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# A NEW CHARACTERIZATION OF THE MAXIMUM GENUS OF A GRAPH

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**0.** By a graph we shall mean a pseudograph in the sense of [2] and [5] (multiple edges and loops are permitted). A graph $G$ is determined if and only if we know its vertex set $V(G)$, edge set $E(G)$, and its incidence relation between vertices and edges (note that both $V(G)$ and $E(G)$ are finite and $V(G)$ is nonempty). A graph is called trivial if it has exactly one vertex and its edge set is empty. Let $G$ be a graph; we denote by $C(G)$ the set of its components; moreover, we denote $c(G) = |C(G)|$; the integer $|E(G)| - |V(G)| + c(G)$ is referred to as the Betti number $\beta(G)$ of $G$; obviously, if $G$ is connected, then $c(G) = 1$ and $\beta(G) = |E(G)| - |V(G)| + 1$.

Let $G$ be a connected graph. The minimum integer $i$ such that there exists an embedding of $G$ into the orientable surface (i.e. compact orientable 2-manifold) $S_i$ of genus $i$ is called the genus $\gamma(G)$ of $G$. An embedding of $G$ into an orientable surface is called a 2-cell embedding if every region is topologically homeomorphic to the Euclidean plane (cf. [2], Section 5.2). Youngs [17] proved that every embedding of $G$ into $S_{\gamma(G)}$ is a 2-cell embedding. The maximum integer $j$ such that there exists a 2-cell embedding of $G$ into $S_j$ is called the *maximum genus* $\gamma_M(G)$ of $G$. In [10] it was shown that $\gamma_M(G) \leqq [\beta(G)/2]$, where $[\alpha]$ denotes the maximum integer $m$ with the property that $m \leqq \alpha$. Nordhaus, Ringeisen, Stewart, and White [9] proved that $\gamma_M(G) = 0$ if and only if no two cycles of $G$ have a vertex in common. $G$ is said to be *upper embeddable* if $\gamma_M(G) = [\beta(G)/2]$.

As was proved by Duke [3], for every connected graph $G$ and every integer $m$, $\gamma(G) \leqq m \leqq \gamma_M(G)$, there exists a 2-cell embedding of $G$ into $S_m$. The theory of 2-cell embeddings is a relatively separated but very fruitful branch of graph theory. The study of the genus of a graph has brought remarkably deep results (see [12] or survey [14]). Various results concerning the maximum genus of a graph are also very interesting (see [2], Section 5.3, [7], or survey [11]).

**1.** The maximum genus of a connected graph was determined by Homenko, Ostroverkhy, and Kusmenko [7] and by Xuong [16].

Let $G$ be a connected graph. We denote by $\mathbf{T}(G)$ the set of spanning trees of $G$.

If $T \in \mathbf{T}(G)$, then we denote

$$x_G(T) = |\{F \in C(G - E(T));\ |E(F)| \text{ is odd}\}| .$$

It is clear that $x_G(T) \equiv \beta(G) \pmod 2$, for every $T \in \mathbf{T}(G)$.

**Theorem A** ([16]). *If $G$ is a connected graph, then*

$$\gamma_M(G) = (\beta(G) - \min_{T \in \mathbf{T}(G)} x_G(T))/2 .$$

The formula for the maximum genus of a connected graph given in [7] reads differently but in substance both results are the same.

The following characterization of upper embeddable graphs was given by Jungerman [8], Xoung [16], and for the connected graphs of even Betti number also by Homenko [6]; see also Theorems 1 and 2 in [7].

**Theorem B.** *A connected graph $G$ is upper embeddable if and only if there exists $T \in \mathbf{T}(G)$ such that $x_G(T) \leqq 1$.*

Note that Theorem B is a special case of Theorem A.

**2.** In the present paper a new way of determining the maximum genus of a connected graph will be shown.

If $H$ is a graph, then we denote

$$B(H) = \{F \in C(H);\ \beta(F) \text{ is odd}\}$$

and $b(H) = |B(H)|$. If $G$ is a connected graph and $A \subseteq E(G)$, then we denote

$$y_G(A) = b(G - A) + c(G - A) - |A| - 1 .$$

**Proposition.** *Let $G$ be a connected graph, and let $A \subseteq E(G)$. Then*

(1) $$y_G(A) \equiv \beta(G) \pmod 2 .$$

Proof. We have

$$\beta(G) + y_G(A) =$$
$$= |E(G)| - |V(G)| + 1 + b(G - A) + c(G - A) - |A| - 1 =$$
$$= b(G - A) + \sum_{F \in C(G - A)} \beta(F) .$$

This means that $\beta(G) + y_G(A) \equiv 0 \pmod 2$, and thus (1) follows.

The following theorem is the main result of the present paper. It will be proved in the next sections.

**Theorem 1.** *Let $G$ be a connected graph. Then*

$$\min_{T \in \mathbf{T}(G)} x_G(T) = \max_{A \subseteq E(G)} y_G(A) .$$

Combining Theorems A and 1 we get an alternative formula for the maximum genus of a connected graph:

**Theorem 2.** *If G is a connected graph, then*

$$\gamma_M(G) = (\beta(G) - \max_{A \subseteq E(G)} y_G(A))/2 .$$

Therefore, we have two complementary tools for determining the maximum genus of a connected graph $G$. Consider an integer $i$ with the properties that $0 \leqq i \leqq \beta(G)$ and $i \equiv \beta(G) \pmod 2$. If we have found a spanning tree $T$ of $G$ such that $x_G(T) = i$, then we know that $\gamma_M(G) \geqq (\beta(G) - i)/2$. On the other hand, if we have found a subset $A$ of $E(G)$ such that $y_G(A) = i$, then we know that $\gamma_M(G) \leqq (\beta(G) - i)/2$.

Combining Theorems B and 1 we get an alternative characterization of upper embeddable graphs:

**Theorem 3.** *A connected graph G is upper embeddable if and only if*

$$b(G - A) + c(G - A) - 2 \leqq |A|\,, \quad \textit{for every} \quad A \subseteq E(G)\,.$$

Theorems B and 3 are complementary in the following sense: if we wish to show that a connected graph is or is not upper embeddable we can use Theorem B or Theorem 3, respectively.

**3.** For every connected graph $G$, we denote

$$x_G = \min_{T \in \mathbf{T}(G)} x_G(T) \quad \text{and} \quad y_G = \max_{A \subseteq E(G)} y_G(A)\,.$$

Theorem 1 asserts that (I) $x_G \geqq y_G$, for every connected graph $G$, and that (II) $x_G \leqq y_G$, for every connected graph $G$. Proving statement (I) is easier than proving statement (II). In this section we shall give two proofs of (I). Two proofs of (II) will be given in Sections 4 and 5.

First proof of (I). Let $G$ be a connected graph. Denote $m = |E(G)|$. If $m = 0$, then $G$ is trivial, and thus $x_G = y_G$. Let $m \geqq 1$. Assume that for every connected graph $G^*$ with the property that $|E(G^*)| < m$, it is proved that $x_{G^*} \geqq y_{G^*}$. We wish to prove that $x_G \geqq y_G$.

There exists $T \in \mathbf{T}(G)$ such that $x_G(T) = x_G$. If for every component $F$ of $G - E(T)$, $|E(F)| \leqq 1$, then no two cycles of $G$ have a vertex in common, and therefore, $x_G(T) = \beta(G) = y_G$. Let there exist a component $F$ of $G - E(T)$ such that $|E(F)| \geqq 2$. Then there exist adjacent edges $e_1$ and $e_2$ of $F$ such that $x_G(T) = x_{G-e_1-e_2}(T)$. Therefore, $x_G \geqq x_{G-e_1-e_2}$. According to the induction assumption, $x_{G-e_1-e_2} \geqq y_{G-e_1-e_2}$. We shall prove that $y_{G-e_1-e_2} \geqq y_G$. Consider $A \subseteq E(G)$ such that $y_G(A) = y_G$. Since $e_1$ and $e_2$ are adjacent edges of $G$, there exists at most one component $F_0$ of $G - A$ with the property that $E(F_0) \cap \{e_1, e_2\} \neq \emptyset$. It is not difficult to show that $y_{G-e_1-e_2}(A - \{e_1, e_2\}) \geqq y_G(A)$. Hence $y_{G-e_1-e_2} \geqq y_G$, which completes the proof of (I).

Second proof of (I). Let $G$ be a connected graph. Consider $A \subseteq E(G)$ such that $y_G(A) = y_G$. Let $\mathbf{T} \in T(G)$. We denote

$$B_{\text{con}} = \{F \in B(G - A);\ \text{the subgraph of } T \text{ induced by } V(F) \text{ is connected}\}.$$

According to the definition of $B(G - A)$, for each $F \in B(G - A)$, $\beta(F)$ is odd. This implies that for each $F \in B_{\text{con}}$, $|E(F) - E(T)|$ is odd. Therefore, it is not difficult to see that for at least $|B_{\text{con}}| - |A - E(T)|$ components $H$ of $G - E(T)$, $|E(H)|$ is odd. Hence, $x_G(T) \geqq |B_{\text{con}}| - |A - E(T)|$. It is clear that $c(T - A) \geqq c(G - A) + |B(G - A) - B_{\text{con}}|$. Since $T$ is connected, we have that $|E(T) \cap A| \geqq c(T - A) - 1$, and thus $0 \geqq |B(G - A) - B_{\text{con}}| + c(G - A) - 1 - |A \cap E(T)|$. We have that

$$x_G \geqq x_G(T) \geqq |B_{\text{con}}| - |A - E(T)| \geqq$$
$$\geqq b(G - A) + c(G - A) - 1 - |A| = y_G(A) = y_G.$$

**4.** Let $G$ be a connected graph. Since $y_G(\emptyset) = b(G)$, $y_G \geqq 0$. We denote by MAX$(G)$ the set of $A \subseteq E(G)$ such that $y_G(A) = y_G$ and for every $A' \subseteq E(G)$, if $y_G(A') = y_G$, then $A$ is not a proper subset of $A'$. It is clear that

(2) if $A \in$ MAX$(G)$ and $F$ is a component of $G - A$ such that $\beta(F)$ is even, then $F$ is trivial.

**Lemma 1.** *Let $G$ be a connected graph, $A \in$ MAX$(G)$, let $F$ be a component of $G - A$ such that $\beta(F)$ is odd, and let $e$ be an edge of $F$. Then $F - e$ is connected and $y_{F-e} = 0$.*

Proof. Since $A \cup \{e\} \notin$ MAX$(G)$, it follows that $F - e$ is connected. Consider an arbitrary $Z \subseteq E(F - e)$. We have that

$$b(G - (A \cup \{e\} \cup Z) = b(G - A) + b((F - e) - Z) - 1,$$

and

$$c(G - (A \cup \{e\} \cup Z) = c(G - A) + c((F - e) - Z) - 1.$$

This implies that

$$y_G(A \cup \{e\} \cup Z) = y_G(A) + y_{F-e}(Z) - 2.$$

Since $A \in$ MAX$(G)$, $y_G(A \cup \{e\} \cup Z) < y_G(A)$. Therefore, $y_{F-e}(Z) < 2$. It follows from (1) that $y_{F-e}(Z) \leqq 0$. Hence $y_{F-e} = 0$, which completes the proof of Lemma 1.

First proof of (II). Let $G$ be a connected graph. Denote $m = |E(G)|$. The case $m = 0$ is obvious. Let $m \geqq 1$. Assume that for every connected graph $G^*$ with the property that $E(G^*) < m$, it is proved that $x_{G^*} \leqq y_{G^*}$. We wish to prove that $x_G \leqq y_G$. We distinguish two cases.

Case 1. Assume that $G$ contains a bridge. Let $e_0$ be a bridge in $G$, and let $G_1$ and $G_2$ be the components of $G - e_0$. According to the induction assumption, $x_{G_1} \leqq y_{G_1}$

and $x_{G_2} \leqq y_{G_2}$. It is clear that $x_G = x_{G_1} + x_{G_2}$. Consider $A_1 \subseteq E(G_1)$ and $A_2 \subseteq E(G_2)$ such that $y_{G_1}(A_1) = y_{G_1}$ and $y_{G_2}(A_2) = y_{G_2}$. Clearly, $y_{G_1}(A_1) + y_{G_2}(A_2) = y_G(\{e_0\} \cup A_1 \cup A_2)$. Hence $y_{G_1} + y_{G_2} \leqq y_G$. This implies that $x_G \leqq y_G$.

Case 2. Assume that $G$ is bridgeless. If $G$ is a cycle, then $x_G = 1 = y_G$. We shall assume that $G$ is not a cycle. We distinguish two subcases.

Subcase 2.1. Assume that for every $A \in \mathrm{MAX}(G)$ and every component $F$ of $G - A$, $|E(F)| \leqq 1$.

We first assume that $G$ contains no loop. Consider an arbitrary $A' \in \mathrm{MAX}(G)$. We have that $b(G - A') = 0$. It follows from (2) that $c(G - A') = |V(G)|$ and $A' = E(G)$. Therefore, $y_G = |V(G)| - |E(G)| - 1 = -\beta(G)$. Since $y_G \geqq 0$, $\beta(G) = 0$, and thus $G$ is a tree. Since $m \geqq 1$, $G$ has a bridge, which is a contradiction.

We now assume that there exists a loop $e_1$ of $G$. We denote by $u$ the vertex of $G$ incident with $e_1$. Since $G$ is bridgeless and different from a cycle, we get that $m \geqq 2$ and for every edge $e$ adjacent to $e_1$ in $G$, $G - e_1 - e$ is connected. We shall show that there exists an edge $e_2$ which is adjacent to $e_1$ in $G$ and such that $y_{G-e_1-e_2} \leqq y_G$. Consider an arbitrary edge $e_0$ adjacent to $e_1$ in $G$. Assume that $y_G < y_{G-e_1-e_0}$. Since $\beta(G - e_1 - e_0) \equiv \beta(G) \pmod 2$, it follows from (1) that $y_G \leqq y_{G-e_1-e_0} - 2$. Consider $A_0 \subseteq E(G - e_1 - e_0)$ such that $y_{G-e_1-e_0}(A_0) = y_{G-e_1-e_0}$. It is clear that $y_G(A_0 \cup \{e_0, e_1\}) = y_{G-e_1-e_0}(A_0) - 2$, and thus $y_G(A_0 \cup \{e_0, e_1\}) = y_G$. This means that there exists $A \in \mathrm{MAX}(G)$ such that $e_1 \in A$. Let $F_2$ be the component of $G - A$ which contains $u$. According to the assumption in Subcase 2.1, $|E(F_2)| \leqq 1$. If $\beta(F_2)$ is even, then $y_G(A - \{e_1\}) > y_G$, which is a contradiction. Therefore, we assume that $\beta(F_2)$ is odd. This means that there exists a loop $e_2$ of $G$ such that $E(F_2) = \{e_2\}$. Since $e_1$ and $e_2$ are adjacent loops of $G$, it is clear that $y_{G-e_1-e_2} \leqq y_G$.

According to the induction hypothesis, $x_{G-e_1-e_2} \leqq y_{G-e_1-e_2}$. Consider $T \in \mathbf{T}(G - e_1 - e_2)$ such that $x_{G-e_1-e_2}(T) = x_{G-e_1-e_2}$. It is easy to see that $x_G(T) \leqq x_{G-e_1-e_2}(T)$, and thus $x_G \leqq y_G$.

Subcase 2.2. Assume that there exists $A \in \mathrm{MAX}(G)$ such that for at least one component $F'$ of $G - A$, $|E(F')| \geqq 2$. We denote $B = B(G - A)$ and $D = C(G - A) - B$. As follows from (2), $F' \in B$. We denote by $H$ the graph obtained from $G - (E(G) - A)$ in such a way that (i) for each $F \in B \cup D$, the vertices of $F$ are identified into one vertex, say $v_F$, and (ii) for each $F \in B$, one new loop, say $e_F$, incident with $v_F$ is added. Clearly, $V(H) = \{v_F; F \in B \cup D\}$. Denote $E_0 = \{e_F; F \in B\}$. Obviously, $E(H) = A \cup E_0$. If $Z_0 \subseteq E(H)$ and $e_0 \in E_0$, then it is easy to see that $y_H(Z_0 \cup \{e_0\}) \leqq y_H(Z_0)$. This implies that

$$y_H = \max_{Z \subseteq A} y_H(Z)\,. \tag{3}$$

Let $Z$ be an arbitrary subset of $A$. There exists a one-to-one mapping $h$ of $C(G - Z)$ onto $C(H - Z)$ with the property that for every $G_1 \in C(G - Z)$ and every $F \in C(G - A)$, if $V(F) \cap V(G_1) \neq \emptyset$, then $v_F$ belongs to $h(G_1)$. Hence $c(H - Z) =$

$= c(G - Z)$. Let $G' \in C(G - Z)$; compare $\beta(G')$ and $\beta(h(G'))$; obviously, for every $F \in B$, $|E(F)| - |V(F)| \equiv 0 \pmod 2$, and – according to (2) – for every $F \in D$, $|E(F)| - |V(F)| = -1$; it follows from the definition of the Betti number of a graph that $\beta(G') \equiv \beta(h(G')) \pmod 2$. This implies that $b(H - Z) = b(G - Z)$, and thus $y_H(Z) = y_G(Z)$. Combining the result of this observation with (3) and with the fact that $y_G(A) = y_G$, we get that

$$y_H = y_H(A) = y_G\,. \tag{4}$$

Consider an arbitrary $T \in \mathbf{T}(H)$ such that $x_H(T) = x_H$. Denote $A_1 = A - E(T)$. It is easy to see that $x_H(T) \geqq |E_0| - |A_1|$. Since $b(H - A) = |E_0|$, $c(H - A) = |V(H)|$, $|E(T)| = |V(H)| - 1$, and $y_G(A) = y_G$, we have that $y_G = |E_0| - |A_1|$. It follows from the assumption in Subcase 2.2 that $|E(H)| < m$. According to the induction hypothesis, $x_H \leqq y_H$. Since $y_H(T) = y_H$ and $y_G = |E_0| - |A_1| \leqq x_H(T)$, it follows from (4) that

$$x_H(T) = |E_0| - |A_1| = y_G\,. \tag{5}$$

It is not difficult to see that there exists a one-to-one mapping $\omega$ of $A_1$ onto a subset of $E_0$ such that for every $e \in A_1$, the edges $e$ and $\omega(e)$ are adjacent in $H$.

For every $F \in B$, we choose one of the edges of $F$, say $e(F)$, as follows: if there exists an edge $e \in A_1$ such that $\omega(e) = e_F$, then there exists an edge $e_0$ of $F$ such that the edges $e$ and $e_0$ are adjacent in $G$; in this case we put $e(F) = e_0$; otherwise, let $e(F)$ be an arbitrary edge of $F$. Let $F \in B$; it follows from Lemma 1 that $F - e(F)$ is connected and $y_{F-e(F)} = 0$; according to the induction hypothesis, $x_{F-e(F)} = 0$. For each $F \in B$, we consider $T_F \in \mathbf{T}(F - e(F))$ such that $x_{F-e(F)}(T_F) = 0$.

Let $T_G$ be the subgraph of $G$ induced by

$$E(T) \cup \bigcup_{F \in B} E(T_F)\,.$$

It follows from (2) that for each $F \in D$, $E(F) = \emptyset$. Clearly, $T_G \in \mathbf{T}(G)$. Since $x_{F-e(F)}(T_F) = 0$, for each $F \in B$, it follows from (5) that $x_G(T_G) \leqq |E_0| - |A_1| = x_H(T)$, and thus $x_G \leqq y_G$, which completes the proof of (II).

**5.** The proof of (II) given in Section 4 was purely graph-theoretical. In the present section we shall give an alternative proof of (II); that proof depends on a matroid theoretical theorem.

Let $E$ be a finite set. We denote by $\exp E$ the set of subsets of $E$. We shall say that an integer-valued function $f$ defined on $\exp E$ is a rank function of $E$ if for every $A$, $A^* \subseteq E$ it holds that

(i) $f(A) \geqq 0$,
(ii) $f(A) \leqq |A|$,
(iii) if $A^* \subseteq A$, then $f(A^*) \leqq f(A)$, and
(iv) $f(A \cup A^*) + f(A \cap A^*) \leqq f(A) + f(A^*)$.

Note that a finite set $E$ together with a rank function of $E$ form a matroid (cf. Theorem 30A in [15]).

We shall say that a set $E$ is partitioned into sets $E_1, \ldots, E_n$ ($n \geqq 1$) if $E_1 \cup \ldots \cup E_n = E$ and the sets $E_1, \ldots, E_n$ are mutually disjoint.

Theorem 1c in [4] may be reformulated as follows:

**Theorem C** (Edmonds and Fulkerson). *Let $E$ be a finite set, and let $f_1, \ldots, f_n$ ($n \geqq 1$) be rank functions of $E$. Assume that*

$$f_1(A) + \ldots + f_n(A) \geqq |A|, \quad \textit{for every} \quad A \subseteq E .$$

*Then $E$ can be partitioned into sets $E_1, \ldots, E_n$ such that*

$$f_1(E_1) + \ldots + f_n(E_n) = |E| .$$

Let $H$ be a graph, and let $W \subseteq V(H)$. We denote by $r_W(H)$ the number of components $F$ of $H$ with the properties that $F$ is a tree and $V(F) \subseteq W$.

Theorem C will be used in the proof of the following lemma:

**Lemma 2.** *Let $G$ be a connected graph, let $m \geqq 0$ be an integer, and let $W \subseteq V(G)$. Assume that*

$$(6) \qquad c(G - A) + r_W(G - A) - (m + 1) \leqq |A|, \quad \textit{for every} \quad A \subseteq E(G) .$$

*Then $E(G)$ can be partitioned into sets $E_1$ and $E_2$ such that $G - E_1 \in \mathbf{T}(G)$ and $r_W(G - E_2) \leqq m$.*

Proof. Let first $r_W(G) \neq 0$. Then $G$ is a tree and $W = V(G)$. If we put $A = E(G)$, then we can see from (6) that $m \geqq |V(G)|$. Clearly, $\emptyset$ and $E(G)$ are the desired sets $E_1$ and $E_2$, respectively.

Let now $r_W(G) = 0$. We denote by $f_1$ and $f_2$ the mappings of $\exp E(G)$ into the set of integers defined as follows:

$$f_1(A) = |A| - c(G - A) + 1 \quad \text{and} \quad f_2(A) = |A| - r_W(G - A)$$

for every $A \subseteq E(G)$. It is easy to see that both $f_1$ and $f_2$ fulfil (i), (ii), and (iii). It follows from a result of Tutte (see [13], p. 225) that $f_1$ fulfils (iv). It can be proved similarly that $f_2$ fulfils (iv); details of the proof will be left to the reader. Therefore, both $f_1$ and $f_2$ are rank functions of $E(G)$. It follows from (6) that $m + f_1(A) + f_2(A) \geqq |A|$ for every $A \subseteq E(G)$. We denote by $f_0$ the integer-valued function on $\exp E(G)$ defined as follows:

$$f_0(A) = \min(m, |A|) \quad \text{for every} \quad A \subseteq E(G) .$$

Since both $f_1$ and $f_2$ fulfil (i), we have that $f_1(A) + f_2(A) \geqq 0$, and therefore, $|A| + f_1(A) + f_2(A) \geqq |A|$ for every $A \subseteq E(G)$. This implies that

$$f_0(A) + f_1(A) + f_2(A) \geqq |A| \quad \text{for every} \quad A \subseteq E(G) .$$

Since $f_0$ is a rank function of $E(G)$, it follows from Theorem C that $E(G)$ can be partitioned into sets $\tilde{E}_0$, $\tilde{E}_1$ and $E'_2$ such that $f_0(\tilde{E}_0) + f_1(\tilde{E}_1) + f_2(E'_2) = |E(G)|$. We put $E'_1 = \tilde{E}_0 \cup \tilde{E}_1$. Since $f_1(E'_1) \geqq f_1(\tilde{E}_1)$ and $f_0(\tilde{E}_0) \leqq m$, we have that $f_1(E'_1) + f_2(E'_2) \geqq |E(G)| - m$. Since $f_1(E'_1) = |E'_1| - c(G - E'_1) + 1$ and $f_2(E'_2) = |E'_2| - r_W(G - E'_2)$, we have that

$$c(G - E'_1) - 1 + r_W(G - E'_2) \leqq m .$$

This means that $E(G)$ can be partitioned into sets $E''_1$ and $E''_2$ with the properties that $E'_1 \subseteq E''_1$, $c(G - E''_1) = c(G - E'_1)$, and that every component of $G - E''_1$ is a tree. Since $E''_2 \subseteq E'_2$, $r_W(G - E''_2) \leqq r_W(G - E'_2)$. Therefore, $c(G - E''_1) - 1 + r_W(G - E''_2) \leqq m$. Since $G$ is connected, there exists $Q \subseteq E''_1$ such that $|Q| = c(G - E''_1) - 1$ and $c(G - (E''_1 - Q)) = 1$. It follows from (iii) that $r_W(G - (E''_2 \cup Q)) = |E''_2 \cup Q| - f_2(E''_2 \cup Q) \leqq |E''_2| - f_2(E''_2) + |Q| = r_W(G - E''_2) + |Q| = r_W(G - E''_2) + c(G - E''_1) - 1 \leqq m$. If we put $E_1 = E''_1 - Q$ and $E_2 = E''_2 \cup Q$, we get the desired result, and thus the lemma is proved.

If $H$ is a graph and $W \subseteq V(H)$, then we denote by $s_W(H)$ the number of trivial components $F$ of $H$ with the property that the only vertex of $F$ belongs to $W$.

We shall prove one more lemma:

**Lemma 3.** *Let $G$ be a connected graph, let $m \geqq 0$ be an integer, and let $W \subseteq V(G)$. Assume that*

$$(7) \qquad c(G - A) + s_W(G - A) - (m + 1) \leqq |A| , \quad \textit{for every} \quad A \subseteq E(G) .$$

*Then $E(G)$ can be partitioned into sets $E_1$ and $E_2$ such that $G - E_1$ is a tree and $r_W(G - E_2) \leqq m$.*

Proof. We wish to prove that (6) holds. On the contrary, we assume that there exists $A' \subseteq E(G)$ such that

$$(8) \qquad c(G - A') + r_W(G - A') - (m + 1) > |A'| .$$

As follows from (7) and (8), $s_W(G - A') < r_W(G - A')$. Denote

$$R = \{F \in C(G - A');\ F \text{ is a nontrivial tree and } V(F) \subseteq W\} .$$

Since $s_W(G - A') < r_W(G - A')$, $R \neq \emptyset$. Denote

$$E' = \bigcup_{F \in R} E(F) .$$

It is clear that $s_W(G - (A' \cup E')) = r_W(G - (A' \cup E')) = r_W(G - A') + |E'|$. Moreover, $c(G - (A' \cup E')) = c(G - A') + |E'|$. It follows from (8) that

$$c(G - (A' \cup E')) + s_W(G - (A' \cup E')) - (m + 1) > |A'| + |E'| ,$$

which is a contradiction. This means that (6) holds. The desired result follows from Lemma 2.

We are now prepared to give the second proof of (II).

Second proof of (II). Let $G$ be a connected graph. We wish to prove that $x_G \leqq y_G$; the structure of this proof (together with Lemma 1 and with the used parts of the first proof of (II) is derived partially from the structure of Anderson's proof [1] of Tutte's theorem on the existence of a 1-factor.

If $G$ is a trivial graph, then $x_G = 0 = y_G$. Let now $G$ be non trivial. Assume that for every connected graph $G$ with $|E(G^*)| < |E(G)|$, it is proved that $x_G \leqq y_G$. Consider $A \in \mathrm{MAX}(G)$. Let $B$, $D$, $H$, and $E_0$ be defined in the same way as in the first proof of (II). We denote $W = \{v_F;\ F \in B\}$ and $J = H - E_0$. Then $W \subseteq V(J)$ and $A = E(J)$. Obviously, $|W| = |E_0| = |B|$.

We now make the following observation. Consider an arbitrary $T' \in \mathbf{T}(J)$. It follows from the definitions of $x_H(T')$ and $r_W(J - E(T'))$ that $x_H(T') \geqq r_W(J - E(T')) \geqq |W| - |E(J - E(T'))|$. If $x_H(T') > r_W(J - E(T'))$, then it is not difficult to see that $|E(J - E(T')| + r_W(J - E(T')) > |W|$, and therefore, $r_W(J - E(T')) > |W| - |E(J - E(T'))|$. This implies that if $r_W(J - E(T')) \leqq |W| - |E(J - E(T'))|$, then $x_H(T') = |W| - |E(J - E(T'))|$.

Let $Q$ be an arbitrary subset of $A$; clearly, $c(J - Q) = c(G - Q)$; moreover, we have that $s_W(J - Q) \leqq b(G - Q)$; since $y_G(Q) \leqq y_G$, $c(J - Q) + s_W(J - Q) - (y_G + 1) \leqq |Q|$. It follows from Lemma 3 that $A$ can be partitioned into sets $A_1$ and $A_2$ such that $J - A_1$ is a tree and $r_W(J - A_2) \leqq y_G$. Denote $T = J - A_1$. Since $E(J) = A$, $T \in \mathbf{T}(J)$, and $|A| = |V(J)| - 1 + |W| - y_G$, we get that $|W| - |A_1| = y_G$. Since $r_W(J - E(T)) \leqq y_G = |W| - |A_1|$, it follows from our observation that (5) holds.

Let $E_0 = \emptyset$. Since $|A_1| = |E_0| - y_G$, $A = E(T)$. It follows from (2) that $G$ is a tree, and thus $x_G = 0 = y_G$.

Let $E_0 \neq \emptyset$. Then the final part of the proof is identical with that of the first proof of (II).

Remark. In the present paper the proofs of (I) and (II) are not arranged chronologically. In fact, the second proofs were found before the first ones.

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