general transformation of second-order linear differential equations as considered and developed by Borůvka [6]; we do not consider such aspects of the Liouville transformation in this paper but it is clear that there is an interesting field for development of the methods of [6] to the general equation (2.11), *i.e.* $M[y] = \lambda\, S[y]$ on I, for real values of the parameter λ. We comment again on this point in section 6 below.

Following the general notation used in earlier sections we introduce a new dependent variable X into the differential equation (4.34) as follows: let $k \in I$, choose $K \in R$ and define $l(\cdot) : I \to R$ by

$$X = l(x) = K + \int_k^x \{w(t)/p(t)\}^{1/2}\, \mathrm{d}t \quad (x \in I) \tag{4.36}$$

and

$$A = K - \int_a^k \{w/p\}^{1/2}, \quad B = K + \int_k^b \{w/p\}^{1/2}.$$

Then $l(k) = K$ and $-\infty \leqq A < B \leqq \infty$; define the interval J of R to have end-points A and B, noting that conditions (4.35) imply $A\ (B) \in J$ if and only if $a\ (b) \in I$.

Clearly $l(\cdot) \in C^{(2)}(I)$ and is strictly increasing on I, and so $l(\cdot)$ has an inverse function, say, $L(\cdot) : J$ onto I with $L(\cdot) \in C^{(2)}(J)$.

Define a new dependent variable Y in the transformation of the equation (4.34) by

$$\begin{aligned} Y(X) &= \{p(x)\, w(x)\}^{1/4}\, y(x) \quad (x \in I) \\ &= \{p(L(X))\, w(L(X))\}^{1/4}\, y(L(X)) \quad (X \in J). \end{aligned} \tag{4.37}$$

If we introduce both of the new variables X and Y into (4.34) we obtain, after a lengthy calculation which is omitted,

$$-Y''(X) + Q(X)\, Y(X) = \lambda\, Y(X) \quad (X \in J) \tag{4.38}$$

where the coefficient Q is given explicitly by (here we follow the form for Q given by Eastham [9, section 4.1], but note the correction required on the right-hand side of [9, (4.1.2)] with $q(x) \to \{s(x)\}^{-1}\, q(x)$),

$$Q(X) = w(x)^{-1}\, q(x) - \{w(x)^{-3}\, p(x)\}^{1/4}\, (p(x)\, (\{p(x)\, w(x)\}^{-1/4})')' \quad (x \in I). \tag{4.39}$$

(We recall the notation used here; primes on p and w denote differentiation with respect to $x \in I$.)

It follows from the conditions (4.35) that the right-hand side of (4.39) is in $L_{\mathrm{loc}}(I)$ and this implies that $Q \in L_{\mathrm{loc}}(J)$. This argument requires the full strength of the conditions (4.35) on the original coefficients p, q and w.

As in section 4.2 we now introduce symmetric differential expressions M, with domain $D(M)$, associated with (4.34), and N, with domain $D(N)$, associated with (4.38); the definitions of the quasi-derivatives are given by the corresponding formulae used in earlier sections.

In the case of the Liouville transformation the relationship between the quasi-derivatives of the differential expressions M and N is more complicated than in the two previous cases in sections 4.1 and 4.2; in both the previous transformations we obtained the result $F^{[0]}\overline{G^{[1]}} = f^{[0]}\overline{g^{[1]}}$ (see (4.9) and (4.32a)) between the original and transformed derivatives; this result fails to hold for the Liouville transformation and this is one of the reasons for the lack of some structural properties of this transformation in comparison with the two previous cases considered in this paper.

We define the mapping $U : C(I) \to C(J)$ by

$$\begin{aligned} F(X) = (Uf)(X) &= \{p(L(X))\, w(L(X))\}^{1/4} f(L(X)) \quad (X \in J) \\ &= \{p(x)\, w(x)\}^{1/4} f(x) \quad (x \in I) \end{aligned} \tag{4.40}$$

and then

$$(U^{-1}F)(x) = \{p(x)\,w(x)\}^{1/4}\,F(l(x)) = f(x) \quad (x \in I).$$

We have then, if $F = Uf$ and using the notation of quasi-derivatives,

$$F^{[0]} = F = \{pw\}^{1/4} f = \{pw\}^{1/4} f^{[0]}, \tag{4.41}$$

$$F' = (\{pw\}^{1/4} f)'\; L' = p^{1/2} w^{-1/2} (\{pw\}^{1/4} f)'$$

$$= \{pw\}^{-1/4}\, pf' + \tfrac{1}{4}\{pw\}^{-1/4}\, w^{-1}(pw)' f$$

i.e.

$$F^{[1]} = \{pw\}^{-1/4} f^{[1]} + \tfrac{1}{4}\{pw\}^{-1/4}\, w^{-1}(pw)' f^{[0]}. \tag{4.42}$$

These results yield, if also $G = Ug$,

$$F^{[0]}\overline{G^{[1]}} = f^{[0]}\overline{g^{[1]}} + \tfrac{1}{4} w^{-1}(pw)'\, \overline{g^{[0]}} f^{[0]} \tag{4.43}$$

and so

$$F^{[0]}\overline{G^{[1]}} - F^{[1]}\overline{G^{[0]}} = f^{[0]}\overline{g^{[1]}} - f^{[1]}\overline{g^{[0]}}. \tag{4.44}$$

We note that, as in previous transformations

$$f \in D(M) \quad \textit{if and only if} \quad F = Uf \in D(N). \tag{4.44a}$$

A more detailed calculation shows that the relationship between the differential expressions M and N is given by

$$N[F](X) = \{pw^{-3}\}^{1/4}(x)\; M[f](x) \quad (x \in I) \tag{4.45}$$

valid for all $f \in D(M)$ with $F = Uf$. This gives, in comparison with (4.12a), for all $[\alpha, \beta] \subseteq I$,

$$\int_{l(\alpha)}^{l(\beta)} N[F]\,\bar{G} = \int_{\alpha}^{\beta} M[f]\,\bar{g} \quad (f, g \in D(M)).$$

The transformation of the Dirichlet integral is more complicated due to the term in $f^{[0]}$ which appears in (4.42); however, if we multiply out a term such as $F^{[1]}G^{[1]}$ and integrate by parts the integrals involving $f^{[1]}g^{[0]}$ and $f^{[0]}g^{[1]}$, we obtain the following result, after a calculation similar to [11, section 7]

$$\int_{l(\alpha)}^{l(\beta)} \{F'\bar{G}' + QF\bar{G}\} = \tfrac{1}{4}[w^{-1}(pw)' f\bar{g}]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \{pf'\bar{g}' + qf\bar{g}\} \tag{4.46}$$

for all $[\alpha, \beta] \subseteq I$ and all $f, g \in D(M)$. We note the presence of an additional term in (4.46) in comparison with the corresponding result for the two previous transformations, *e.g.* (4.23).

The results (4.41 to 4.46) should be compared with the results given in [10, section 9] and [11, sections 4 to 8, see in particular (4.6a) (with correction $L[f]$ to $L[F]$ and (6.1)] although in both references $w(x) = l(x \in I)$.

In this case of the Liouville transformation the associated integrable-square

Hilbert function spaces are $L^2_w(I)$ for (4.34) and $L^2(J)$ for the transformed equation; note that in $L^2(J)$ there is a unit weight function on the interval J. The operator U is an isometric map of $L^2_w(I)$ onto $L^2(J)$; similarly for U^{-1}.

Passing now to consideration of invariance properties we remark that, in contradiction to the corresponding property of the two previous transformations, a regular end-point of a (b) of the equation (4.34) may be transformed to a singular end-point A (B) of (4.38). Examples below will confirm this statement; it is also seen from the definition of the coefficient Q in (4.39) that there is no way of deciding, in general, if Q is integrable L in a neighbourhood of one or other of the end-points A or B. The examples show that a regular end-point of (4.34) can be transformed by the Liouville transformation to a singular LC end-point of (4.38).

This picture is completed by noting, however, that a singular LP end-point a (b) of (4.34) is always transformed by Liouville into a singular LP end-point A (B) of (4.38); this property is remarked on in [11, section 4]. To prove this result we note that the conditions (4.35) imply that the positivity condition (2.18) is satisfied by w on I for (4.34); the corresponding property is also satisfied for (4.38). Hence the criterion (3.8) may be applied to determine the LP condition for both equations. Let $\Delta(T)$ be defined as in (3.7) (see also (4.21)) for (4.34), and equivalently $\Delta(S)$ for (4.38); then if $f \in \Delta(T)$ and $F = Uf$ we have $F \in D(N)$, $F = L^2(J)$ and

$$\int_{l(\alpha)}^{l(\beta)} |N[F]|^2 = \int_{\alpha}^{\beta} w|w^{-1} M[f]|^2$$

for all $[\alpha, \beta] \subseteq I$; thus $N[F] \in L^2(J)$ and $F \in \Delta(S)$. From the identity (4.44) it now follows that

$$\lim_{a} (f^{[0]}\overline{g^{[1]}} - f^{[1]}\overline{g^{[0]}}) = 0 \quad (f, g \in \Delta(T))$$

if and only if

$$\lim_{A} (F^{[0]}\overline{G^{[1]}} - F^{[1]}\overline{G^{[0]}}) = 0 \quad (F, G \in \Delta(S)).$$

This gives the required LP results at a and A; similarly for b and B.

This LP result also follows from the criterion (3.5); if the solution $\phi(\cdot, \lambda)$ of (4.34) satisfies the initial conditions (3.1) at the point $k \in I$ then Φ defined by

$$\Phi(X, \lambda) = \{p(k)\, w(k)\}^{1/4}\, \{p(x)\, w(x)\}^{1/4}\, \phi(x, \lambda) \quad (x \in I)$$

satisfies (4.38) and the initial conditions $\Phi^{[0]}(K, \lambda) = 0$, $\Phi^{[1]}(K, \lambda) = 1$ for all $\lambda \in C$. Also

$$\int_{l(\alpha)}^{l(\beta)} |\Phi(X, \lambda)|^2\, \mathrm{d}X = \{p(k)\, w(k)\}^{1/2} \int_{\alpha}^{\beta} w(x)\, |\phi(x, \lambda)|^2\, \mathrm{d}x$$

and we may now apply (3.5) to give the LP result.

We may also show, by the methods of section 4.2, that the maximal operators T and S, defined as in sections 4.1 and 4.2, are unitarily equivalent under the isometric map U.

In general there are no other invariants under the Liouville transformation in comparison with those previously described in sections 4.1 and 4.2. The form of the results (4.43) and (4.46) prevent, in general, any attempt to assess the equations (4.34) and (4.38) for the invariance of the *SLP*, *D*, *CD* and *WD* properties under this transformation. However, no example are available to illustrate the range of possibilities for these properties. In [11, sections 9 and 10] general conditions are given, again with $w(x) = 1$ $(x \in I)$, when both (4.34) and (4.38) are *SLP* and *D* at a singular end-point.

The following two examples illustrate some of the points made above; in particular the Liouville transformation of regular points to *LC* singular points:

(i) (see Everitt and Zettl [18] for further details)

$$a = 0\,, \quad b = \infty\,, \quad p(x) = w(x) = x^{1/2}\,, \quad q(x) = 0 \quad (x \in (0, \infty))$$

i.e.

$$-(x^{1/2}\, y'(x))' = \lambda x^{1/2}\, y(x) \quad (x \in [0, \infty))\,;$$

here the origin is a regular point of the equation since p^{-1}, $w \in L_{\mathrm{loc}}[0, \infty)$; although the conditions (4.35) are not satisfied at the end-points 0 we can apply the Liouville transformation with $k = 0 = K$ to give

$$X = l(x) = \int_0^x 1\, \mathrm{d}t = x\,, \quad Y(X) = x^{1/4}\, y(x) \quad (x \in [0, \infty))$$

with $A = 0$ and $B = \infty$; the transformed equation is

$$-Y''(X) - \tfrac{3}{16} X^{-2}\, Y(X) = \lambda\, Y(X) \quad (X \in (0, \infty))$$

which has a singular point at 0; this equation is *LC* at 0 from Titchmarsh [29, section 4.8 with $\nu = 1/4$, or section 5.15 with $\delta = 3/16$].

(ii) $a = 0$, $b = \infty$ and

$$p(x) = 1\,, \quad q(x) = 0\,, \quad w(x) = (x - 1)^4 \quad (x \in [0, \infty))$$

i.e.

$$-y''(x) = \lambda (x - 1)^4\, y(x) \quad (x \in [0, \infty))\,;$$

here the equation is regular on $[0, \infty)$ but $w(1) = 0$ and so the conditions (4.35) are not satisfied, in this respect, at the point 1; however, we can look at the equation on the disjoint intervals $[0, 1)$ and $(1, \infty)$ and apply the Liouville transformation separately; we have

$$X = l(x) = \tfrac{1}{3}[1 - (1 - x)^3]\,, \quad Y(X) = |1 - x|\, y(x) \quad (x \in [0, \infty))\,;$$

the transformed equation is

$$-Y''(X) - \tfrac{2}{9}(X - \tfrac{1}{3})^{-2}\, Y(X) = \lambda\, Y(X) \quad (X \in [0, \tfrac{1}{3}) \cup (\tfrac{1}{3}, \infty))$$

which has a singular point at $\frac{1}{3}$; this equation is *LC* at $\frac{1}{3}+$ and $\frac{1}{3}-$, again from Titchmarsh [29, section 5.15 with $\delta = \frac{2}{9}$].

We summarize this section; the Liouville transformation does reduce the generalized Sturm-Liouville equation to a canonical form involving only one coefficient but this is at the expense of restrictions on the original coefficients and a possible loss of invariance of some of the classification properties of the original equation.

4.4. A special case of the Liouville transformation. It is clear from the results (4.43) and (4.46) that the terms which cause difficulty for the invariance properties of the Liouville transformation would not appear if we choose the coefficients p and w of the differential equation (4.34) so that $p(x)\,w(x) = 1$ (say) for all $x \in I$, *i.e.* so that $(pw)' = 0$ on I. This suggests that the differential equation

$$(4.47) \qquad -(w^{-1}y')' + qy = \lambda wy \quad \text{on} \quad I$$

should transform by Liouville to the form (4.38) and yet leave the classification properties invariant; this is, in fact, the case.

Let $q, w : I \to R$ and satisfy the conditions:

(i) $q \in L_{\text{loc}}(I)$

(ii) $w \in L_{\text{loc}}(I)$ and $w(x) > 0$ (almost all $x \in I$).

Now apply the Liouville transformation to (4.47);

$$X = l(x) = K + \int_k^x \{w^2\}^{1/2} = K + \int_k^x w(t)\,\mathrm{d}t \quad (x \in I)$$

$$Y(X) = \{w(x)^{-1}\,w(x)\}^{1/4}\,y(x) = y(x) \quad (x \in I)\,.$$

We have $Y' = y'\ L' = w^{-1}y'$; thus $Y'' = (w^{-1}y')'\ w^{-1}$ and so (4.47) transforms to

$$(4.48) \qquad -Y'' + QY = \lambda Y \quad \text{on} \quad J$$

with $Q(X) = w(L(X))^{-1}\ q(L(X))\ (X \in J)$ and J is the interval determined as in section 4.3. Note that $Q \in L_{\text{loc}}(J)$ and that no differentiability conditions are required on w.

In this special case all the classification properties are invariant under the Liouville transformation, as for the transformations in sections 4.1 and 4.2.

4.5. Inverse transformations in the right-definite case. The Ahlbrandt transformation of section 4.1 can always be applied in the oppositive direction. If we are given a generalized Sturm-Liouville equation

$$(4.49) \qquad -(py')' + qy = \lambda wy \quad \text{on} \quad I$$

and a complex-valued coefficient $R : I \to C$ with $R \in L_{\text{loc}}(I)$, then this equation can be transformed isometrically to

$$-(P(Y' - RY))' - \bar{R}P(Y' - RY) + QY = \lambda WY \quad \text{on} \quad J$$

where $J = I$; the coefficients P, Q and W can be readily calculated in terms of p, q and w, *e.g.* $P = |\mu|^{-2}\,p$ with μ determined as in section 4.1, but with r replaced by R.

The transformation in section 4.2 can be applied in the opposite direction in certain circumstances only; starting with the equation

$$-y'' + qy = \lambda wy \quad \text{on} \quad I$$

and given an interval J, with end-points A and B, a non-negative coefficient P on J with $P^{-1} \in L_{\mathrm{loc}}(J)$, $k \in I$ and $K \in J$ such that

$$a = k - \int_A^K P^{-1}, \quad b = k + \int_K^B P^{-1}$$

then the equation can be transformed isometrically to the differential equation

$$-(PY')' + QY = \lambda WY \quad \text{on} \quad J$$

where, for example, $Q = P^{-1}q$.

In general the Liouville transformation cannot be applied in the converse direction; given the relationship (4.39) and a coefficient Q the determination of coefficients p, q and w, and a suitable interval, seems to present considerable analytical difficulties. Even if $w = 1$ and $q = 0$, the resulting non-linear differential equation for p presents unusual problems.

4.6. Transformations of the Legendre equation. Finally, in this consideration of transformations of the right-definite case of the general differential equation (2.11), we remark that the application of the transformations considered in earlier sections does not necessarily simplify the original differential equation; indeed, in some cases it is more appropriate to leave the equation as it stands when considering specific properties.

As an example consider the Legendre equation in the form which has polynomial solutions, *i.e.*

$$-((1 - x^2)\, y'(x))' + \tfrac{1}{4}\, y(x) = \lambda\, y(x) \quad (x \in (-1, 1))\,; \tag{4.50}$$

for $\lambda = (n + \frac{1}{2})^2$ $(n = 0, 1, 2, \ldots)$ this equation has the Legendre polynomial $P_n(\cdot)$ as a solution. This equation has singular end-points at ± 1 which are both in the LC case.

Firstly since $p(x) = 1 - x^2 > 0$ $(x \in (-1, 1))$ we can apply the transformation of section 4.2; it may be verified that the transformed equation takes the form

$$-Y''(X) + (4 \cosh^2 X)^{-1}\, Y(X) = \lambda(\cosh^2 X)^{-1}\, Y(X) \quad (X \in (-\infty, \infty))$$

which no longer has $P_n(\cdot)$ but $P_n(\tanh(\cdot))$ on $(-\infty, \infty)$ as a direct solution for $\lambda = (n + \frac{1}{2})^2$.

Secondly since $p(x) = (1 - x^2)$, $q(x) = \frac{1}{4}$ and $w(x) = 1$ $(x \in (-1, 1))$ all satisfy the conditions (4.35) we can apply the Liouville transformation of section 4.3; the transformed equation takes the form

$$-Y''(X) - \tfrac{1}{4}(\sec^2 X)\, Y(X) = \lambda\, Y(X) \quad (X \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi))$$

as in Titchmarsh [29, section 4.3 and (4.5.1)]. This equation has $\{\cos(\cdot)\}^{1/2}$ . . $P_n(\sin(\cdot))$ on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ as a solution for $\lambda = (n+\frac{1}{2})^2$.

In working with the Legendre equation it is often best to keep to the original form (4.50) although the Liouville form is used in the analysis given in Titchmarsh [29, sections 4.4 and 4.5].

5. TRANSFORMATION: LEFT-DEFINITE CASE

This section is concerned with the differential equation (2.11) in the left-definite case, *i.e.*

$$M[y] = \lambda\, S[y] \quad \text{on} \quad I \tag{2.11}$$

or, equivalently,

$$-(p(y'-ry))' - \bar{r}p(y'-ry) + qy = \lambda\{\mathrm{i}(\varrho y)' + \mathrm{i}\varrho y' + wy\} \quad \text{on} \quad I \tag{5.1}$$

where now the conditions (2.16) on the coefficients are taken to hold, *i.e.*

$$p(x) \geqq 0 \quad \text{and} \quad q(x) \geqq 0 \quad (\text{almost all } x \in I). \tag{2.16}$$

We recall that the basic conditions (2.1 to 3) and (2.7 to 8) on the coefficients p, q, w, r and ϱ are satisfied, and that there is no sign restriction on w in this left-definite case.

The quasi-derivatives of (5.1) are given in (2.13), *i.e.*

$$y_\lambda^{[0]} = y\,, \quad y_\lambda^{[1]} = p(y'-ry) + \mathrm{i}\lambda\varrho y \quad \text{on} \quad I\,, \tag{2.13}$$

and boundary conditions in (3.4).

The solution ϕ, used in the classification of (2.11) at singular end-points, is determined by the initial conditions (3.1) which we write here in the form, for some $k \in I$,

$$\phi_\lambda^{[0]}(k,\lambda) = 0\,, \quad \phi_\lambda^{[1]}(k,\lambda) = 1 \quad (\lambda \in C)\,. \tag{5.2}$$

In the left-definite case the associated Hilbert function space $H^2_{p,q,r}(I)$ is defined in (2.16) with norm

$$\int_I \{p\,|f' - rf|^2 + q|f|^2\}\,.$$

As indicated at the end of section 3.2 we are concerned here, in this the left-definite case, only with the transformation of the differential equation (2.11), and not with the transformation of any associated differential operators. Under any transformation we write the transformed equation as

$$N[Y] = \lambda\, T[Y] \quad \text{on} \quad J \tag{5.3}$$

where the symmetric differential expressions $N[\cdot]$ and $T[\cdot]$ are, up to multiplicative factors, the transformations of the original differential expressions M and S. We note that in this section S and T represent differential expressions and not, as in section 4, differential operators in a Hilbert function space.

In sections 5.1 and 5.2 we transform the equation (2.11) by the same transformations as used in the corresponding sections 4.1 and 4.2. Section 5.3 is concerned with the effect of the Liouville transformation on (2.11), and general remarks on the left-definite case are made in section 5.4.

5.1. Transformation to real left-definite form. It is somewhat remarkable that the Ahlbrandt transformation of section 4.1 is equally applicable to the general differential equation (5.1) even though there is no connection between the coefficient ϱ and the other coefficient r.

With the same notation given in (4.1 and 2), (4.5) and (4.13) we find that the Ahlbrandt transformation

$$X = x \quad Y(X) = \{\mu(x)\}^{-1}\, y(x) \quad (x \in I) \tag{4.13}$$

gives

$$M[y] = \bar{\mu}^{-1} N[Y] \quad \text{on} \quad J$$

where $N[\cdot]$ is defined in (4.5). Recall that in this case $J = I$ and $'$ represents differentiation with respect to either x or X.

For the corresponding transformation of $S[\cdot]$ we have

$$\begin{aligned} \bar{\mu}\, S[y] &= \bar{\mu}\{\mathrm{i}(\varrho y)' + \mathrm{i}\varrho y' + wy\} \\ &= \bar{\mu}\{\mathrm{i}(\varrho\mu Y)' + \mathrm{i}\varrho(\mu Y)' + w\mu Y\} \\ &= \mathrm{i}(|\mu|^2\, \varrho Y)' + \mathrm{i}|\mu|^2\, \varrho Y' + |\mu|^2\, wY \\ &\quad + \mathrm{i}\varrho\mu'\bar{\mu}Y - \mathrm{i}\varrho\mu\bar{\mu}'Y \\ &= \mathrm{i}(RY)' + \mathrm{i}RY' + W_0 Y \\ &= T[Y] \quad (\text{say}), \end{aligned} \tag{5.4}$$

where

$$R(X) = |\mu(X)|^2\, \varrho(X) \quad (X \in J),$$

$$W_0(X) = W(X) - 2\, R(X)\, \mathrm{im}\, [r(X)] \quad (X \in J). \tag{5.5}$$

Here $T[\cdot]$ is a first-order symmetric differential expression of the same form as given by (2.9) with ϱ replaced by R and w by W_0. We note that R and W_0 satisfy the required conditions (2.8) and (2.7), respectively, in particular $R \in AC_{\mathrm{loc}}(J)$ and $W_0 \in L_{\mathrm{loc}}(J)$.

Thus the differential equation (2.11) is transformed by (4.13) to

$$N[Y] = \lambda\, T[Y] \quad \text{on} \quad J \tag{5.6}$$

or, equivalently,

$$-(PY')' + QY = \lambda\{\mathrm{i}(RY)' + \mathrm{i}RY' + W_0 Y\} \quad \text{on} \quad J. \tag{5.7}$$

(We note that this transformation is valid for the general equation (2.11) with the coefficients p, q, w, r and ϱ satisfying only the basic conditions of section 2; the non-negative condition (2.16) is not in any way essential to the application of the Ahlbrandt transformation.)

The left-definite condition (2.16) is invariant under this transformation, *i.e.* from (4.2)

$$P(X) \geqq 0 \quad \text{and} \quad Q(X) \geqq 0 \quad (\text{almost all } X \in J).$$

Given any solution y of (5.1) let Y be the corresponding solution of (5.7), *i.e.* $Y(X) = \{\mu(x)\}^{-1} y(x)$ $(x \in I)$; then the relationship between the quasi-derivatives for (5.1) and (5.7) may be calculated from (2.13) to obtain, with $K = k$,

$$Y_\lambda^{[0]}(K) = \mu(k)^{-1} y_\lambda^{[0]}(k), \quad Y_\lambda^{[1]}(k) = \bar{\mu}(k)\, y_\lambda^{[1]}(k).$$

From this it follows that boundary conditions of the form (3.4) are transformed to conditions of the same form; Dirichlet and Neumann boundary conditions are left invariant.

If the end-point a (b) of I is a regular end-point of (5.1) then A (B) of J is a regular end-point of (5.7); this follows, as in section (4.1), from the definition of the transformed coefficients.

If Φ is the solution of (5.7) which satisfies the same initial conditions (5.2), at $K = k$, that ϕ satisfies at k then $\Phi(X, \lambda) = \{\bar{\mu}(k)\, \mu(x)\}^{-1} \phi(x, \lambda)$ $(x \in I)$; a calculation then shows that for all $[\alpha, \beta] \subseteq I$ and all $\lambda \in C_\pm$

$$\int_\alpha^\beta \{p|\phi' - r\phi|^2 + q|\phi|^2\} = |\mu(k)|^2 \int_\alpha^\beta \{P|\Phi'|^2 + Q|\Phi|^2\}.$$

From this result it follows, see the definition in section 3.2, that the transformed equation (5.7) is $LP_\pm$ $(LC_\pm)$ at a singular end-point A (B) if and only if the original equation is $LP_\pm$ $(LC_\pm)$ at the singular end-point a (b).

Thus every left-definite symmetric equation of the general form (5.1) can be transformed by the Ahlbrandt transformation into a symmetric equation of the form (5.7) with real left-definite terms; boundary conditions and the $LP_\pm$ $(LC_\pm)$ classification are invariant under this transformation.

5.2. Transformation to leading coefficient unity. We consider now the differential equation

(5.8) $$-(py')' + qy = \lambda\{\mathrm{i}(\varrho y)' + \mathrm{i}\varrho y' + wy\} \quad \text{on} \quad I$$

where p, q, w and ϱ satisfy the basic conditions of section 2 and the non-negative left-definite condition

(2.16) $$p(x) \geqq 0 \quad \text{and} \quad q(x) \geqq 0 \quad (\text{almost all } x \in I).$$

Now apply the transformation considered in section 4.2, *i.e.* from (4.29) and (4.33),

$$X = t(x) = K + \int_k^x p^{-1} \quad (x \in I), \quad Y(X) = y(T(X)) \quad (X \in J)$$

where the transformed interval J has end-points A and B as defined in (4.29).

As in section 4.2 we obtain

$$p\, M[y] = N[Y]$$

with $N[\cdot]$ defined by (4.32) with coefficient $Q = pq$; from (2.16) it follows that $Q \geqq 0$ almost everywhere on J so that $N[\cdot]$ is left-definite on J.

For the corresponding transformation of $S[\cdot]$ we have

$$\begin{aligned} p\, S[y] &= p\{i(\varrho y)' + i\varrho y' + wy\} \\ &= p\{2i\varrho y' + i\varrho' y + wy\} \\ &= p\{2i\varrho Y' p^{-1} + i(d\varrho/dX)\, p^{-1} Y + wY\} \\ &= 2iRY' + iR'Y + WY = T[Y] \quad \text{(say)} \end{aligned}$$

where $W = pw$, as in section 4.2, and

$$R(X) = \varrho(T(X)) \quad (X \in J)\,;$$

here, as before, R' denotes the derivative of R on the interval J. Clearly $T[\cdot]$ is a first-order symmetric differential expression of the same form as given by (2.9) with ϱ replaced by R and w by W. We note that R and W satisfy the required conditions (2.8) and (2.7), respectively, on J; the result $R \in AC_{\mathrm{loc}}(J)$ follows from the argument in section 4.2 leading to (4.32a).

Thus the differential equation (5.8) is transformed to a left-definite equation of the form

$$-Y'' + QY = \lambda\{i(RY)' + iRY' + WY\} \quad \text{on} \quad J\,. \tag{5.9}$$

Given any solution y of (5.8) let Y be the corresponding solution of (5.9), *i.e.* $Y(X) = y(T(X))$ $(X \in J)$; then the quasi-derivatives for (5.8) and (5.9) satisfy, with $K = t(k)$,

$$Y_\lambda^{[0]}(K) = y_\lambda^{[0]}(k)\,, \quad Y_\lambda^{[1]}(K) = y_\lambda^{[1]}(k)\,.$$

Boundary conditions are transformed as before, and regular end-points of I pass to regular end-points of J.

If ϕ and Φ are the solutions of (5.8) and (5.9) satisfying the initial conditions (5.2), at k and K respectively, then $\Phi(X, \lambda) = \phi(x, \lambda)$ for all $\lambda \in C_\pm$; also, for all $[\alpha, \beta] \subseteq I$,

$$\int_\alpha^\beta \{p|\phi'|^2 + q|\phi|^2\} = \int_{t(\alpha)}^{t(\beta)} \{|\Phi'|^2 + Q|\Phi|^2\} \quad (\lambda \in C_\pm)$$

so that we have the $LP_\pm$ $(LC_\pm)$ invariance at singular end-points, as in section 5.1.

We may summarize the results of sections 5.1 and 5.2 by recording the result that every left-definite differential equation of the general form (5.1) can be transformed to the left-definite form (5.9) wihout any additional constraints on the coefficients p, q, w, r and ϱ. The combined transformation leaves invariant the form of initial and boundary conditions, regular and singular end-points, and the $LP_+(LC_\pm)$ classification at singular end-points.

5.3. The Liouville transformation. Again it is somewhat remarkable that, under the appropriate additional conditions on the coefficients p and w, the differential equatiton (5.8) (or, equivalently, the original equation (5.1) through the Ahlbrandt transformation) can be transformed by the Liouville transformation to a symmetric form involving only two coefficients Q and R.

Let the coefficients q and ϱ satisfy the basic conditions of section 2; let the coefficients p and w satisfy the Liouville conditions (4.35); let the new variables X and Y be defined for the Liouville transformation by (4.36) and (4.37) respectively. Then we find

$$\{pw^{-3}\}^{1/4}(-(py') + qy) = -Y'' + QY$$

where the coefficient Q is defined by (4.39); here, as before, $Q \in L_{\mathrm{loc}}(J)$.

Also, we omit the calculations,

$$\{pw^{-3}\}^{1/4}\{\mathrm{i}(\varrho y)' + \mathrm{i}\varrho y' + wy\} = \{\mathrm{i}(RY)' + \mathrm{i}RY' + Y\}$$

where

$$R(X) = \{p(L(X))\,w(L(X))\}^{-1/2}\,\varrho(L(X)) \quad (X \in J)\,. \tag{5.10}$$

Thus, under these conditions, the differential equation

$$-(py')' + qy = \lambda\{\mathrm{i}(\varrho y)' + \mathrm{i}\varrho y' + wy\} \quad \text{on} \quad I \tag{5.8}$$

is transformed by the Liouville transformation to the form

$$-Y'' + QY = \lambda\{\mathrm{i}(RY)' + \mathrm{i}RY' + Y\} \quad \text{on} \quad J \tag{5.11}$$

with Q given by (4.39) and R by (5.10); we have $Q \in L_{\mathrm{loc}}(J)$ and, following previous arguments, $R \in AC_{\mathrm{loc}}(I)$.

In general, however, even though the coefficient q satisfies the left-definite condition $q \geqq 0$ (almost everywhere on I) it is not the case that the transformed equation (5.11) always inherits this property. For (5.11) to be left-definite we have to have $Q \geqq 0$ (almost everywhere on J) or, equivalently, that the original coefficients p, q and w satisfy

$$q \geqq \{pw\}^{1/4}\,(p(\{pw\}^{-1/4})')' \quad \text{(almost everywhere on } I)\,;$$

it is possible to construct examples which do not satisfy this condition.

All the difficulties over invariance properties indicated in section 4.3 for the right-definite case, continue for the left-definite case when the Liouville transformation is applied. It is not even clear if the $LP_\pm$ classification remains invariant under Liouville in the left-definite case.

Seemingly there is little advantage, in general, in applying the Liouville transformation to the left-definite case; at the expense of additional constraints on the coefficients and the loss of invariance properties, the only advantage over the transformation in section 5.2 is to replace the coefficient w by unity, and this has little significance in the left-definite case. It is possible, however, that the Liouville transformation is of interest when considering examples of the left-definite case, particularly if the coefficients are of a simple analytical character.

5.4. General remarks on the left-definite case. The special case of the Liouville transformation given in section 4.4 applies to the left-definite case also; the differential equation

$$-(w^{-1}y')' + qy = \lambda\{i(\varrho y)' + i\varrho y' + wy\} \quad \text{on} \quad I$$

is transformed to

$$-Y'' + QY = \lambda\{i(RY)' + iRY' + Y\} \quad \text{on} \quad J$$

with, as before, $Q = w^{-1}q$ but now with $R(X) = \varrho(L(X))$ $(X \in J)$.

The remarks on the inverse transformations in section 4.5 hold good for the left-definite case.

6. GENERAL REMARKS

The motivation for the results presented in this paper stem from the idea of the original Liouville transformation of second-order differential equations; for a historical reference see Neuman [24, reference [3]]. The paper is concerned with the transformation theory of the general second-order linear differential equation

$$M[y] = \lambda\, S[y] \quad \text{on} \quad I$$

where M and S are symmetric (formally symmetric or formally self-adjoint) differential expressions of the second-order and first-order respectively, λ is a complex-valued parameter and I is an arbitrary interval of the real line.

The transformations considered in this paper are specific to the above mentioned differential equation. In Dunford and Schwartz there is a discussion on the general transformation theory of certain ordinary scalar differential equations with particular reference to isometric (unitary) transformations; see [7].

There would seem to be scope for relating the results of this paper to the work on general transformations of Borůvka in [6], particularly in relation to the properties of the Schwarz derivative and the Kummer transformation.

Finally, in the paper Neuman [25] results are obtained connecting the Kummer transformation and the limit-point limit-circle classification of the second-order differential equation

$$-(py')' + qy = 0 \quad \text{on} \quad [a, b).$$

These results suggest that interest would be sustained in the study of the Kummer transformation and the generalized Sturm-Liouville equation given by (1.2) above; work in this direction has been started by Neuman in [24].

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