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A NOTE ON DIMENSION OF P_3^n

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The dimension of a (symmetric loopless) graph G is the minimal n for which G is an induced subgraph of the product of n complete graphs. The dimension of G is denoted by dim G. (Let us note that every graph G is embeddable into the product of sufficiently many complete graphs, and hence the number dim G is well defined.) The notion of the dimension was introduced and studied in several papers ([1], [2], [4]). For information about results see [3].

We will use the following notation. The product $G \times H$ of two graphs G and H is defined by

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{((a_1, a_2), (b_1, b_2));\ (a_1, b_1) \in E(G),\ (a_2, b_2) \in E(H)\}.$$

The n-th power G^n of a graph G is the product $G \times G \times \ldots \times G$ of n copies of G. The sum of n copies of a graph G is denoted by nG.

The three-path P_3 is defined by

$$V(P_3) = \{0, 1, 2, 3\}, \quad E(P_3) = \{(0, 1), (1, 2), (2, 3)\}.$$

The complete graph with the vertex set $n = \{0, 1, \ldots, n - 1\}$ is denoted by K_n. In this note we prove the following theorems.

Theorem 1. *For any integer n,* $\dim P_3^n = 2n$.

Theorem 2. *For any connected component F_n of P_3^n we have* $\dim F_n = n + 1$.

For a bipartite graph G, the number bid G is the minimal n for which G is an induced subgraph of P_3^n. The number bid G was introduced in [5]. Theorems 1 and 2 imply

Corollary.

$$\dim G \leqq 1 + \operatorname{bid} G \quad \textit{for any connected bipartite graph } G,$$

$$\dim G \leqq 2 \,.\, \operatorname{bid} G \quad \textit{for any bipartite graph } G,$$

and these bounds are the best possible.

We will use the following facts to prove the above theorems.

Proposition 1 (for the proof see [2]).

$$\dim pK_2 = 1 + \{\log_2 p\}$$

(*where the symbol* { } *means the upper integral approximation*).

Proposition 2. *For each n the graph* P_3^n *consists of* 2^{n-1} *isomorphic components; any of them is denoted by* F_n.

Proposition 3. *The maximal integer p such that* pK_2 *is an induced subgraph of* F_n *is exactly*

$$\binom{n}{\left[\frac{n}{2}\right]}.$$

The proofs of Propositions 2 and 3 are given in [5].

From the Stirling formula or by easy induction one obtains the following

Proposition 4.

$$\binom{n}{\left[\frac{n}{2}\right]} \geqq \frac{2^n}{n} \quad \text{for} \quad n \geqq 2\,.$$

Proof of Theorem 1. Since $P_3 \leqq K_2 \times K_3$, i.e. P_3 is an induced subgraph of $K_2 \times K_3$, we obtain

$$P_3^n \leqq (K_2 \times K_3)^n$$

and hence

$$\dim P_3^n \leqq 2n\,.$$

In order to prove the converse inequality we show that

$$\lim_{n\to\infty} \frac{\dim P_3^n}{2n} = 1\,. \tag{1}$$

By Propositions 2 and 3

$$\binom{n}{\left[\frac{n}{2}\right]} 2^{n-1} K_2 \leqq P_3^n\,,$$

hence by Proposition 4

$$\frac{2^{2n-1}}{n} K_2 \leqq P_3^n$$

and by Proposition 1

$$2n \geqq \dim P_3^n \geqq 2n - \log_2 n .$$

Thus, (1) is proved.

Now, suppose that there exists k such that

$$\dim P_3^k \leqq 2k - 1 .$$

Then

$$\lim_{n \to \infty} \frac{\dim (P_3^k)^n}{2kn} \leqq \lim_{n \to \infty} \frac{n(2k - 1)}{2kn} < 1 ,$$

which contradicts (1).

Proof of Theorem 2. Let us consider the component F_n of P_3^n containing the vertex $(0, 0, \ldots, 0)$. This component F_n is the following graph:

$$V(F_n) = V_0 \cup V_1 , \quad \text{where} \quad V_0 = \{(a_1, a_2, .., a_n);\ \forall_i\ a_i \in \{0, 2\}\} ,$$

$$V_1 = \{(a_1, a_2, \ldots, a_n);\ \forall_i\ a_i \in \{1, 3\}\} ,$$

$$((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) \in E(F_n) \quad \text{iff} \quad |a_i - b_i| = 1 \quad \text{for all } i .$$

Let us define the system of homomorphisms $\varphi_i : F_n \to K_3$, $i = 1, 2, \ldots, n$, by putting

$$\varphi_i(a_1, a_2, \ldots, a_n) = \begin{cases} 0 & \text{for} \quad a_i = 0, 3 , \\ 1 & \text{for} \quad a_i = 1 , \\ 2 & \text{for} \quad a_i = 2 , \end{cases}$$

and homomorphism $\psi : F_n \to K_2$ (a 2-coloring)

$$\psi(a) = \begin{cases} 0 & \text{for} \quad a \in V_0 , \\ 1 & \text{for} \quad a \in V_1 . \end{cases}$$

One can easily check that the product of homomorphisms

$$\psi \times \varphi_1 \times \varphi_2 \times \ldots \times \varphi_n$$

gives an embedding of F_n into $K_2 \times K_3^n$, and hence

$$\dim F_n \leqq n + 1 .$$

Now, suppose that there exists k such that

$$\dim F_k \leqq k ,$$

i.e. there is an embedding $\varphi : F_k \to K_r^k$ for some r. By Proposition 2

$$P_3^k \cong 2^{k-1} F_k \cong \sum(F_k^{(A)}; A \subset \{1, 2, \ldots, k-1\}), \tag{2}$$

where $F_k^{(A)}$ denotes the A-th copy of F_k. Let us define a system of homomorphisms ψ_i, $i = 1, 2, \ldots, k-1$,

$$\psi_i : \sum F_k^{(A)} \to K_2$$

by putting

$$\psi_i(x^{(A)}) = \begin{cases} 0 & \text{for} \quad x \in V_0\,, \quad i \in A \quad \text{or} \quad x \in V_1, \quad i \notin A\,, \\ 1 & \text{for} \quad x \in V_0\,, \quad i \notin A \quad \text{or} \quad x \in V_1\,, \quad i \in A\,, \end{cases}$$

and a homomorphism $\psi_k : \sum F_k^{(A)} \to K_r^k$ by putting

$$\psi_k(x^{(A)}) = \varphi(x)\,.$$

One can easily check that

$$\psi_1 \times \psi_2 \times \ldots \times \psi_{k-1} \times \psi_k : \sum F_k^{(A)} \to K_2^{k-1} \times K_r^k$$

is an embedding. Hence by (2)

$$\dim P_3^k \leqq 2k - 1\,,$$

which contradicts Theorem 1.

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