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DIRECT DECOMPOSABILITY OF TOLERANCES ON LATTICES, SEMILATTICES AND QUASILATTICES

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In the paper [2] the authors considered tolerances on direct products of monoids and distributive lattices in order to obtain conditions under which a tolerance is a direct product of tolerances on direct factors. Actually, the following result is proved:

Theorem. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two monoids or two distributive lattices with greatest and least elements. Then the following two implications are equivalent*:

(1) $T \in LT(\mathfrak{A} \times \mathfrak{B}) \Rightarrow$ *there exist* $T_1 \in LT(\mathfrak{A})$, $T_2 \in LT(\mathfrak{B})$ *such that* $T = T_1 \times T_2$.

(2) $\langle a, b\rangle \in T \Rightarrow T_A(pr_1 a, pr_1 b) \times T_B(pr_2 a, pr_2 b) \subseteq T$.

The aim of this paper is twofold:

– to extend the above result for algebras in the title;
– to prove that (2) of Theorem holds automatically in lattices, i.e. lattices have directly decomposable tolerances without any constraints.

0. BASIC CONCEPTS

Let $\mathfrak{A} = (A, F)$ be an algebra. By a *tolerance* T (or *tolerance relation*) on $\mathfrak{A}$ we mean a reflexive and symmetric binary relation on A with the *Substitution Property* with respect to F, i.e. T is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$. The set of all tolerances on an algebra $\mathfrak{A}$ constitutes an algebraic lattice $LT(\mathfrak{A})$ [1], and the meet in $LT(\mathfrak{A})$ coincides with the set intersection. We denote the join in $LT(\mathfrak{A})$ by $\vee_A$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras of the same type, $\mathfrak{A} \times \mathfrak{B}$ their direct product and $T \in LT(\mathfrak{A} \times \mathfrak{B})$. T is called *directly decomposable* if there exist $T_1 \in LT(\mathfrak{A})$ and $T_2 \in LT(\mathfrak{B})$ such that $T = T_1 \times T_2$. If every tolerance on $\mathfrak{A} \times \mathfrak{B}$ is directly decomposable, we say that $\mathfrak{A} \times \mathfrak{B}$ has directly decomposable tolerances. If $\mathscr{C}$ is a class of algebras such that for every pair $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$, $\mathfrak{A} \times \mathfrak{B}$ has directly decomposable tolerances, $\mathscr{C}$ is said to have *directly decomposable tolerances.*

Let a and b be two elements of an algebra $\mathfrak{A}$. $T_A(a, b)$ denotes the *least tolerance* *on* $\mathfrak{A}$ *collapsing the pair* $\langle a, b\rangle$, i.e. $T_A(a, b) = \bigcap\{T \mid T \in LT(\mathfrak{A})$ and $\langle a, b\rangle \in T\}$. Thus $T_A(a, b)$ is a generalization of the concept of a principal congruence.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras of the same type and x an element of $\mathfrak{A} \times \mathfrak{B}$. When $x_1 = pr_1 x$ and $x_2 = pr_2 x$, $[x_1, x_2]$ is a componentwise denotation for x. Further, if $T_1 \in LT(\mathfrak{A})$ and $T_2 \in LT(\mathfrak{B})$, we have $\langle x, y\rangle \in T_1 \times T_2$ if $\langle x_1, y_1\rangle \in T_1$ and $\langle x_2, y_2\rangle \in T_2$. As noted in [2], the direct product of two tolerances is a tolerance on the direct product of the corresponding algebras.

1. TOLERANCES ON DIRECT PRODUCTS

The aim of this section is to give conditions under which the identity

$$(1) \qquad (T_1 \times T_2) \vee_{A \times B} (S_1 \times S_2) = (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$$

is valid for two algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type and for every $T_1, S_1 \in LT(\mathfrak{A})$ and every $T_2, S_2 \in LT(\mathfrak{B})$. It is worth noting that (1) holds for congruences T_1, T_2, S_1 and S_2 on any algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, see [3].

An algebra is called *idempotent* if for every m-ary polynomial $q(x_1, \ldots, x_m)$ over $\mathfrak{A}$ and for every $a \in A$, $q(a, \ldots, a) = a$. $\mathfrak{A}$ is called *superidempotent* if it is idempotent and for every m-ary polynomial q and every two elements a and b of $\mathfrak{A}$ there are elements c and d such that $q(k, \ldots, k, a, k, \ldots, k) = a$ and $q(k, \ldots, k, b, k, \ldots$ $\ldots, k) = b$, where k is c or d according to the following rule: if k on ith place is c (d) in the expression for a then k on the ith place is also c (d) in the expression for b, and vice versa. When $\mathfrak{A}$ is a lattice, it is superidempotent: the elements c and d corresponding to given a and b are $a \vee b$ and $a \wedge b$.

Theorem 1. *Let* $\mathscr{C}$ *be a class of superidempotent algebras of the same type, where for every* $\mathfrak{A} \in \mathscr{C}$ *and every* $T, S \in LT(\mathfrak{A})$, $\langle u, v\rangle \in T \vee_A S$ *if and only if there exists a positive integer* N *such that for every even* $n > N$ *there are elements* u_i, v_i *of* $\mathfrak{A}$ $(i = 1, \ldots, n)$ *and an* n*-ary polynomial* p *over* $\mathfrak{A}$ *with the properties*

(i) $p(u_1, \ldots, u_n) = u$ *and* $p(v_1, \ldots, v_n) = v$;

(ii) $\langle u_i, v_i\rangle \in T$ *for even and* $\langle u_i, v_i\rangle \in S$ *for odd values of* i $(i = 1, \ldots, n)$.

Then the identity (1) *is valid for all* $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ *and every* $T_1, S_1 \in LT(\mathfrak{A})$, $T_2, S_2 \in$ $\in LT(\mathfrak{B})$.

Proof. The proof is a modification of the proof of [2, Thm. 1]. Evidently $T_1 \times T_2$, $S_1 \times S_2 \subseteq (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$, whence it remains to prove the inclusion $(T_1 \vee_A S_1) \times (T_2 \vee_B S_2) \subseteq (T_1 \times T_2) \vee_{A \times B} (S_1 \times S_2)$.

Let $a_1, b_1 \in A$ and $a_2, b_2 \in B$ be elements such that $\langle [a_1, a_2], [b_1, b_2]\rangle \in$ $\in (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$. Then $\langle a_1, b_1\rangle \in T_1 \vee_A S_1$ and $\langle a_2, b_2\rangle \in T_2 \vee_B S_2$. According to the assumption, there are two positive integers N_1 and N_2 such that for every even integer $n > \max(N_1, N_2)$ there exist elements $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$

of $\mathfrak{A}$, $u'_1, \ldots, u'_n$ and $v'_1, \ldots, v'_n$ of $\mathfrak{B}$, and n-ary polynomials p and q over $\mathfrak{A}$ and $\mathfrak{B}$ having the properties

(i) $p(u_1, \ldots, u_n) = a_1$, $p(v_1, \ldots, v_n) = b_1$, $q(u'_1, \ldots, u'_n) = a_2$ and $q(v'_1, \ldots, v'_n) = b_2$;

(ii) $\langle u_i, v_i\rangle \in T_1$ and $\langle u'_i, v'_i\rangle \in T_2$ for even and $\langle u_i, v_i\rangle \in S_1$ and $\langle u'_i, v'_i\rangle \in S_2$ for odd values of i, $i = 1, \ldots, n$.

We define now an n^2-ary polynomial r as follows: $r(y_1, \ldots, y_{n^2}) = p(q(x_{11}, \ldots \ldots, x_{1n}), \ldots, q(x_{n1}, \ldots, x_{nn}))$, where $x_{k,j} = y_{(k-1).n+j}$. Let c_i and d_i be elements of $\mathfrak{A}$ such that $q(k_i, \ldots, k_i, u_i, k_i, \ldots, k_i) = u_i$ and $q(k_i, \ldots, k_i, v_i, k_i, \ldots, k_i) = v_i$, where k_i is c_i or d_i, $i = 1, \ldots, n$. Furthermore, $\langle c_i, c_i\rangle$, $\langle d_i, d_i\rangle \in S_1, T_1$ for every value of i. Accordingly, we can now write the following scheme:

- t and s are indices, $t, s = 1, \ldots, n$;
- for each separate value of t, $z_s = w_s = k_t$ when $s \neq t$, and $z_s = u_t$, $w_s = v_t$ when $s = t$;
- $\langle [z_s, u'_t], [w_s, v'_t]\rangle \in T_1 \times T_2$ when t is even and $\langle [z_s, u'_t], [w_s, v'_t]\rangle \in S_1 \times S_2$ when t is odd.

Moreover, $r([z_1, u'_1], [z_2, u'_1], \ldots, [z_n, u'_n]) = [r(z_1, \ldots, z_n, z_1, \ldots, z_n, \ldots, z_n), r(u'_1, \ldots, u'_1, u'_2, \ldots, u'_2, u'_3, \ldots, u'_n)] = [r(u_1, k_1, \ldots, k_1, k_2, u_2, k_2, \ldots, k_2, k_3, \ldots \ldots, k_n, u_n), r(u'_1, \ldots, u'_1, u'_2, \ldots, u'_n)] = [a_1, a_2]$ and similarly $r([w_1, v'_1], [w_2, v'_1], \ldots \ldots, [w_n, v'_n]) = [b_1, b_2]$. Obviously, $\mathfrak{A} \times \mathfrak{B}$ is superidempotent when $\mathfrak{A}$ and $\mathfrak{B}$ are, whence one can derive from the polynomial r a new polynomial r^* such that r^* is $(n^2 + j)$-ary, where j is even and $r^*(y_1, \ldots, y_{n^2+j}) = pr_1(r(y_1, \ldots, y_{n^2}), y_{n^2+1}, \ldots \ldots, y_{n^2+j})$. But then, according to the superidempotency of $\mathfrak{A} \times \mathfrak{B}$ with respect to r^*, $[a_1, a_2]$ and $[b_1, b_2]$, we have two sequences of elements and a polynomial r^* over $\mathfrak{A} \times \mathfrak{B}$ such that (i) and (ii) hold for every even $m > n^2 - 2$ and thus $\langle [a_1, a_2], [b_1, b_2]\rangle \in (T_1 \times T_2) \vee_{A\times B} (S_1 \times S_2)$. This completes the proof.

A *join-semilattice* $\mathfrak{S} = (S, \vee)$ is called *down directed*, if for any two elements $a, b \in S$ there is a common lower bound c of a and b in $\mathfrak{S}$. An *up directed meet-semilattice* is defined dually. A *quasilattice* $\mathfrak{Q} = (Q, \vee, \wedge)$ is a structure, where $\vee$ and $\wedge$ are commutative, associative and idempotent (see Płonka [4]), i.e. $\mathfrak{Q}$ is a join-semilattice with respect to $\vee$ and a meet-semilattice with respect to $\wedge$. $\mathfrak{Q}$ is a lattice if and only if the absorption laws hold in $\mathfrak{Q}$. $\mathfrak{Q}$ is down directed, if it is down directed as a join-semilattice, and up directed, if it is up directed as a meet-semilattice. Obviously, down directed join-semilattices and quasilattices as well as up directed meet-semilattices and quasilattices are superidempotent.

Theorem 2. *Let $\mathscr{C}$ be one of the following classes of algebras*:

(i) *the class of all lattices*;

(ii) *the class of all down directed join-semilattices*;

(iii) *the class of all up directed meet-semilattices*;

(iv) *the class of all down directed quasilattices;*
(v) *the class of all up directed quasilattices.*
Then (1) *is true for each* $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ *and for every* $T_1, S_1 \in LT(\mathfrak{A})$, $T_2, S_2 \in LT(\mathfrak{B})$.

Proof. We have to show that when $\mathfrak{A} \in \mathscr{C}$, $T, S \in LT(\mathfrak{A})$ and $u, v \in A$, then $\langle u, v\rangle \in T \vee_A S$ if and only if there exists an even positiev integer N such that for every even integer $n > N$ there are elements $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ and an n-ary polynomial p over $\mathfrak{A}$ such that (i) and (ii) of Theorem 1 hold. After proving this the assertion of the theorem follows from Theorem 1. We shall present the proof only for lattices; the proofs for (ii)–(v) are analogous and hence we omit them.

As proved in [1, Thm. 2], $\langle u, v\rangle \in T \vee_A S$ if and only if there is a polynomial $p^*(y_1, \ldots, y_m)$ and elements $u_1^*, \ldots, u_m^*$ and $v_1^*, \ldots, v_m^*$ such that $\langle u_i^*, v_i^*\rangle \in T$ or $\langle u_i^*, v_i^*\rangle \in S$, $i = 1, \ldots, m$, $p^*(u_1^*, \ldots, u_m^*) = u$ and $p^*(v_1^*, \ldots, v_m^*) = v$. Thus if the conditions (i) and (ii) of Theorem 1 hold, then $\langle u, v\rangle \in T \vee_A S$. So it remains to show the converse and we shall do it by modifying the polynomial $p^*(y_1, \ldots, y_m)$ and the sequences $u_1^*, \ldots, u_m^*$ and $v_1^*, \ldots, v_m^*$ in a suitable manner.

Let us denote $u_1^* \wedge \ldots \wedge u_m^* \wedge v_1^* \wedge \ldots \wedge v_m^*$ by a^*. Trivially, $\langle a^*, a^*\rangle \in T, S$, $u_i^* \vee a^* = u_i^*$ and $v_i^* \vee a^* = v_i^*$ for each i, $i = 1, \ldots, m$. If $\langle u_1^*, v_m^*\rangle \in S$, we put $u_1 = u_1^*$ and $v_1 = v_1^*$, and if $\langle u_1^*, v_1^*\rangle \notin S$, we put $u_1 = v_1 = a^*$, $u_2 = u_1^*$ and $v_2 = v_1^*$; clearly then $\langle u_2, v_2\rangle \in T$. Assume that $\langle u_1^*, v_1^*\rangle \in S$, whence $u_1 = u_1^*$ and $v_1 = v_1^*$. If now $\langle u_2^*, v_2^*\rangle \in T$, we put $u_2 = u_2^*$ and $v_2 = v_2^*$, and if $\langle u_2^*, v_2^*\rangle \notin T$, then we put $u_2 = v_2 = a^*$. In that case $\langle u_2, v_2\rangle \in T$ and because then $\langle u_2^*, v_2^*\rangle \notin S$, we put $u_2^* = u_3$ and $v_2^* = v_3$. So from $u_1^*, \ldots, u_m^*$, from $v_1^*, \ldots, v_m^*$ and from a^* we can easily construct two new sequences $u_1, \ldots, u_{2k}$ and $v_1, \ldots, v_{2k}$ such that $\langle u_i, v_i\rangle \in T$ for even and $\langle u_i, v_i\rangle \in S$ for odd values of i, $i = 1, \ldots, 2k$. Assume that $\langle u_1^*, v_1^*\rangle \notin S$, and so $u_1 = v_1 = a^*$, $u_2 = u_1^*$ and $v_2 = v_1^*$. Then we replace y_1 in the polynomial $p^*(y_1, \ldots, y_m)$ by the expression $x_1 \vee x_2$ and obtain a new polynomial $p'(x_1, x_2, y_2, \ldots, y_m)$. After performing all similar necessary modifications in the polynomial p^* we have a new one: $p(x_1, \ldots, x_{2k})$. Because $u_i^* \vee a^* = u_i^*$ and $v_i^* \vee a^* = v_i^*$, $p(u_1, \ldots, u_{2k}) = u$ and $p(v_1, \ldots, v_{2k}) = v$. Now we may put $N = 2k - 2$ and if $n > N$, we put $u_i = v_i = a^*$ for $i = 2k + 1, \ldots, n$, and moreover $p(x_1, \ldots, x_n) = p(x_1, \ldots, x_{2k}) \vee x_{2k+1} \vee \ldots \vee x_n$. In this case the conditions (i) and (ii) of Theorem 1 also hold, and the required result follows from [1, Thm. 2].

It is proved in [2] that the identity (1) implies a similar identity for an arbitrary number of tolerances on direct factors, i.e.

$$(2) \qquad \bigvee_{A\times B}\{T_\gamma \times S_\gamma \mid \gamma \in \Gamma\} = \bigvee_A\{T_\gamma \mid \gamma \in \Gamma\} \times \bigvee_B\{S_\gamma \mid \gamma \in \Gamma\}$$

(Γ is an arbitrary index set) in the class of all distributive lattices with a least and a greatest element as well as in the class of all monoids with a unit element. In the following we extend this result. The proof follows from that of [2, Thm. 2], where the unit element is substituted by a lower bound (by an upper bound) of the elements under consideration and the operation $\circ$ by $\vee$ (by $\wedge$). Hence the proof is omitted.

Theorem 3. *Let $\mathscr{C}$ be one of the classes* (i)–(v) *of algebras in Theorem* 2. *Then* (2) *is valid for each pair* $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ *and for every* $T_\gamma \in LT(\mathfrak{A})$, $S_\gamma \in LT(\mathfrak{B})$, *where* Γ *is an arbitrary index set.*

2. DIRECT DECOMPOSABILITY

Theorem 5 of [2] can be generalized in the following way:

Theorem 4. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two algebras of the same type satisfying* (2). *Then the following conditions are equivalent*:

(1) $\mathfrak{A} \times \mathfrak{B}$ *has directly decomposable tolerances*;

(2) $\langle a, b\rangle \in T$ *implies* $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T$ *for each* $T \in LT(\mathfrak{A})$.

Proof. (1) $\Rightarrow$ (2). The equality $T_{A\times B}(a, b) = T_1 \times T_2$ evidently implies that $T_A(a_1, b_1) \subseteq T_1$, $T_B(a_2, b_2) \subseteq T_2$, and thus $\langle a, b\rangle \in T$ implies that $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T_1 \times T_2 \subseteq T$.

(2) $\Rightarrow$ (1). Let

$T_1 = \{\langle a_1, b_1\rangle |$ there exist a_2, b_2 of $\mathfrak{B}$ such that

$\langle [a_1, a_2], [b_1, b_2]\rangle \in T\}$ and

$T_2 = \{\langle a_2, b_2\rangle |$ there exist a_1, b_1 of $\mathfrak{A}$ such that

$\langle [a_1, a_2], [b_1, b_2]\rangle \in T\}$.

By Theorem 14 in [1], $T_1 = \bigvee_A\{T_A(a_1, b_1) \mid \langle a, b\rangle \in T\}$ and $T_2 = \bigvee_B\{T_B(a_2, b_2) \mid \langle a, b\rangle \in T\}$. Then it follows from (2) that $T_1 \times T_2 = (\bigvee_A\{T_A(a_1, b_1) \mid \langle a, b\rangle \in T\}) \times (\bigvee_B\{T_B(a_2, b_2) \mid \langle a, b\rangle \in T\}) \subseteq \bigvee_{A\times B}\{T_A(a_1, b_1) \times T_B(a_2, b_2) \mid \langle a, b\rangle \in T\} \subseteq T$. The converse inclusion is evident. Because $T_1 \in LT(\mathfrak{A})$ and $T_2 \in LT(\mathfrak{B})$, (1) is proved.

Next we shall prove two lemmas, by means of which we can prove the second result from the introduction.

Lemma 1. *Let* $\mathfrak{A} = (A, F)$ *be an algebra and* $a, b \in A$. $\langle x, y\rangle \in T_A(a, b)$ *if and only if there exists a binary algebraic function* φ *over* $\mathfrak{A}$ *such that* $x = \varphi(a, b)$ *and* $y = \varphi(b, a)$.

Proof. Clearly the set of all pairs $\langle x, y\rangle$ for all binary algebraic functions φ from the theorem over $\mathfrak{A}$ constitute a reflexive and symmetric binary relation T having the Substitution Property and collapsing $\langle a, b\rangle$, i.e. $T_A(a, b) \subseteq T$. The converse inclusion is evident.

Lemma 2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two lattices. Then* $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T_{A\times B}(a, b)$ *for every pair* $\langle a, b\rangle$ *of elements of* $\mathfrak{A} \times \mathfrak{B}$.

Proof. Let $\langle x, y\rangle \in T_A(a_1, b_1) \times T_B(a_2, b_2)$. Then $\langle x_1, y_1\rangle \in T_A(a_1, b_1)$, $\langle x_2, y_2\rangle \in T_B(a_2, b_2)$ and, according to Lemma 1, there exist $(2 + n)$-ary and $(2 + m)$-ary polynomials p and q such that $x_1 = p(a_1, b_1, c_1, \ldots, c_n)$, $y_1 = p(b_1, a_1, c_1, \ldots, c_n)$, $x_2 = q(a_2, b_2, d_1, \ldots, d_m)$ and $y_2 = q(b_2, a_2, d_1, d_2, \ldots, d_m)$. Let $s = \max(m, n)$ and let us put $c_i = c_n$ and $d_j = d_m$ for $i = n, \ldots, s$ and $j = m, \ldots, s$. Now we can construct a $(4 + s)$-ary polynomial r as follows: $r(x, y, k_1, \ldots, k_s, e_1, e_2) = (e_1 \wedge p(x, y, k_1, \ldots, k_s)) \vee (e_2 \wedge q(x, y, k_1, \ldots, k_s))$. But then $p(x, y, c_1, \ldots, c_n) = r(x, y, c_1, \ldots, c_s, h, g)$ and $q(x, y, d_1, \ldots, d_m) = r(x, y, d_1, \ldots, d_s, g, h)$, where $h = x \vee y \vee c_1 \vee \ldots \vee c_n \vee d_1 \vee \ldots \vee d_m$ and $g = x \wedge y \wedge c_1 \wedge \ldots \wedge c_n \wedge d_1 \wedge \ldots \wedge d_m$. Further, $\langle x, y\rangle = \langle [x_1, x_2], [y_1, y_2]\rangle = \langle [r(a_1, b_1, c_1, \ldots, c_s, h, g), r(a_2, b_2, d_1, \ldots, d_s, g, h)], [r(b_1, a_1, c_1, \ldots, c_s, h, g), r(b_2, a_2, d_1, \ldots, d_s, g, h)]\rangle = \langle r(a, b, [c_1, d_1], \ldots, [c_s, d_s], [h, g], [g, h]), r(a, b, [c_1, d_1], \ldots, [c_s, d_s], [h, g], [g, h])\rangle = \langle \varphi(a, b), \varphi(b, a)\rangle$, where $\varphi(x, y) = r(x, y, [c_1, d_1], \ldots, [c_s, d_s], [h, g], [g, h])$. According to Lemma 1, $\langle x, y\rangle \in T_{A \times B}(a, b)$. This completes the proof.

Now we can prove

Theorem 5. *The class of all lattices has directly decomposable tolerances.*

Proof. By Theorem 3, the class from the theorem satisfies the identity (2), and thus Theorem 4 can be used. According to Lemma 2, 2) of Theorem 4 holds, whence the proof is a direct consequence of Theorem 4.

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