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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 2, 197–207

Persistent URL: http://dml.cz/dmlcz/101796

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

A NEW METHOD FOR OBTAINING EIGENVALUES OF VARIATIONAL INEQUALITIES: OPERATORS WITH MULTIPLE EIGENVALUES

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(Received March 2, 1979)

We shall consider a real Hilbert space H, a closed convex cone K in H with its vertex at the origin and a linear symmetric completely continuous operator $A : H \to \to H$. The inner product in H is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$. The following eigenvalue problem for a variational inequality will be studied:

(I) $$u \in K,$$

(II) $$\langle \lambda u - Au, v - u \rangle \geqq 0 \text{ for all } v \in K.$$

We shall say that a real number λ is an eigenvalue of the variational inequality (I), (II) if there exists a corresponding eigenvector of (I), (II), i.e. a nontrivial $u \in H$ satisfying the conditions (I), (II). Analogously as in the papers [3], [4], we shall prove the existence of an eigenvalue of the variational inequality lying between given eigenvalues $\lambda^{(1)}, \lambda^{(0)}$ of a certain type of A.

More precisely, it was proved in [4] that if $\lambda^{(1)}, \lambda^{(0)}$ $(0 < \lambda^{(1)} < \lambda^{(0)})$ are simple eigenvalues of A and each of them has an eigenvector in the interior of K, then there exists an eigenvalue of (I), (II) in $(\lambda^{(1)}, \lambda^{(0)})$ having the corresponding eigenvector on the boundary of K. Moreover, it was proved that there exists a closed connected (in a certain sense) and unbounded in ε set of triplets $[\lambda, u, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying the penalty equation $\lambda u - Au + \varepsilon\beta u = 0$, starting with $\varepsilon = 0$ at $\lambda^{(0)}$ in the direction of the corresponding eigenvector $u^{(0)} \notin K$ of A. The mentioned eigenvalue and eigenvector of (I), (II) were obtained by the limiting process $\varepsilon \to +\infty$ along this set. The theory was further developed in [5] in order to obtain bifurcation points of a more general problem.

The aim of this paper is to extend these results to the case of eigenvalues $\lambda^{(0)}, \lambda^{(1)}$ of arbitrary multiplicities. For a given couple of eigenvalues $\lambda^{(1)}, \lambda^{(0)}$ $(0 < \lambda^{(1)} < < \lambda^{(0)})$ such that each of them has at least one corresponding eigenvector in the interior of K, we shall approximate the operator A by operators A_n such that $\lambda^{(1)}, \lambda^{(0)}$ are simple eigenvalues of A_n. The existence of branches of solutions of the equation with the penalty with the mentioned properties for A_n will follow from [4] and we shall show that the analogous branch for A can be defined by a suitable limiting process. Under certain assumptions, the theory ensures the existence of infinitely many eigenvalues of the variational inequality having the corresponding eigenvectors on the boundary of K.

The branch of solutions of the penalty equations in [4] was in fact obtained as a global bifurcation branch for a certain equation in $\mathbb{R} \times H$ (an extension of the penalty equation) and the present result can be viewed also as a global bifurcation result for a special equation.

In the connection with the eigenvalue problem for variational inequalities, we must mention the results of E. Miersemann [7], [8], who has proved by another method the existence of a finite number (depending of the character of the problem) of bifurcation points of a more general variational inequality. Further references are given in [4].

Some definitions and modifications of the results from [4] are recalled in Section 1. Particularly, a small correction to [4] is given in Remark 1.2. Main results of the present paper are contained in Theorems 2.1, 2.2 (Section 2).

1. TERMINOLOGY AND REMARKS TO SOME FORMER RESULTS

In the whole paper, K will be a closed convex cone in H with its vertex at the origin and A will be a linear completely continuous symmetric operator in H. We shall denote by K^0 and ∂K the interior and the boundary of K, respectively. The set of all eigenvalues and the set of all eigenvectors of the operator A will be denoted by Λ_A and E_A, respectively. The set of all eigenvalues and eigenvectors of the variational inequality (I), (II) will be denoted by Λ_V and E_V, respectively. Moreover, $E_A(\lambda)$ will be the set of all eigenvectors of A corresponding to a given eigenvalue $\lambda \in \Lambda_A$ and $E_V(\lambda)$ will be the set of all eigenvectors of (I), (II) corresponding to a given eigenvalue $\lambda \in \Lambda_V$. Analogously, we shall write $\Lambda_{A_n}, E_{A_n}, \Lambda_{V_n}, E_{V_n}, E_{A_n}(\lambda), E_{V_n}(\lambda)$ if the operator A is replaced by A_n and (I), (II) is replaced by (I),

$$(\mathrm{II}_n) \qquad (\lambda u - A_n u, v - u) \geqq 0 \quad \text{for all} \quad v \in K .$$

The strong and the weak convergence is denoted by $\to$ and $\rightharpoonup$, respectively.

Definition 1.1. *We shall write*

$$\begin{aligned}
&\lambda \in \Lambda_i && \text{if} \quad \lambda \in \Lambda_A \quad \text{and} \quad E_A(\lambda) \cap K^0 \neq \emptyset ;\\
&\lambda \in \Lambda_b && \text{if} \quad \lambda \in (\Lambda_A \setminus \Lambda_i) \quad \text{and} \quad E_A(\lambda) \cap \partial K \neq \emptyset ;\\
&\lambda \in \Lambda_{V,b} && \text{if} \quad \lambda \in \Lambda_V \quad \text{and} \quad E_V(\lambda) \subset \partial K ;\\
&\lambda \in \Lambda_e && \text{if} \quad \lambda \in \Lambda_A \quad \text{and} \quad E_A(\lambda) \cap K = \emptyset .
\end{aligned}$$

The elements of Λ_i, Λ_b and Λ_e are called the *interior eigenvalues*, *boundary eigenvalues* and *external eigenvalues*, respectively, *of the operator A*. The elements of $\Lambda_{V,b}$ are called the *boundary eigenvalues of the variational inequality* (I), (II).

Remark 1.1. The basic properties of and relations between the sets Λ_i, Λ_b, $\Lambda_{V,b}$, Λ_e are explained and illustrated by examples in [4, Section 1]. Let us mention only that $\lambda \in \Lambda_i$ if and only if $\lambda \in \Lambda_V$ with $E_V(\lambda) \cap K^0 \neq \emptyset$. Thus, we can also speak about interior eigenvalues of (I), (II) but they coincide with interior eigenvalues of A. Moreover, the following assertion si true:

Lemma 1.1. (see [4, Lemma 1.1]). *If* $\lambda \in \Lambda_i$ *then* $E_A(\lambda) \cap K = E_V(\lambda)$.

In the sequel we shall consider a nonlinear completely continuous operator $\beta : H \to H$ satisfying the following assumptions:

(P) $\beta u = 0$ if and only if $u \in K$, $\langle \beta u, u \rangle > 0$ for all $u \notin K$ (i.e. β is the penalty operator corresponding to K);

(H) $\beta(tu) = t\beta u$ for all $t > 0$, $u \in H$ (i.e. β is positive homogeneous);

(M) $\langle \beta u - \beta v, u - v \rangle \geqq 0$ for all $u, v \in H$ (i.e. β is monotone);

(β, K) if $u \in K^0$, $v \notin K$, then $\langle \beta v, u \rangle < 0$;

(β, ∂K) if $u \in \partial K$, then there exists a neighborhood U of u such that $(\beta v, u) = 0$ for all $v \in U$.

Remark 1.2. The assumptions (P), (H), (M) were used also in [4], (β, K) is a slight modification of (β, K^0) from [4] (where $\neq 0$ was writen instead of < 0). These assumptions are fulfilled in all examples discussed in [4]. In [4], additional assumptions (CC), (SC′) were introduced, but they were not necessary as we shall explain in Remarks 1.3, 1.4. There is a mistake in [4, Remark 2.1] where it is stated that (CC) is fulfilled in the case of the penalty operator

$$(1.1) \qquad \langle \beta u, v \rangle = -\int_I u^-(x)\, v(x)\, dx \quad \text{for all} \quad u, v \in H$$

(the penalty operator corresponding to the cones of the type $K = \{u \in H;\ u \geqq 0$ on $I\}$, where H is a subspace of $W_2^k(0,1)$, I is a subinterval of $\langle 0,1 \rangle$). This assumption is satisfied for the operators of the type

$$(1.2) \qquad \langle \beta u, v \rangle = -\sum_{i=1}^{n} u^-(x_i)\, v(x_i) \quad \text{for all} \quad u, v \in H$$

only (the penalty operators corresponding to the cones of the type $\{u \in H;\ u(x_i) \geqq 0,$ $i = 1, \ldots, n\}$, where H is as above, $x_i \in (0,1)$, $i = 1, \ldots, n$ are given points). Nonetheles all the assertions concerning the examples in [4, Section 4] are true because it is possible to use Theorem 1.1 formulated below instead of Theorem 2.3 from [4].

The last assumption (β, ∂K) was not considered in [4] but will be used in the study of multiple eigenvalues in Section 2. Unfortunately, (β, ∂K) is fulfilled for the penalty operators of the type (1.2) only.

Definition 1.2. We shall denote by Z the closure (in $\mathbb{R} \times H \times \mathbb{R}$) of the set of all $[\lambda, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying the conditions $\varepsilon \neq 0$ and

(a) $\|v\|^2 = \dfrac{\varepsilon}{1 + \varepsilon}$,

(b) $\lambda v - Av + \varepsilon\beta v = 0$.

If A is replaced by A_n then we shall write Z_n instead of Z.

Remark 1.3. The assumption (CC) in [4] was used in the proof of the following implication only:

(1.3) if $[\lambda_n, u_n, \varepsilon_n]$ satisfy (b), $u_n \notin K$ $(n = 1, 2, \ldots)$, $\lambda_n \to \lambda > 0$, $u_n \rightharpoonup u$, $\varepsilon_n \to +\infty$, then $u_n \to u$.

This implication follows directly from (P), (β, K) (without using (CC)) in the following way. We have

$$\lambda_n\langle u_n, u_n\rangle - \langle Au_n, u_n\rangle + \varepsilon_n\langle\beta u_n, u_n\rangle = 0\,,$$
$$\lambda_n\langle u_n, u\rangle - \langle Au_n, u\rangle + \varepsilon_n\langle\beta u_n, u\rangle = 0$$

and this gives

$$\lambda \limsup \|u_n\|^2 - \lambda\|u\|^2 = \limsup \varepsilon_n\langle\beta u_n, u\rangle - \liminf \varepsilon_n\langle\beta u_n, u_n\rangle\,.$$

But $\beta u_n \to 0$ (because $\{\varepsilon_n\beta u_n\}$ is bounded by (b)) and it follows from here by the standard procedure that $u \in K$ (for details see [4], proof of Lemma 2.4 or [5, Remark 3.3]). The assumptions (P), (β, K) imply $\langle\beta u_n, u_n\rangle \geqq 0$, $\langle\beta u_n, u\rangle \leqq 0$ and therefore we obtain

$$\limsup \|u_n\| \leqq \|u\|\,.$$

This implies $u_n \to u$ and (1.3) is proved. Hence, the assumption (CC) in [4] can be omitted.

Remark 1.4. The assumption (SC′) in [4] was necessary in Lemma 2.2. But Lemma 2.2 was used for the special sequences $\{[\lambda_n, u_n, \varepsilon_n]\}$ with $\varepsilon_n \to 0$ only. In fact, Lemma 2.2 in [4] can be replaced by the following weaker Lemma 1.2 in which (SC′) is not assumed. Hence, the assumption (SC′) in [4] can be omitted.

Lemma 1.2 (cf. [4, Lemma 2.2]). *Let* $[\lambda_n, u_n, \varepsilon_n]$, $[\lambda_0, u_0, 0]$ *satisfy* (b), $\|u_0\| \neq 0$ $(n = 1, 2, \ldots)$, $[\lambda_n, u_n, \varepsilon_n] \to [\lambda_0, u_0, 0]$ *in* $\mathbb{R} \times H \times \mathbb{R}$ *and let* (P), (M) *be fulfilled. Then*

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_0}{\varepsilon_n} = -\frac{\langle\beta u_0, u_0\rangle}{\|u_0\|^2} \leqq 0\,.$$

If $u_0 \notin K$, *then the last expression is negative.*

Proof. We have

$$\lambda_n u_n - Au_n + \varepsilon_n\beta u_n = 0\,,$$
$$\lambda_0 u_0 - Au_0 = 0$$

and it follows from here (using the symmetry of A) that

$$(\lambda_n - \lambda_0)\langle u_n, u_0\rangle + \varepsilon_n\langle \beta u_n, u_0\rangle = 0 .$$

This together with (M), (P) implies the assertion.

Remark 1.5. The condition (a) cannot be fulfilled with $\varepsilon \in (-1, 0)$. Hence, if Z_0 is a connected subset of Z containing a point of the type $[\lambda, 0, 0]$, then $\varepsilon \geqq 0$ for all $[\lambda, v, \varepsilon] \in Z_0$.

Remark 1.6. If $[\lambda, 0, 0] \in Z$, then $\lambda \in \Lambda_A$. Moreover, if $[\lambda_n, v_n, \varepsilon_n] \in Z$, $[\lambda_n, v_n, \varepsilon_n] \to [\lambda, 0, 0]$, $v_n/\|v_n\| \rightharpoonup u$, then $u \in E_A(\lambda)$ and $v_n/\|v_n\| \to u$. Indeed, we have

$$\lambda_n u_n - Au_n + \varepsilon_n \beta u_n = 0$$

for $u_n = v_n/\|v_n\|$; using the complete continuity of A, β, we obtain from here $u_n \to u$ and $\lambda u - Au = 0$.

Remark 1.7. It follows from Remark 1.6 and the assumption (P) that for each $\lambda_0 \in \Lambda_A$ there exists $\delta > 0$ such that $\varepsilon > 0$ and $v \notin K$ for all $[\lambda, v, \varepsilon] \in Z$ with $\lambda \neq \lambda_0$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (We have used the fact that the eigenvalues of A are isolated.)

Remark 1.8. In the following, we shall investigate connected subsets Z_0 of Z starting at a given point $[\lambda^{(0)}, 0, 0]$, $\lambda^{(0)} \in \Lambda_i$ and such that the following conditions are fulfilled for all $[\lambda, v, \varepsilon] \in Z_0$:

(c) if $[\lambda, v, \varepsilon] \neq [\tilde{\lambda}, 0, 0]$ for all $\tilde{\lambda} \in \Lambda_A$, then $v \notin K$;

(d) if $[\lambda, v, \varepsilon] \neq [\lambda^{(0)}, 0, 0]$, then $\lambda \in (\lambda^{(1)}, \lambda^{(0)})$.

Let us remark that the sets Z and Z_0 can be obtained from the sets S and S_0 considered in [4] by the transformation $[\lambda, u, \varepsilon] \to [\lambda, v, \varepsilon]$ with $v = \varepsilon/(1 + \varepsilon)\, u$. The conditions (c), (d) are natural modifications of (c), (d) considered in [4] for the set S_0. The set Z seems to be more advantageous than S from [4] because we can consider connected subsets Z_0 of Z, while the corresponding sets S_0 in [4] had a disconnectedness at the points of the type $[\lambda, u, 0]$, $\tilde{\lambda} \in \Lambda_A$ (see [4], Remark 2.2) and the description of this situation was formally complicated (see [4, Theorem 2.3]). The mentioned disconnectedness vanishes by the transformation of S_0 onto Z_0.

The following theorem represents a slight modification of Theorem 2.3 from [4] and will be of basic importance for the proof of the main result of the present paper.

Theorem 1.1 (cf. [4, Theorem 2.3]). *Let $\lambda^{(1)}, \lambda^{(0)} \in \Lambda_i$ be simple, $0 < \lambda^{(1)} < \lambda^{(0)}$, $(\lambda^{(1)}, \lambda^{(0)}) \cap (\Lambda_b \cup \Lambda_i) = \emptyset$. Assume that there exists a completely continuous operator β satisfying the conditions* (P), (H), (M), (β, K). *Then there exists an unbounded closed connected subset $Z_0 \subset Z$ containing the point $[\lambda^{(0)}, 0, 0]$ and such that the implications* (c), (d) *hold for all $[\lambda, v, \varepsilon] \in Z_0$. If $\{[\lambda_n, v_n, \varepsilon_n]\} \subset Z_0$, $\varepsilon_n \to +\infty$ *), then there exists a subsequence of indices $\{r_n\}$ such that $r_n \to +\infty$, $\lambda_{r_n} \to \lambda_\infty$, $v_{r_n} \to v$, where $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$, $v_\infty \in \partial K \cap E_V(\lambda_\infty) \setminus E_A(\lambda_\infty)$.*

*) It follows from (d) that Z_0 is unbounded in ε.

Proof. Let S_0 be the set from [4, Theorem 2.3]. (All the assumptions are fulfilled with the exception of (CC), (SC′), but these can be omitted by Remarks 1.3, 1.4). Define

$$Z_0 = \left\{[\lambda, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}\,; \quad v = \frac{\varepsilon}{1 + \varepsilon}\, u\,, \ [\lambda, u, \varepsilon] \in S_0\right\}.$$

It follows from the assertion of [4, Theorem 2.3] that Z_0 has all the properties mentioned in Theorem 1.1.

The following Lemmas give information about the properties of the equation with the penalty and will be useful for the proof of the main result. Analogous assertions were used also in [4].

Lemma 1.3 (see [4, Lemma 2.1]). *If $\lambda \in A_i$ and the condition* (β, K) *is fulfilled, then*

$$\lambda u - Au + \varepsilon \beta u \neq 0$$

for all $u \notin K$, $\varepsilon > 0$.

Lemma 1.4 (cf. [4, Lemma 2.4]). *Let $\lambda^{(1)}, \lambda^{(0)} \in A_i$, $0 < \lambda^{(1)} < \lambda^{(0)}$ and let the assumptions* (P), (M), (β, K) *and* (β, ∂K) *be fulfilled. Suppose that there exist $\lambda_n, u_n, \varepsilon_n$ $(n = 1, 2, \ldots)$ satisfying the conditions*

(a′) $\|u_n\| = \dfrac{\varepsilon_n}{1 + \varepsilon_n}$, $n = 1, 2, \ldots$, $\varepsilon_n \to +\infty$,

(b′) $\lambda_n u_n - Au_n + \varepsilon_n \beta u_n = 0$, $n = 1, 2, \ldots$,

(c′) $u_n \notin K^0$, $n = 1, 2, \ldots$,

(d′) $\lambda_n \in (\lambda^{(1)}, \lambda^{(0)})$, $n = 1, 2, \ldots$,

and such that $\lambda_n \to \lambda_\infty$, $u_n \rightharpoonup u_\infty$, $\varepsilon_n \to +\infty$ for some λ_∞, u_∞. Then $\lambda_\infty \in A_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$, $u_n \to u_\infty$ and $u_\infty \in E_V(\lambda_\infty) \cap \partial K$.

Proof. The assertion of Lemma 1.4 is the same as that of Lemma 2.4 in [4], but the assumption (CC) is omitted, (β, K^0) is replaced by (formally) stronger (β, K) and the simplicity of $\lambda^{(0)}, \lambda^{(1)}$ is replaced by the assumption (β, ∂K). We have explained in Remark 1.3 how Lemma 2.4 from [4] can be proved without the assumption (CC). Realizing this, we can prove $\lambda_\infty \in A_{V,b}$, $u_n \to u$, $u_\infty \in E_V(\lambda_\infty) \cap \partial K$ in Lemma 1.4 analogously as in Lemma 2.4 from [4]. It was clear in [4] that $\lambda_\infty \in (\lambda^{(1)}, \lambda^{(0)})$ because neither $\lambda = \lambda^{(1)}$ nor $\lambda = \lambda^{(0)}$ was possible as a consequence of the assumption that $\lambda^{(1)}, \lambda^{(0)} \in A_i$ are simple. In the case of the present Lemma 1.4, the assertion $\lambda_\infty \in (\lambda^{(1)}, \lambda^{(0)})$ follows from (β, ∂K). Indeed, if $\lambda = \lambda^{(1)}$, then Lemma 1.1 implies that $u_\infty \in E_A(\lambda^{(1)})$ and therefore

$$\lambda^{(1)} u_\infty - Au_\infty = 0\,.$$

This together with (b′) and the symmetry of A implies

$$(\lambda_n - \lambda^{(1)}) \langle u_n, u_\infty \rangle + \varepsilon_n \langle \beta u_n, u_\infty \rangle = 0\,,$$

(β, ∂K) gives $\langle \beta u_n, u_\infty \rangle = 0$ for n sufficiently large and this is not possible by (d'). Analogously, $\lambda = \lambda^{(0)}$ cannot occur.

Remark 1.9. If A is replaced by A_n in Theorem 1.1, then we write $Z_{0,n}$ instead of Z_0.

2. EIGENVALUES OF THE VARIATIONAL INEQUALITY CORRESPONDING TO MULTIPLE EIGENVALUES OF THE OPERATOR

Theorem 2.1. *Let $\lambda^{(0)}, \lambda^{(1)} \in \Lambda_i$, $0 < \lambda^{(1)} < \lambda^{(0)}$, $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_i = \emptyset$. Assume that there exists a completely continuous operator β satisfying the conditions* (P), (H), (M), (β, K), (β, ∂K). *Then there exists $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$.*

Remark 2.1. We have $\Lambda_b \subset \Lambda_{V,b}$ and therefore the assertion of Theorem 2.1 is trivial if $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_b \neq \emptyset$. In the case $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_b = \emptyset$ it follows from the following theorem.

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be fulfilled and let $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_b = \emptyset$. Then there exists an unbounded closed connected subset Z_0 of Z containing the point $[\lambda^{(0)}, 0, 0]$ and such that the implications* (c), (d) *from Remark* 1.8 *hold for all $[\lambda, v, \varepsilon] \in Z_0$. If $\{[\lambda_n, v_n, \varepsilon_n]\} \subset Z_0$, $\varepsilon_n \to +\infty$ *), then there exists a sequence of indices $\{r_n\}$ such that $r_n \to +\infty$, $\lambda_{r_n} \to \lambda_\infty$, $v_{r_n} \to v_\infty$, where $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$ and $v_\infty \in \partial K \cap E_{V,b}(\lambda_\infty) \setminus E_A(\lambda_\infty)$, $\|v_\infty\| = 1$.*

Remark 2.2. Let the assumptions of Theorem 2.2 be fulfilled. We shall choose orthonormal bases $\{u_1^{(1)}, \ldots, u_r^{(1)}\}$ and $\{u_1^{(0)}, \ldots, u_s^{(0)}\}$ of $E_A(\lambda^{(1)})$ and $E_A(\lambda^{(0)})$, respectively, such that $u_1^{(0)}, u_1^{(1)} \in K^0$. Introduce operators A_n $(n = 1, 2, \ldots)$ by

$$A_n u = Au - \frac{1}{n} \sum_{i=2}^{r} \langle u_i^{(1)}, u \rangle\, u_i^{(1)} + \frac{1}{n} \sum_{j=2}^{s} \langle u_j^{(0)}, u \rangle\, u_j^{(0)} . \tag{2.1}$$

It is easy to see that

$$A_n \to A \quad \text{in the operator norm} . \tag{2.2}$$

Further, $\lambda^{(1)}, \lambda^{(0)}$ are simple interior eigenvalues of A_n, $\Lambda_{A_n} \cap (\lambda^{(1)}, \lambda^{(0)}) = \Lambda_A \cap (\lambda^{(1)}, \lambda^{(0)})$, $E_{A_n}(\lambda) = E_A(\lambda)$ for all $\lambda \in \Lambda_A \cap (\lambda^{(1)}, \lambda^{(0)})$ and therefore the assumptions of Theorem 1.1 are fulfilled for A_n (with an arbitrary fixed $n = 1, 2, \ldots$).

Remark 2.3. Let $Z_{0,n}$ denote the set from Theorem 1.1 for the operator A_n from Remark 2.2 (see Remark 1.9). Introduce the set Z_L as the set of all $[\lambda, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ such that there exist a sequence $\{r_n\}$ of indices $(r_n \to \infty)$ and a sequence $\{[\lambda_n, v_n, \varepsilon_n]\}$ such that $[\lambda_n, v_n, \varepsilon_n] \in Z_{0,r_n}$, $[\lambda_n, v_n, \varepsilon_n] \to [\lambda, v, \varepsilon]$ in $\mathbb{R} \times H \times \mathbb{R}$. We have

$$\lambda_n v_n - A_{r_n} v_n + \varepsilon_n \beta v_n = 0 \tag{b'}$$

*) It follows from (d) that Z_0 is unbounded in ε.

for such points and thus it follows by (2.2) that $[\lambda, v, \varepsilon] \in Z$, i.e. $Z_L \subset Z$. The set Z_L is not connected in general. We shall denote by Z_0 the component of Z_L containing the point $[\lambda^{(0)}, 0, 0]$. Our aim is to prove that Z_0 has all the properties described in Theorem 2.2.

Lemma 2.1 (see [9]). *Let K be a compact metric space and A, B disjoint closed subsets of K. Then either*

(2.3) *there exists a closed connected subset of K meeting both A and B*

or

(2.4) $K = K_A \cup K_B$, *where K_A, K_B are disjoint compact subsets of K, $A \subset K_A$, $B \subset K_B$.*

Lemma 2.2. *The set Z_0 from Remark 2.3 is unbounded.*

Proof. The sets $Z_{0,n}$ from Remark 2.3 are unbounded (in ε) by Theorem 1.1 and it follows from here that also Z_L is unbounded. Let us suppose that Z_0 is bounded. Then there exists $R > 0$ such that $Z_0 \subset B_R$, $(Z_L \setminus Z_0) \cap \partial B_R \neq \emptyset$, where B_R denotes the open ball in $\mathbb{R} \times H \times \mathbb{R}$ with the centre at the origin and with the radius R, ∂B_R denotes its boundary. It is easy to see that Z_L is locally compact in $\mathbb{R} \times H \times \mathbb{R}$ and therefore $K = \bar{B}_R \cap Z_L$ is a compact metric space under the induced topology from $\mathbb{R} \times H \times \mathbb{R}$. If we set $A = Z_0$, $B = (Z_L \setminus Z_0) \cap \partial B_R$, then A, B are disjoint closed subsets of K. The case (2.3) from Lemma 2.1 cannot occur because $A = Z_0$ is a component of K. Hence, Lemma 2.1 implies that there exist disjoint compact sets K_A, K_B such that $Z_0 \subset K_A$, $(Z_L \setminus Z_0) \cap \partial B_R \subset K_B$, $Z_L \cap \bar{B}_R = K_A \cup K_B$. Denote the distance between K_A, K_B by η. We have $\eta > 0$ and it follows from the definition of Z_L (Remark 2.3) and from the connectedness of $Z_{0,n}$ that there exists a bounded sequence $\{[\lambda_n, v_n, \varepsilon_n]\} \subset \bar{B}_R$ such that $[\lambda_n, v_n, \varepsilon_n] \in Z_{0,r_n}$, $r_n \to +\infty$ and

$$\operatorname{dist}([\lambda_n, v_n, \varepsilon_n], Z_0) \geqq \frac{\eta}{4}, \quad \operatorname{dist}([\lambda_n, v_n, \varepsilon_n], Z_L \cap \bar{B}_R \setminus Z_0) \geqq \frac{\eta}{4}.$$

We can assume that $\lambda_n \to \lambda$, $v_n \rightharpoonup v$, $\varepsilon_n \to \varepsilon$. The condition (b'), the complete continuity of β and (2.2) imply $v_n \to v$ and therefore $[\lambda, v, \varepsilon] \in Z_L$. But simultaneously we obtain

$$\operatorname{dist}([\lambda, v, \varepsilon], Z_0) \geqq \frac{\eta}{4}, \quad \operatorname{dist}([\lambda, v, \varepsilon], Z_L \setminus Z_0) \geqq \frac{\eta}{4}$$

and this a contradiction.

Lemma 2.3. *The conditions* (c), (d) *from Remark* 1.8 *are fulfilled for all* $[\lambda, v, \varepsilon] \in Z_0$, *where Z_0 is the set from Remark* 2.3.

Proof. It follows from Theorem 1.1 and from the definition of Z_0 that for all $[\lambda, v, \varepsilon] \in Z_0$ we have

(2.5) $$v \notin K^0\,,$$

(2.6) $$\lambda \in \langle \lambda^{(1)}, \lambda^{(0)} \rangle\,.$$

Hence, if (c) is not fulfilled then there exists $[\lambda, v, \varepsilon] \in Z_0$ with $v \in \partial K$, $\|v\| > 0$. The equation (b) together with (P) implies

$$\lambda v - Av = 0 . \tag{2.7}$$

The case $\lambda \in (\lambda^{(1)}, \lambda^{(0)})$ is impossible due to the assumption $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_b = \emptyset$. The definition of Z_0 ensures the existence of $[\lambda_n, v_n, \varepsilon_n] \in Z_{0,r_n}$ (r_n a suitable sequence of indices, $r_n \to \infty$) such that $[\lambda_n, v_n, \varepsilon_n] \to [\lambda, v, \varepsilon]$. The points $[\lambda_n, v_n, \varepsilon_n]$ satisfy (b') and this together with (2.7) and the symmetry of A implies

$$(\lambda_n - \lambda) \langle v_n, v \rangle - \langle A_{r_n} v_n, v \rangle + \langle A v_n, v \rangle + \varepsilon_n \langle \beta v_n, v \rangle = 0 .$$

But $\langle \beta v_n, v \rangle = 0$ for n sufficiently large by (β, ∂K) and using (2.1) we obtain

(2.8)
$$(\lambda_n - \lambda) \langle v_n, v \rangle + \frac{1}{r_n} \sum_{i=2}^{r} \langle u_i^{(1)}, v_n \rangle \langle u_i^{(1)}, v \rangle - \frac{1}{r_n} \sum_{j=2}^{s} \langle u_j^{(0)}, v_n \rangle \langle u_j^{(0)}, v \rangle = 0 ,$$

where $u_i^{(1)}, u_j^{(0)}$ were introduced in Remark 2.2. If $\lambda = \lambda^{(0)}$, then $\langle u_i^{(1)}, v \rangle = 0$ $(i = 1, \ldots, r)$ and $\langle v_n, v \rangle \to \|v\|^2 > 0$, $\sum_{j=2}^{s} \langle u_j^{(0)}, v_n \rangle \langle u_j^{(0)}, v \rangle \to \sum_{j=2}^{s} \langle u_j^{(0)}, v \rangle^2 > 0$ because $v \in E_A(\lambda^{(0)})$, $v \neq c u_1^{(0)}$ for all $c \in \mathbb{R}$. Further, $\lambda_n < \lambda^{(0)}$ and therefore the left hand side in (2.8) is negative, which is a contradiction. Analogously the case $\lambda = \lambda^{(1)}$ leads to the contradiction and (c) for all $[\lambda, v, \varepsilon] \in Z_0$ is proved.

Now, let us suppose that (d) is not fulfilled. Then there exists $[\lambda, v, \varepsilon] \in Z_0$ such that $[\lambda, v, \varepsilon] \neq [\lambda^{(0)}, 0, 0]$ and either $\lambda = \lambda^{(0)}$ or $\lambda = \lambda^{(1)}$. Let $[\lambda, v, \varepsilon] = [\lambda^{(1)}, 0, 0]$. Then it follows from the connectedness of Z_0 and Remark 1.7 that there exist $[\lambda_n, v_n, \varepsilon_n] \in Z_0$ such that $\lambda_n > \lambda^{(1)}$, $\varepsilon_n > 0$ $(n = 1, 2, \ldots)$, $[\lambda_n, v_n, \varepsilon_n] \to [\lambda^{(1)}, 0, 0]$. This is not possible by Lemma 1.2 and therefore $[\lambda, v, \varepsilon] \neq [\lambda^{(1)}, 0, 0]$. Hence, $v \notin K$, $\varepsilon > 0$ by (c), (a) and this contradicts Lemma 1.3.

Proof of Theorem 2.2 follows directly from Remark 2.3 and Lemmas 2.2, 2.3, 1.4.

Theorem 2.3. *Assume that there exists a completely continuous operator β satisfying the conditions* (P), (H), (M), (β, K), (β, ∂K). *If Λ_i is an infinite sequence, then $\Lambda_{V,b}$ contains an infinite sequence converging to zero. If, moreover, Λ_i contains an infinite sequence of couples $\lambda_k^{(1)}, \lambda_k^{(0)}$ such that $(\lambda_k^{(1)}, \lambda_k^{(0)}) \cap \Lambda_b = \emptyset$ $(k = 1, 2, \ldots)$, then the set $\Lambda_{V,b}$ contains an infinite sequence of eigenvalues of* (I), (II) *converging to zero such that the corresponding eigenvectors are not eigenvectors of the operator A.*

Proof follows immediately from Theorems 2.1, 2.2 and Remark 2.1.

Example 2.1. Let Ω be a bounded domain in $\mathbb{R}^2$ with the lipschitzian boundary $\partial\Omega$. The points from Ω will be denoted by $x = [x_1, x_2]$. Let H be the Sobolev space

$\mathring{W}_2^2(\Omega)$ with the inner product defined by

$$\langle u, v \rangle = \int_\Omega \Delta u \, \Delta v \, dx \quad \text{for all} \quad u, v \in H .$$

Consider the cone

$$K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, \ldots, n\} ,$$

where $x^{(i)} \in \Omega$ $(i = 1, \ldots, n)$ are given points, and the operator A defined by

$$\langle Au, v \rangle = \int_\Omega \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \quad \text{for all} \quad u, v \in H .$$

Then K is a closed convex cone in H and A is a linear symmetric completely continuous operator in H. (We use the fact that the space $W_2^2(\Omega)$ is continuously imbedded into the space of functions continuous on $\bar{\Omega}$ and into the space $W_2^1(\Omega)$.) Let us remark that the eigenvalues and eigenvectors of A are eigenvalues and eigenvectors of the boundary value problem

(2.9) $$\lambda \, \Delta^2 u + \Delta u = 0 \quad \text{on} \quad \Omega ,$$

(2.10) $$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega ,$$

and the variational inequality (I), (II) in our case corresponds to the problem with fixed obstacles from below at the points $x^{(i)}$ (cf. [4, Section 4]). We can use the penalty operator defined by

$$\langle \beta u, v \rangle = - \sum_{i=1}^{n} u^-(x_i) \, v(x_i) \quad \text{for all} \quad u, v \in H ,$$

where u^- denotes the negative part of u. All the assumptions of our theory are fulfilled. Let us remark that $\lambda \in \Lambda_i$ if and only if there exists a corresponding eigenvector u of (2.9), (2.10) satisfying $u(x^{(j)}) > 0$ for all $j = 1, \ldots, n$; $\lambda \in \Lambda_b$ if and only if $\lambda \notin \Lambda_i$ and there exists a corresponding eigenvector u of (2.9), (2.10) satisfying $u(x^{(j)}) \geqq 0$ for $j = 1, \ldots, n$, $u(x^{(k)}) = 0$ for at least one k; $\lambda \in \Lambda_e$ if and only if for each corresponding eigenvector of (2.9), (2.10), we have $u(x^{(j)}) > 0$ for at least one j and $u(x^{(k)}) < 0$ for at least one k; $\lambda \in \Lambda_{V,b}$ if and only if for each corresponding eigenvector of (I), (II), we have $u(x^{(j)}) \geqq 0$, $j = 1, \ldots, n$ and $u(x^{(k)}) = 0$ for at least one k.

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