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CHARACTERIZATIONS OF CERTAIN CLASSES OF POSETS HAVING GS-LATTICES OF A RELATIVELY SMALL SIZE

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0. INTRODUCTION

A set of initial segments (of o-ideals) in a poset P is said to be a *generating system* on P if it is closed under arbitrary nonempty intersections and contains P as well as all principal initial segments in P. The complete lattice $\mathrm{Gs}(P)$ of all generating systems on P, ordered by inclusion, is called a *gs-lattice* on P. According to Funayama, [2], the generating systems on P are, up to isomorphisms, exactly all the σ-dense completions of P.

The problem of constructing a σ-dense completion of a given poset with prescribed properties appears in various branches of mathematics. A solution of a problem of this kind was used by D. Scott and R. Solovay, [3], for a general development of the method of forcing in the set-theory and by the author, [5], for a description of connections between certain properties of elements of alphabets of formal languages. Hence it may be useful to get some more information concerning the structure of $\mathrm{Gs}(P)$ on a given poset P and the relations between P and the gs-lattice on P.

In the paper [6] a certain special class $\mathscr{P}_S$ of all the so called simple posets was studied. For an arbitrary $P \in \mathscr{P}_S$ an easy construction of $\mathrm{Gs}(P)$ was found showing that the cardinal number $|\mathrm{Gs}(P)|$ is proportional to $2^{|P|}$. The class $\mathscr{P}_C$ of all posets with complemented gs-lattices was proved to be a subclass of $\mathscr{P}_S$ and a superclass of the class $\mathscr{P}_T$ of all posets with a one-element gs-lattice.

In this work we denote by $\mathscr{M}_S$, $\mathscr{M}_C$, $\mathscr{M}_T$ the classes of all posets P from $\mathscr{P}_S$, $\mathscr{P}_C$, $\mathscr{P}_T$, respectively, such that every ordinally indecomposable subposet of P satisfies the Ascending Chain Condition. An internal description of $\mathscr{M}_S$, $\mathscr{M}_C$, $\mathscr{M}_T$ is given and the classes $\mathscr{M}_S$, $\mathscr{M}_C$ are also characterized by introducing a finite list of "forbidden" subposets. The last theorem says that none of these two methods can be used for a characterization of any of the classes $\mathscr{P}_S$, $\mathscr{P}_C$, $\mathscr{P}_T$.

Let P be a poset. We denote

$$\omega_P a = \{b \in P;\ b \leqq a\}\,,\quad \omega_P^- a = \omega_P a - \{a\}\,,\quad \bar{\varepsilon}_P a = \{b \in P;\ a \nleqq b\}$$

for an arbitrary $a \in P$. If A is a nonempty subset of P then we put $\alpha_P[A] = \{\alpha_P a;\ a \in A\}$ for $\alpha = \omega, \omega^-, \bar{\varepsilon}$ and define $\omega_P A = \bigcup \omega_P[A]$. We reserve the symbol $\mathfrak{O}_P$ for the set of all initial segments in P and $\mathfrak{N}_P$ for the least element of Gs(P). The generating system $\mathfrak{N}_P$ is called a *normal* or a *MacNeille completion* of P. Clearly,

$$\mathfrak{N}_P = \{P\} \cup \{\bigcap \omega_P[X];\ \emptyset \subset X \subseteq P\}\,.$$

In the whole paper we fix P for the notation of a poset and write α instead of α_P for $\alpha = \omega,\ \omega^-, \bar{\varepsilon}, \mathfrak{O}, \mathfrak{N}$.

1.1. Lemma. *The following assertions* (i), (ii) *hold for an arbitrary P.*

(i) *If $A \in \mathfrak{N}$ and $a \in P - A$ then there is an upper bound b of A such that $a \nleqq b$.*

(ii) *If $A \in \omega^-[P]$ and $a \in P - A$ is not an upper bound of A then there is an upper bound b of A satisfying $a \nleqq b$.*

Proof. The assertion (i) is true trivially. If $A \in \omega^-[P]$ then there is $b \in P$ such that $A = \omega^- b$. For an arbitrary $a \in P - A$ it holds that $a \nless b$. If, moreover, a is not an upper bound of A then $a \neq b$ and we have $a \nleqq b$.

In accordance with Theorems 3.19, 4.3, 4.6 from [6] we define

$$\begin{Bmatrix} P \in \mathscr{P}_S \\ P \in \mathscr{P}_C \\ P \in \mathscr{P}_T \end{Bmatrix} \quad \text{if} \quad \begin{Bmatrix} \mathfrak{O} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \cup \omega^-[P] \\ \mathfrak{O} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \\ \mathfrak{O} \subseteq \mathfrak{N} \end{Bmatrix}.$$

For definitions of some further concepts and symbols which we shall use here without defining them the reader is refered to [6].

If $\mathscr{P}$ is an arbitrary class of posets then we denote by $\mathbf{O}\mathscr{P}$ the least superclass of $\mathscr{P}$ closed under the formation of isomorphic images and ordinal sums. One can easily see that $\mathbf{O}\mathscr{P}$ is exactly the class of all P for which there are a chain I and a set $\{P_i;\ i \in I\} \subseteq \mathscr{P}$ with the property $P \cong \sum_{i \in I} P_i$.

The least infinite ordinal number will be denoted by ω_0. Hence ω_0 is the set $\{0, 1, \ldots\}$ ordered in the natural way.

If μ is an arbitrary ordinal number then we say that $(a_\gamma)_{\gamma < \mu}$ is an *ascending, non-descending, descending* chain in P whenever $\{a_\gamma;\ \gamma < \mu\} \subseteq P$ and $a_\gamma < a_\delta$, $a_\gamma \leqq a_\delta$, $a_\gamma > a_\delta$, respectively, for all $\gamma < \delta < \mu$. An ascending (nondescending, descending) chain $(a_\gamma)_{\gamma < \mu}$ is said to be finite whenever $\mu < \omega_0$. In this case we can write $(a_0, a_1, \ldots \ldots, a_{\mu - 1})$ instead of $(a_\gamma)_{\gamma < \mu}$.

We say that P satisfies the *Ascending Chain Condition* (the ACC) if every ascending chain in P is finite. It is well known that if P satisfies the ACC then for each $Q \in \mathfrak{O}$ there is an antichain A in P such that $Q = \omega A$. At the same time, the following is true. Whenever P is a chain then P satisfies the ACC iff $P = \{a_\gamma; \gamma < \mu\}$ where $(a_\gamma)_{\gamma<\mu}$ is a descending chain.

A nonmepty poset P is said to be *ordinally indecomposable* if $P = Q + R \Rightarrow \Rightarrow Q = \emptyset$ or $R = \emptyset$. Clearly, if P is ordinally indecomposable and there are at least two different elements in P then for each $a \in P$ we can find $b \in P$ fulfilling $a \parallel b$ (a is incomparable with b).

1.2. Lemma. *If $P = Q + R$ and $Q \neq \emptyset$ then each minimal element of P is in Q.*

1.3. Lemma. *Every P can be represented in the form $P \cong \sum_{i\in I} P_i$ where I is a chain and P_i is an ordinally indecomposable poset for each $i \in I$. This representation is unique in the following sense: If $P \cong \sum_{j\in J} Q_j$ is another representation of P with the same properties then there is an isomorphism f of I onto J such that $P_i \cong Q_{f(i)}$ for every $i \in I$.*

Proof. This is a consequence of Theorem 2.12, [4].

1.4. Definition. We denote by $\mathscr{M}$ the class of all P such that each ordinally indecomposable subposet of P satisfies the ACC. We put $\mathscr{M}_X = \mathscr{M} \cap \mathscr{P}_X$ for $X = S, C, T$.

1.5. Lemma. *If $P = \sum_{i\in I} P_i$ then the following assertions* (i)–(iv) *are true.*

(i) $P \in \mathscr{M} \Leftrightarrow P_j \in \mathscr{M}$ *for all* $j \in I$.

(ii) $P \in \mathscr{P}_S \Leftrightarrow P_j \in \mathscr{P}_S$ *for all* $j \in I$.

(iii) $P \in \mathscr{P}_C \Leftrightarrow P_j \in \mathscr{P}_C$ *for all* $j \in I$.

(iv) $P \in \mathscr{P}_T \Leftrightarrow$ *the assertions* (a), (b) *hold for all* $j \in I$.

(a) $\mathfrak{O}_{P_j} \subseteq \mathfrak{N}_{P_j} \cup \{\emptyset\}$.

(b) *P_j has a least element $\Rightarrow$ there is $k \in I$ such that $k \prec j$ and P_k has a greatest element.*

Proof. (1) The statement (i) is true trivially.

(2) Assume $P \in \mathscr{P}_S$ and choose $j \in I$, $A_j \in \mathfrak{O}_{P_j}$ arbitrarily. In the case $A_j = P_j$ we have $A_j \in \mathfrak{N}_{P_j}$. If $\emptyset \subset A_j \subset P_j$ then we put $A = \sum_{i<j} P_i + A_j$. It is clear that $A \in \mathfrak{O} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \cup \omega^-[P]$ and $A_j = P_j \cap A$. By this and 2.5 [6] we obtain $A_j \in \mathfrak{N}_{P_j} \cup \bar{\varepsilon}_{P_j}[P_j] \cup \omega^-_{P_j}[P_j]$. If $A_j = \emptyset$ then $A_j \in \mathfrak{N}_{P_j} \cup \bar{\varepsilon}_{P_j}[P_j] \cup \omega^-_{P_j}[P_j]$ according to 2.7 [6]. Hence $\mathfrak{O}_{P_j} \subseteq \mathfrak{N}_{P_j} \cup \bar{\varepsilon}_{P_j}[P_j] \cup \omega^-_{P_j}[P_j]$ and $P_j \in \mathscr{P}_S$ for all $j \in I$.

Conversely, $P_j \in \mathscr{P}_S$ for all $j \in I \Rightarrow \mathfrak{D}_{P_j} \subseteq \mathfrak{N}_{P_j} \cup \bar{\varepsilon}_{P_j}[P_j] \cup \omega_{P_j}^-[P_j]$ for all $j \in I \Rightarrow \Rightarrow \mathfrak{D} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \cup \omega^-[P]$ by 2.5, 2.7 [6] $\Rightarrow P \in \mathscr{P}_S$.

(3) $P \in \mathscr{P}_C \Leftrightarrow P_j \in \mathscr{P}_C$ for all $j \in I$ can be proved by the method from (2).

(4) Suppose $P \in \mathscr{P}_T$ and take $j \in I$ arbitrarily.

Let $A_j \in \mathfrak{D}_{P_j}$. If $A_j = P_j$ then $A_j \in \mathfrak{N}_{P_j}$. In the case $\emptyset \subset A_j \subset P_j$ denote $A = \sum_{i<j} P_i + A_j$. Then $A_j = P_j \cap A$, $A \in \mathfrak{D} \subseteq \mathfrak{N}$ and it follows that $A_j \in \mathfrak{N}_{P_j}$ according to 2.5 (i) [6]; this proves (a).

If P_j has a least element o then $\omega^- o \in \mathfrak{D} \subseteq \mathfrak{N}$. By this and 2.6 [6] it follows that $\omega^- o$ has a greatest element i. Thus there is $k \prec j$ such that i is a greatest element in P_k and we have proved (b).

Assume that the conditions (a), (b) hold for all $j \in I$ and choose $A \in \mathfrak{D}$ arbitrarily.

If there is $j \in I$ satisfying $\emptyset \subset A_j \subset P_j$ for $A_j = P_j \cap A$ then $\emptyset \subset A_j \in \mathfrak{D}_{P_j}$ and we have $A_j \in \mathfrak{N}_{P_j}$ by (a). This and 2.5 (i) [6] imply $A \in \mathfrak{N}$. In the case $P_i \cap A \in \{\emptyset, P_i\}$ for all $i \in I$ suppose that P_j has a least element o and $A = \omega^- o$ for some $j \in I$. Then, by (b), there is $k \in I$ such that $k \prec j$ and P_k has a greatest element i. It is obvious that i is a greatest element in A. Hence $A \in \mathfrak{N}$ according to 2.6 [6].

1.6. Lemma. *If Q is a final segment in P then the following assertions* (i), (ii), (iii) *hold.*

(i) *P satisfies the* ACC $\Rightarrow$ *Q satisfies the* ACC.

(ii) $P \in \mathscr{P}_S \Rightarrow Q \in \mathscr{P}_S$.

(iii) $P \in \mathscr{P}_C \Rightarrow Q \in \mathscr{P}_C$.

Proof. (1) The statement (i) is true trivially.

(2) Suppose $P \in \mathscr{P}_S$ and take $A \in \mathfrak{D}_Q$ arbitrarily. If $A = \emptyset$ then $A \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ by 2.7 [6]. In the case $A \neq \emptyset$ put $B = (P - Q) \cup A$. Then, obviously, $B \in \mathfrak{D} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \cup \omega^-[P]$ and we get $A \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ according to 2.1 [6]. Hence $\mathfrak{D}_Q \subseteq \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$, which proves $Q \in \mathscr{P}_S$ and also (ii).

(3) The assertion (iii) can be verified in the same way as (ii).

1.7. Definition. For an arbitrary P and m, $0 < m < \omega_0$, we denote by ${}^mP$ the set of all m-element antichains in P ordered in the following way: If $A, B \in {}^mP$ then $A \leqq B$ whenever for each $a \in A$ there is $b \in B$ with the property $a \leqq b$.

If there is m satisfying $0 < m < \omega_0$ and ${}^mP = \emptyset$ then we put

$$bP = \begin{cases} \text{the greatest } m \text{ such that } {}^mP \neq \emptyset \text{ in the case } P \neq \emptyset, \\ 0 \text{ otherwise}. \end{cases}$$

The number bP is called the *breadth* of P.

$bP = m \Rightarrow P$ is a union of m chains by Theorem 1.1 [1]. It is obvious that, conversely, if P is a union of m chains then $bP \leqq m$.

1.8. Lemma. *If P satisfies the* ACC *then ${}^mP$ satisfies the* ACC *for all m, $0 < m < \omega_0$.*

Proof. Let P satisfy the ACC. We prove the assertion "${}^mP$ satisfies the ACC for all m, $0 < m < \omega_0$" by induction.

(1) Assume that there is an ascending chain $(A_i)_{i<\omega_0}$ in ${}^1P$. If we denote $A_i = \{a_i\}$ each $i < \omega_0$ then $(a_i)_{i<\omega_0}$ is an ascending chain in P.

(2) Let us take n, $1 \leqq n < \omega_0$, arbitrarily and let the following implication hold: P satisfies the ACC $\Rightarrow {}^nP$ satisfies the ACC.

Suppose that $(A_i)_{i<\omega_0}$ is an ascending chain in ${}^{n+1}P$. Then we can find a nondescending chain $(a_i)_{i<\omega_0}$ in P such that $a_i \in A_i$ for all $i < \omega_0$. If for every $i < \omega_0$ there exists $j < \omega_0$ with $i < j$, $a_i < a_j$ then it is possible to select an infinite ascending chain from $(a_i)_{i<\omega_0}$. Otherwise there is $i_0 < \omega_0$ with the property $a_i = a_{i_0}$ for all i, $i_0 < i < \omega_0$. Now it can be easily seen that $(B_i)_{i<\omega_0}$, where $B_i = A_{i_0+i} - \{a_{i_0}\}$ for each $i < \omega_0$, is an ascending chain in ${}^nP$.

Each of these two conclusions implies that P does not satisfy the ACC.

2. CHARACTERIZATIONS OF THE CLASS $\mathscr{M}_S$

We first consider an important subset of $\mathscr{M}_S$.

2.1. Definition. We say that a nondescending chain $(\alpha_\gamma)_{\gamma<\mu+1}$ of ordinal numbers is a *description* whenever (i) or (ii) or (iii) is true:

(i) $\mu = 1$, $\alpha_0 = 0$, $\alpha_1 = 1$.

(ii) $1 < \mu < \omega_0$, $\alpha_0 = 0 = \alpha_1$ and $\alpha_\gamma < \alpha_\mu \leqq \omega_0$ for all $\gamma < \mu$.

(iii) $\mu = \omega_0$, $\alpha_0 = 0 = \alpha_1$ and α_μ is the least ordinal number fulfilling $\alpha_\gamma < \alpha_\mu \leqq \leqq \omega_0$ for all $\gamma < \mu$.

A description $(\alpha_\gamma)_{\gamma<\mu+1}$ is said to be *finite* whenever $\mu < \omega_0$ and $\alpha_\mu < \omega_0$.

2.2. Definition. Let $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ be a description. We put $\nu = \alpha_\mu$, $L_\pi = \{\ell_\gamma;\ \gamma < \mu\}$, $R_\pi = \{r_\delta;\ \delta < \nu\}$ and $P_2\pi = L_\pi \cup R_\pi$. We define an ordering on $P_2\pi$ in such a way that $(\ell_\gamma)_{\gamma<\mu}$ and $(r_\delta)_{\delta<\nu}$ are descending chains in $P_2\pi$ and that

$$\begin{Bmatrix} \ell_\gamma \leqq r_\delta \\ r_\delta \leqq \ell_\gamma \end{Bmatrix} \text{ iff } \begin{Bmatrix} \delta < \alpha_\gamma \\ \alpha_{\gamma+1} < \delta \end{Bmatrix} \quad \text{for all} \quad \gamma < \mu\,, \quad \delta < \nu\,.$$

If the description π is finite then we put $P_3\pi = P_2\pi \cup \{s\}$ and define an ordering on $P_3\pi$ in the following way. $P_3\pi$ is an extension of $P_2\pi$ and $s \parallel x$ for all $x \in \in \{\ell_{\mu-1}, r_{\nu-1}\}$, $s < x$ for all $x \in P_2\pi - \{\ell_{\mu-1}, r_{\nu-1}\}$.

Whenever $\alpha_\mu = 1$ in a description π, a finite description π, then one can write P_2^μ, P_3^μ instead of $P_2\pi$, $P_3\pi$, respectively.

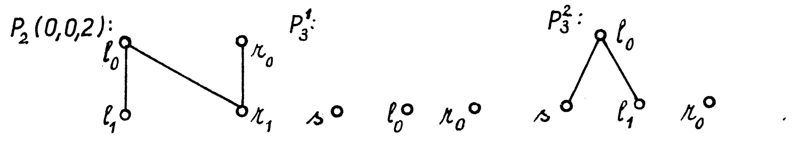

Figure 1

2.3. Definition. We put

$$\Gamma_S = \{1\} \cup \{P_2\pi;\ \pi \text{ is a description}\} \cup \{P_3\pi;\ \pi \text{ is a finite description}\}\ .$$

2.4. Lemma. *Every poset Q from Γ_S satisfies the* ACC *and is ordinally indecomposable.*

Proof. (a) If $Q = 1$ then both these statements are true obviously.

(b) Let there exist a description π such that $Q = P_2\pi$ and denote $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$, $\nu = \alpha_\mu$. Using $Q = L_\pi \cup R_\pi$ and the fact that L_π, R_π satisfy the ACC, we can easily show the validity of the ACC for Q.

Q is ordinally indecomposable:

(1) For each $\gamma < \mu$ there is $\delta < \nu$ such that $\ell_\gamma \parallel r_\delta$: It is sufficient to put $\delta = \alpha_\gamma$.

(2) For each $\delta < \nu$ there is $\gamma < \mu$ such that $r_\delta \parallel \ell_\gamma$: Take $\delta < \nu$ arbitrarily and set $G_\delta = \{\gamma;\ \gamma < \mu,\ \delta \leqq \alpha_\gamma\}$. In the case $G_\delta = \emptyset$ we have $\alpha_\gamma < \delta$ for all $\gamma < \mu$. If we suppose $\mu < \omega_0$ then $\alpha_{\mu-1} < \delta < \nu = \alpha_\mu$ and thus $r_\delta \parallel \ell_{\mu-1}$. The supposition $\mu = \omega_0$ implies that $\nu = \alpha_\mu$ is the least ordinal number $\varkappa$ with $\alpha_\gamma < \varkappa$ for all $\gamma < \mu$. Then $\nu \leqq \delta$ and we have a contradiction. In the case $G_\delta \neq \emptyset$ denote by γ_0 the least ordinal number in G_δ. It follows by $\gamma_0 = 0$ that $\delta = \alpha_0 = 0$ and by this we get $r_\delta \parallel \ell_0$. Whenever $\gamma_0 > 0$ then $\alpha_{\gamma_0 - 1} < \delta \leqq \alpha_{\gamma_0}$ and, clearly, $r_\delta \parallel \ell_{\gamma_0 - 1}$.

(3) Let us admit that there are nonempty posets S, T satisfying $Q = S + T$. By means of (1), (2) one can easily verify that $X \cap Y \neq \emptyset$ for arbitrary $X \in \{S, T\}$, $Y \in \{L_\pi, R_\pi\}$. If ℓ_{γ_0}, r_{δ_0} are greatest elements in $S \cap L_\pi$, $S \cap R_\pi$, respectively, then $S = \omega_Q\{\ell_{\gamma_0}, r_{\delta_0}\}$, $\ell_{\gamma_0} \parallel r_\delta \Rightarrow \delta_0 \leqq \delta$ for all $\delta < \nu$ and $\ell_\gamma \parallel r_\delta \Rightarrow \delta < \delta_0$ for all $\delta < \nu$, $\gamma < \gamma_0$. By these implications and by $\ell_{\gamma_0} \parallel r_{\alpha\gamma_0}$, $\ell_{\gamma_0 - 1} \parallel r_{\alpha\gamma_0}$ ($0 < \gamma_0$ because of $T \cap$ $\cap\, L_\pi \neq \emptyset$) it follows that $\delta_0 \leqq \alpha_{\gamma_0}$ on the one hand and $\alpha_{\gamma_0} < \delta_0$ on the other. We have a contradiction.

(c) Assume $Q = P_3\pi$ for a finite description $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ and put $\nu = \alpha_\mu$. Then Q is finite and satisfies the ACC obviously.

Q is ordinally indecomposable: Let $Q = S + T$ and $S \neq \emptyset$. Then $\{s, \ell_{\mu-1}, r_{\nu-1}\} \subseteq$ $\subseteq S$ according to 1.2. This gives $S' = S - \{s\} \neq \emptyset$ and, clearly, $P_2\pi = S' + T$. But then $T = \emptyset$ in virtue of (b).

2.5. Definition. We put $A_\pi(\gamma, \delta) = \omega_{P_2\pi}\{\ell_\gamma, \mathscr{r}_\delta\}$ for an arbitrary description $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$, $\gamma < \mu$, and $\delta < \alpha_\mu$ such that $\alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}$.

If, moreover, π is a finite description then we put $B_\pi(\gamma, \delta) = \omega_{P_3\pi}\{\ell_\gamma, \mathscr{r}_\delta\}$.

2.6. Lemma. *Suppose* $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ *is a description and put* $\nu = \alpha_\mu$. *If* $Q = P_2\pi$ *then*

$$\mathfrak{D}_Q = \{\emptyset\} \cup \omega_Q[Q] \cup \{A_\pi(\gamma, \delta);\ \gamma < \mu,\ \delta < \nu \text{ and } \alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}\}\,.$$

If π *is a finite description and* $Q = P_3\pi$ *then*

$$\mathfrak{D}_Q = \{\emptyset\} \cup \omega_Q[Q] \cup \{B_\pi(\gamma, \delta);\ \gamma < \mu,\ \delta < \nu \text{ and } \alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}\} \cup$$
$$\cup \{\{\mathscr{s}, \ell_{\mu-1}\}, \{\mathscr{s}, \mathscr{r}_{\nu-1}\}, \{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\}\}\,.$$

Proof. Let $Q = P_2\pi$ and $R \in \mathfrak{D}_Q - \{\emptyset\}$. As Q satisfies the ACC by 2.4, there is an antichain A in Q such that $R = \omega_Q A$. If $|A| = 1$ then $R \in \omega_Q[Q]$. Since $bQ = 2$, the remaining possibility is $|A| = 2$. Then, clearly, $A = \{\ell_\gamma, \mathscr{r}_\delta\}$ for some $\gamma < \mu$, $\delta < \nu$. By Definition 2.2, A is an antichain iff $\alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}$. Hence $R = A_\pi(\gamma, \delta)$ and we have proved $\mathfrak{D}_Q \subseteq \{\emptyset\} \cup \omega_Q[Q] \cup \{A_\pi(\gamma, \delta);\ \gamma < \mu,\ \delta < \nu,\ \alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}\}$. The converse inclusion is true obviously.

Let π be a finite description, $Q = P_3\pi$ and $R \in \mathfrak{D}_Q - \{\emptyset\}$. Then by 2.4 there is an antichain A satisfying $R = \omega_Q A$. If $|A| = 1$ then $R \in \omega_Q[Q]$. In the case $|A| = 2$ either $\mathscr{s} \notin A$ or $\mathscr{s} \in A$. If $\mathscr{s} \notin A$ then we can find $\gamma < \mu$, $\delta < \nu$ such that $\alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}$ and $A = \{\ell_\gamma, \mathscr{r}_\delta\}$. Hence $R = B_\pi(\gamma, \delta)$. If $\mathscr{s} \in A$ then either $A = \{\mathscr{s}, \ell_{\mu-1}\} = R$ or $A = \{\mathscr{s}, \mathscr{r}_{\nu-1}\} = R$ because A is an antichain, $\mathscr{s} \parallel x$ iff $x \in \{\ell_{\mu-1}, \mathscr{r}_{\nu-1}\}$ and $\mathscr{s}, \ell_{\mu-1}$, $\mathscr{r}_{\nu-1}$ are minimal in Q. In the case $|A| = 3$ it holds that $A = \{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\} = R$. As $bQ = 3$, we have proved $\mathfrak{D}_Q \subseteq \{\emptyset\} \cup \omega_Q[Q] \cup \{B_\pi(\gamma, \delta);\ \gamma < \mu,\ \delta < \nu,\ \alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}\} \cup \{\{\mathscr{s}, \ell_{\mu-1}\}, \{\mathscr{s}, \mathscr{r}_{\nu-1}\}, \{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\}\}$. The converse inclusion is true obviously.

2.7. Definition. We put

$$\omega A_{i_1 i_2 \ldots i_k} = \omega A - \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$$

for an arbitrary P, $A = \{a_0, a_1, \ldots, a_{m-1}\} \in {}^m P$, $0 < k \leqq m$ and $0 \leqq i_1 < i_2 < \ldots < i_k < m$.

2.8. Lemma. *The following assertions* (i), (ii), (iii) *are equivalent.*

(i) *P is ordinally indecomposable and $P \in \mathscr{M}_S$.*

(ii) *P is ordinally indecomposable, $P \in \mathscr{M}$ and none of the posets $P_2, P_2', P_3, P_3', P_4$ from* Fig. 2 *can be embedded into P.*

(iii) *There is $Q \in \Gamma_S$ such that $P \cong Q$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that P is an ordinally indecomposable element of $\mathscr{M}_S$. Then P satisfies the ACC and $\mathfrak{D} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P] \cup \omega^-[P]$.

(a) If $bP = 1$ then none of the posets $P_2, P'_2, P_3, P'_3, P_4$ can be embedded into P.

(b) Let $bP = 2$. Then neither P_3 nor P'_3, P_4 can be embedded into P.

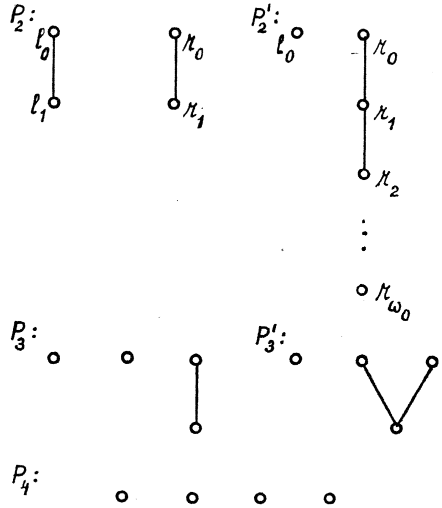

Figure 2

(b1) P_2 cannot be embedded into P: Let us admit that $\iota : P_2 \to P$ is an embedding and denote $a_0 = \iota\ell_1$, $a_1 = \iota\ell_0$, $b_0 = \iota r_1$, $b_2 = \iota r_0$. Then (a_0, a_1), (b_0, b_1) are ascending chains with $a_0 \nleqq b_1$, $b_0 \nleqq a_1$.

Assume that $1 < k < \omega_0$ and $(a_i)_{i<k}$, $(b_i)_{i<k}$ are ascending chains in P satisfying $a_{i-1} \nleqq b_i$, $b_{i-1} \nleqq a_i$ for all i, $0 < i < k$. Clearly, $A = \{a_{k-2}, b_{k-2}\} \in {}^2P$.

If $\omega A \in \bar{\varepsilon}[P]$ then there is $a \in P$ such that $\omega A = \bar{\varepsilon}a$. Hence $a \nleqq a_{k-2}$, $a \nleqq b_{k-2}$, $a \leqq a_{k-1}$, $a \leqq b_{k-1}$. As simultaneously $a_{k-2} \nleqq b_{k-1}$ and $b_{k-2} \nleqq a_{k-1}$, it holds that $a_{k-2} \nleqq a$ and $b_{k-2} \nleqq a$. Thus $A \cup \{a\} \in {}^3P$, contrary to $bP = 2$.

If $\omega A \in \omega^-[P]$ then $\omega A = \omega^- b$ for some $b \in P$. We immediately obtain $\{a_{k-1}, b_{k-1}, b\} \in {}^3P$ which is a contradiction.

In the case $\omega A \in \mathfrak{N}$ there is an upper bound b_k of ωA satisfying $a_{k-1} \nleqq b_k$ by 1.1 (i). As $b_k \nleqq a_{k-1}$ is also true, we have $a_{k-1} \parallel b_k$; this fact, $bP = 2$ and $a_{k-1} \parallel b_{k-1}$ give $b_k \leqq b_{k-1}$ or $b_{k-1} < b_k$. Since $b_k \leqq b_{k-1}$ implies an invalid assertion $a_{k-2} \leqq$ $\leqq b_{k-1}$, it holds that $b_{k-1} < b_k$. Similarly we can show that there is $a_k \in P$ with the properties $b_{k-1} \nleqq a_k$, $a_{k-1} < a_k$. Hence $(a_i)_{i<k+1}$, $(b_i)_{i<k+1}$ are ascending chains in P such that $a_{i-1} \nleqq b_i$, $b_{i-1} \nleqq a_i$ for all i, $0 < i < k + 1$.

By induction it follows that there are ascending chains $(a_i)_{i<\omega_0}$, $(b_i)_{i<\omega_0}$ in P – a contradiction.

(b2) P_2' cannot be embedded into P: Let us admit that there is an embedding $\iota : P_2' \to P$. If we put $Q = \{a \in P;\ a \parallel \iota\ell_0 \text{ and } a < \iota r_i \text{ for all } i < \omega_0\}$ then $\iota r_{\omega_0} \in Q$ and thus $Q \neq \emptyset$. Q is a chain in virtue of $bP = 2$ and Q satisfies the ACC trivially. These facts imply that there is a greatest element a in Q. Now let A be the antichain $\{a, \iota\ell_0\}$.

If $\omega A \in \bar{\varepsilon}[P]$ then there is $b \in P$ with $\omega A = \bar{\varepsilon} b$. Obviously, $b < \iota r_i$ for all $i < \omega_0$ and $b \nleqq a$, $b \nleqq \iota\ell_0$. If $a < b$ then $b \parallel \iota\ell_0$ and we have a contradiction with the choice of a. If we suppose $a \parallel b$ then $b \nparallel \iota\ell_0$; hence $\iota\ell_0 < b$ and we obtain $\iota\ell_0 < \iota r_i$ for all $i < \omega_0$ which is also a contradiction.

Suppose that $\omega A \in \mathfrak{N} \cup \omega^-[P]$. As $\iota\ell_0 \in \omega A$ and $\iota\ell_0 \nleqq \iota r_1$, $\iota r_1 \notin \omega A$ and ιr_1 is not an upper bound of ωA. Hence there is an upper bound b of ωA satisfying $\iota r_1 \nleqq b$ according to 1.1. By this and by $\iota\ell_0 < b$ it follows that $\iota r_i \parallel b$ for $i = 0, 1$, so that $\varkappa = \{(\ell_0, b), (\ell_1, \iota\ell_0), (r_0, \iota r_0), (r_1, \iota r_1)\}$ is an embedding of P_2 into P, contrary to (b1).

(c) Let $bP = 3$. Clearly, there is no embedding of P_4 into P. The poset ${}^3P$ is nonempty and, by 1.8, ${}^3P$ satisfies the ACC. Hence there is a maximal element $A = \{a_0, a_1, a_2\}$ in ${}^3P$. Let us put $P' = P - \omega A_{012}$.

(c1) ${}^3P' = \{A\}$: If $B = \{b_0, b_1, b_2\} \in {}^3P'$ then $b_i \nless a_j$ for all $i < 3$ and $j < 3$ with respect to the minimality of a_0, a_1, a_2 in P'. If there exists $i < 3$ fulfilling $a_i \nleqq b_j$ for $j = 0, 1, 2$ then $B \cup \{a_i\} \in {}^4P$ which contradicts $bP = 3$. Thus $A \leqq B$ and we obtain $B = A$ by the maximality of A in ${}^3P$.

(c2) P_2, P_2', P_3, P_3' cannot be embedded into P' : P_3 and P_3' cannot be embedded into P' according to (c1) and the minimality of a_0, a_1, a_2 in P'. If there is an embedding of P_2, P_2' into P' then we can find $i < 3$ and an embedding of P_2, P_2', respectively, into $P'' = P' - \{a_i\}$. It follows by (c1) and $a_i \in A$ that $bP'' = 2$. Since P'' is a final segment in P, P'' satisfies the ACC and $P'' \in \mathscr{P}_S$ with regard to 1.6. Then neither P_2 nor P_2' can be embedded into P'' by (b1), (b2) – a contradiction.

(c3) $P' = P$: Let us admit $\omega A_{01} \neq \omega a_2$.

If $\omega A_{01} \in \mathfrak{N}$ then, by 1.1 (i), there is an upper bound b_i of ωA_{01} satisfying $a_i \nleqq b_i$ for $i = 0, 1$. In the case $a_0 \nleqq b_1$ or $a_1 \nleqq b_0$, $\{a_0, a_1, b_1\}$ or $\{a_0, a_1, b_0\}$ is an element of ${}^3P$ greater than A – a contradiction. For this reason $a_0 \leqq b_1$ and $a_1 \leqq b_0$. But then $\iota = \{(\ell_0, b_0), (\ell_1, a_1), (r_0, b_1), (r_1, a_0)\}$ is an embedding of P_2 into $P - \omega A_{01}$. By this and by $P - \omega A_{01} \subseteq P'$ we obtain that P_2 can be embedded into P', contrary to (c2).

Since a_0, a_1 are two different minimal elements in $P - \omega A_{01}$, we have $\omega A_{01} \notin \bar{\varepsilon}[P]$.

In the case $\omega A_{01} \in \omega^-[P]$ there is $b \in P$ such that $\omega A_{01} = \omega^- b$. Then $a_0 \nleqq b$, $a_1 \nleqq b$, $a_2 < b$ and we have $A < \{a_0, a_1, b\} \in {}^3P$, which is a contradiction.

Hence the proof of $\omega A_{01} = \omega a_2$ is complete. Similarly we can see that $\omega A_{02} = \omega a_1$ and $\omega A_{12} = \omega a_0$. Now it is clear that $\omega A_{012} = \omega^- a_i$ for $i = 0, 1, 2$. By this and by $bP = 3$ it follows that $P = \omega A_{012} + P'$. This fact, $\emptyset \subset A \subseteq P'$ and the ordinal indecomposability of P give $P' = P$.

(d) Suppose $bP \notin \{1, 2, 3\}$. Then ${}^4P \neq \emptyset$ and ${}^4P$ satisfies the ACC by 1.8. Thus there is a maximal element $A = \{a_0, a_1, a_2, a_3\}$ in ${}^4P$.

Let us admit $\omega A_0 \in \bar{\varepsilon}[P]$. Then $\omega A_0 = \bar{\varepsilon} a$ for some $a \in P$. As a is a least element and a_0 a minimal one in $P - \omega A_0$, $a = a_0$ is true. Consider the set ωA_{01}. Because a_0 and a_1 are two different minimal elements in $P - \omega A_{01}$, we have $\omega A_{01} \notin \bar{\varepsilon}[P]$. Then $\omega A_{01} \in \mathfrak{N} \cup \omega^-[P]$, $a_0 \notin \omega A_{01}$ and a_0 is not an upper bound of ωA_{01} since $a_2 \nleqq a_0$, $a_2 \in \omega A_{01}$. Hence, by 1.1, there is an upper bound b of ωA_{01} such that $a_0 \nleqq b$. Since $a_i < b$ for $i = 2, 3$, it holds that $b \in P - \omega A_0 = P - \bar{\varepsilon} a_0$ and $a_0 \leqq b$ which is a contradiction. In the same way we can prove $\omega A_i \notin \bar{\varepsilon}[P]$ for $i = 1, 2, 3$.

Thus the remaining case is $\omega A_i \in \mathfrak{N} \cup \omega^-[P]$ for $i = 0, 1, 2, 3$. Then, according to the fact that $a_i \in P - \omega A_i$ is not an upper bound of ωA_i and to 1.1, we can find an upper bound b_i of ωA_i with the property $a_i \nleqq b_i$ for each $i < 4$. One can easily see that $B = \{b_0, b_1, b_2, b_3\} \in {}^4P$ and $A < B$; it is a contradiction.

(ii) $\Rightarrow$ (iii): Suppose that P is ordinally indecomposable, $P \in \mathscr{M}$ and $P_2, P_2', P_3, P_3', P_4$ cannot be embedded into P. Then certainly $bP \in \{1, 2, 3\}$.

(a′) If $bP = 1$ then $P \cong 1 \in \Gamma_S$.

(b′) Let $bP = 2$. Then there exist chains A, B with $P = A \cup B$. Since A, B satisfy the ACC and $A \neq \emptyset \neq B$, it is possible to find ordinal numbers $\mu > 0$, $\nu > 0$ such that $A = \{a_\gamma; \gamma < \mu\}$, $B = \{b_\delta; \delta < \nu\}$ and $(a_\gamma)_{\gamma<\mu}$, $(b_\delta)_{\delta<\nu}$ are descending chains.

As P is ordinally indecomposable, it holds that $a_0 \parallel b_0$. If $\mu = 1 = \nu$ then $P \cong P_2^1$. Suppose $\mu > 1$ or $\nu > 1$ and put $Q = P - \{a_0, b_0\}$. If neither a_0 nor b_0 is an upper bound of Q then $\mu > 1$, $\nu > 1$ and $\iota = \{(\ell_0, a_0), (\ell_1, a_1), (\mathscr{r}_0, b_0), (\mathscr{r}_1, b_1)\}$ is an embedding of P_2 into P – a contradiction. If both a_0 and b_0 are upper bounds of Q then $P = Q + \{a_0, b_0\}$ which is also a contradiction. Hence exactly one of the elements a_0, b_0 is an upper bound of Q.

Let a_0 have this property. Then $\mu > 1$ and for each $\gamma < \mu$ there is $\delta < \nu$ such that $a_\gamma \nleqq b_\delta$ with regard to the ordinal indecomposability of P. Denote by α_γ the least δ with $a_\gamma \nleqq b_\delta$ and put $\alpha_\mu = \nu$.

(b′1) $a_\gamma \leqq b_\delta$ iff $\delta < \alpha_\gamma$ for arbitrary $\gamma < \mu$, $\delta < \nu$ follows immediately by the definition of α_γ.

(b′2) $b_\delta \leqq a_\gamma$ iff $\alpha_{\gamma+1} < \delta$ for arbitrary $\gamma < \mu$, $\delta < \nu$:

If $\gamma + 1 = \mu$ and $b_\delta \leqq a_\gamma$ for some $\delta < \nu = \alpha_{\gamma+1}$ then b_δ is comparable with all elements of P, contrary to the ordinal indecomposability of P.

In the case $\gamma + 1 < \mu$ put $b = b_{\alpha_{\gamma+1}}$. Obviously, it is sufficient to prove $b \nleqq a_\gamma$ and $b_\delta \leqq a_\gamma$ for all $\delta > \alpha_{\gamma+1}$. If $b \leqq a_\gamma$ then $P = \omega\{a_{\gamma+1}, b\} + (P - \omega\{a_{\gamma+1}, b\})$

according to (b′1) which contradicts the ordinal indecomposability of P. Admit $b_\delta \nleqq a_\gamma$ for some $\delta > \alpha_{\gamma+1}$. Then $\iota = \{(\ell_0, a_\gamma), (\ell_1, a_{\gamma+1}), (\imath_0, b), (\imath_1, b_\delta)\}$ is an embedding of P_2 into P with respect to $a_{\gamma+1} < a_\gamma$, $b_\delta < b$, $a_{\gamma+1} \nleqq b$ – a contradiction.

(b′3) $(\alpha_\gamma)_{\gamma<\mu+1}$ is a description: Let us take $\gamma < \gamma' < \mu$ arbitrarily. If $\delta < \alpha_\gamma$ then $a_\gamma \leqq b_\delta$. By this and $a_{\gamma'} < a_\gamma$ it follows that $a_{\gamma'} \leqq b_\delta$ and further $\delta < \alpha_{\gamma'}$ so that $\alpha_\gamma \leqq \alpha_{\gamma'}$. Thus $(\alpha_\gamma)_{\gamma<\mu}$ is a nondescending chain. $\alpha_0 = 0$ obviously, $\alpha_1 = 0$ is a consequence of $a_1 \nleqq b_0$ and $\alpha_\gamma < \alpha_\mu$ for all $\gamma < \mu$ follows by $\alpha_\mu = \nu$ and by the definition of α_γ.

(1) $\mu \leqq \omega_0$: Admit $\omega_0 < \mu$ and denote $b = b_{\alpha_{\omega_0}}$, $A = \omega\{a_{\omega_0}, b\}$. In the first case suppose $\alpha_\gamma < \alpha_{\omega_0}$ for all $\gamma < \omega_0$. If we take $\gamma < \omega_0$ arbitrarily then $\gamma + 1 < < \omega_0$ and we obtain $\alpha_{\gamma+1} < \alpha_{\omega_0}$. This gives $b \leqq a_\gamma$ by (b′2) so that a_γ is an upper bound of A in P. If $\delta < \alpha_{\omega_0}$ is arbitrary then $a_{\omega_0} < b_\delta$ and thus b_δ is an upper bound of A in P as well. We have proved $P = A + (P - A)$ which contradicts the ordinal indecomposability of P. In the second case there is $\gamma_0 < \omega_0$ satisfying $\alpha_\gamma = \alpha_{\omega_0}$ for all γ, $\gamma_0 \leqq \gamma < \omega_0$. But then $\iota = \{(\ell_0, b), (\imath_{\omega_0}, a_{\omega_0})\} \cup \{(\imath_\gamma, a_{\gamma_0+\gamma}); \gamma < \omega_0\}$ is an embedding of P_2' into P which is also a contradiction.

(2) If $\mu = \omega_0$ then α_μ is the least ordinal number $\varkappa$ with the property $\alpha_\gamma < \varkappa$ for all $\gamma < \mu$: Assume $\mu = \omega_0$ and choose $\delta < \nu = \alpha_\mu$ arbitrarily. Then there is $\gamma < \mu$ such that $b_\delta \nleqq a_\gamma$ according to the ordinal indecomposability of P. By this, (b′2) and by $\gamma + 1 < \mu$ it follows that $\delta \leqq \alpha_{\gamma+1}$.

(3) $\alpha_\gamma < \omega_0$ for all $\gamma < \mu$: If $\gamma < \mu$ then $\gamma < \omega_0$ by (1). Admit $\alpha_\gamma \geqq \omega_0$ for some $\gamma < \omega_0$ and denote by γ_0 the least such γ. Then, as $\gamma_0 > 0$ is obvious, we have $\alpha_{\gamma_0-1} < \omega_0$. One can easily see that $\iota = \{(\ell_0, a_{\gamma_0}), (\imath_{\omega_0}, b_{\alpha_{\gamma_0}})\} \cup \{(\imath_\gamma, b_{\alpha_{\gamma_0-1}+\gamma}); \gamma < \omega_0\}$ is an embedding of P_2' into P – a contradiction.

The proof of (b′3) is complete. Hence $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ is a description and, with respect to (b′1), (b′2), $\iota = \{(a_\gamma, \ell_\gamma); \gamma < \mu\} \cup \{(b_\delta, \imath_\delta); \delta < \alpha_\mu\}$ is an isomorphism of P onto $P_2\pi$.

(c′) Let $bP = 3$. Take $A = \{a_0, a_1, a_2\} \in {}^3P$ arbitrarily. If $a \in \omega A_{012}$ then $a \leqq a_i$ for some $i < 3$. By this and by the fact that neither P_3 nor P_3' can be embedded into P we obtain $a < a_i$ for $i = 0, 1, 2$. This and $bP = 3$ give $P = \omega A_{012} + + (P - \omega A_{012})$. But then $\omega A_{012} = \emptyset$ with regard to $\emptyset \subset A \subseteq P - \omega A_{012}$ and the ordinal indecomposability of P. Hence A is exactly the set of all minimal elements in P and ${}^3P = \{A\}$ is true obviously.

Let us put $P' = P - A$.

If $P' = \emptyset$ then $P = A \cong P_3^1$.

Suppose $P' \neq \emptyset$ and admit that there are two different elements in A which are not lower bounds of P'. Let for example $a_0 \nleqq b_0$ and $a_1 \nleqq b_1$ for some $b_0, b_1 \in P'$. It follows by $a_0 \nleqq b_1$ that $\{a_0, a_1, b_1\} \in {}^3P - \{A\}$ – a contradiction. For this reason $a_0 \leqq b_1$ and we can prove $a_1 \leqq b_0$ by the same argument. But then $\iota =$

$= \{(\ell_0, b_1), (\ell_1, a_0), (\imath_0, b_0), (\imath_1, a_1)\}$ is an embedding of P_2 into P which is impossible.

Let a_0 be one of the lower bounds of P'. If we put $P'' = P - \{a_0\}$ then $bP'' = 2$ in virtue of $a_0 \in A$ and ${}^3P = \{A\}$. Since (ii) is true for P and $P'' \subseteqq P$, the following two assertions hold. P'' satisfies the ACC and $P_2, P_2', P_3, P_3', P_4$ cannot be embedded into P''. Further, P'' is ordinally indecomposable: If there are nonempty posets Q, R with $P'' = Q + R$ then $a_i \in Q$ for $i = 1, 2$ according to 1.2 so that a_0 is a lower bound of R. Then $P = (Q \cup \{a_0\}) + R$, contrary to the ordinal indecomposability of P. By means of (b') we obtain that there exists a description $\pi = (\alpha_\gamma)_{\gamma < \mu + 1}$ and an isomorphism $\iota : P'' \to P_2\pi$. Since $\iota a_1, \iota a_2$ are two different minimal elements in $P_2\pi$, μ and α_μ are successor ordinals. Then π is a finite description and it is clear that $\iota \cup \{(a_0, \mathscr{o})\}$ is an isomorphism of P onto $P_3\pi$.

(iii) $\Rightarrow$ (i): If $P \cong Q$ for some $Q \in \Gamma_S$ then P satisfies the ACC and is ordinally indecomposable by 2.4. It is sufficient to prove $\mathfrak{D}_Q \subseteqq \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$.

(a'') If $Q = 1$ then the statement is true obviously.

(b'') Let there exist a description π such that $Q = P_2\pi$ and denote $\pi = (\alpha_\gamma)_{\gamma < \mu + 1}$, $\nu = \alpha_\mu$.

As $\emptyset \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ by 2.7 [6] and $\omega_Q[Q] \subseteqq \mathfrak{N}_Q$, it remains to prove that $A_\pi(\gamma, \delta) = \omega_Q\{\ell_\gamma, \imath_\delta\} \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ for all $\gamma < \mu$, $\delta < \nu$ such that $\alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}$ according to 2.6.

Suppose $\delta = \alpha_\gamma$. If $\gamma = 0$ then $\delta = 0$ and $A_\pi(\gamma, \delta) = Q \in \mathfrak{N}_Q$. In the case $\gamma > 0$, $\delta = 0$ it holds that $A_\pi(\gamma, \delta) = \bar{\varepsilon}_Q \ell_{\gamma-1} \in \bar{\varepsilon}_Q[Q]$. If $\delta > 0$ then $0 < \delta \leqq \alpha_{\gamma+1}$ and, regarding $\alpha_1 = 0$, we obtain $\gamma > 0$. Now $\delta - 1 < \alpha_\gamma$ implies $\ell_\gamma \leqq \imath_{\delta-1}$ by 2.2 so that $A_\pi(\gamma, \delta) \subseteqq \omega_Q^- \imath_{\delta-1}$. If $\ell_{\gamma-1} \nleqq \imath_{\delta-1}$ then $A_\pi(\gamma, \delta) = \omega_Q^- \imath_{\delta-1} \in \omega_Q^-[Q]$; if $\ell_{\gamma-1} \leqq \imath_{\delta-1}$ then $\ell_{\gamma-1}$ is a least element in $Q - A_\pi(\gamma, \delta)$. Hence $A_\pi(\gamma, \delta) = \bar{\varepsilon}_Q \ell_{\gamma-1} \in \bar{\varepsilon}_Q[Q]$.

If $\delta = \alpha_\gamma + 1$ then $\alpha_{\gamma+1} > 0$ so that $\gamma > 0$ and we obtain $\imath_\delta \leqq \ell_{\gamma-1}$, $\imath_{\delta-1} \nleqq \ell_{\gamma-1}$. This gives $A_\pi(\gamma, \delta) = \omega_Q^- \ell_{\gamma-1} \in \omega_Q^-[Q]$.

In the case $\delta > \alpha_\gamma + 1$ it holds that $\delta - 1 > \alpha_\gamma$ and thus $\imath_{\delta-1} \leqq \ell_{\gamma-1}$. Then $\imath_{\delta-1}$ is a least element in $Q - A_\pi(\gamma, \delta)$ so that $A_\pi(\gamma, \delta) = \bar{\varepsilon}_Q \imath_{\delta-1}$.

(c'') Assume $Q = P_3\pi$ for a finite description $\pi = (\alpha_\gamma)_{\gamma < \mu + 1}$ and put $\nu = \alpha_\mu$.

$B_\pi(\gamma, \delta) \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ for all $\gamma < \mu$, $\delta < \nu$ such that $\alpha_\gamma \leqq \delta \leqq \alpha_{\gamma+1}$: In the case $\gamma < \mu - 1$ or $\delta < \nu - 1$ put $R = P_2\pi$. It holds that $A_\pi(\gamma, \delta) \in \mathfrak{D}_R - \{\emptyset\}$, $B_\pi(\gamma, \delta) = A_\pi(\gamma, \delta) \cup \{\mathscr{o}\}$ and also $\omega_Q A_\pi(\gamma, \delta) = B_\pi(\gamma, \delta)$. Indeed, $\mathscr{o}$ is a lower bound of $R - \{\ell_{\mu-1}, \imath_{\nu-1}\}$ and $A_\pi(\gamma, \delta) \cap (R - \{\ell_{\mu-1}, \imath_{\nu-1}\}) \neq \emptyset$ by supposition. As, simultaneously, $A_\pi(\gamma, \delta) \in \mathfrak{N}_R \cup \bar{\varepsilon}_R[R] \cup \omega_R^-[R]$ according to (b''), we obtain $B_\pi(\gamma, \delta) \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ by 2.2 [6]. As $B_\pi(\mu - 1, \nu - 1) = \bar{\varepsilon}_Q \mathscr{o} \in \bar{\varepsilon}_Q[Q]$, it remains to prove $A \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q] \cup \omega_Q^-[Q]$ for $A = \{\mathscr{o}, \ell_{\mu-1}\}$, $\{\mathscr{o}, \imath_{\nu-1}\}$, $\{\mathscr{o}, \ell_{\mu-1}, \imath_{\nu-1}\}$ according to 2.7 [6] and 2.6.

By $\mu = 1$ it follows that $\nu = 1$ and $Q = \{\mathscr{s}, \ell_0, \mathscr{r}_0\}$. Then $\{\mathscr{s}, \ell_0\}, \{\mathscr{s}, \mathscr{r}_0\} \in \bar{\varepsilon}_Q[Q]$ and $\{\mathscr{s}, \ell_0, \mathscr{r}_0\} \in \mathfrak{N}_Q$. If $\mu > 1$ then put $Q' = Q - \{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\}$ and consider the cases $\alpha_{\mu-1} = \nu - 1$, $\alpha_{\mu-1} < \nu - 1$.

In the first case $\ell_{\mu-1}$ is a lower bound of Q' and $\mathscr{r}_{\nu-1} \parallel \ell_{\mu-2}$. This implies $\{\mathscr{s}, \ell_{\mu-1}\} = \omega_Q^- \ell_{\mu-2} \in \omega_Q^-[Q]$, $\{\mathscr{s}, \mathscr{r}_{\nu-1}\} = \bar{\varepsilon}_Q \ell_{\mu-1} \in \bar{\varepsilon}_Q[Q]$ and $\{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\} = \omega_Q^- \mathscr{r}_{\nu-2} \in \omega_Q^-[Q]$ whenever $\nu > 1$, $\{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\} = \bar{\varepsilon}_Q \ell_{\mu-2} \in \bar{\varepsilon}_Q[Q]$ whenever $\nu = 1$.

In the second case $\alpha_{\mu-1} \leqq \nu - 2 < \nu = \alpha_\mu$ and $\mathscr{r}_{\nu-1}$ is a lower bound of Q' with regard to $\mathscr{r}_{\nu-1} < \ell_{\mu-2}$. These facts imply $\ell_{\mu-1} \parallel \mathscr{r}_{\nu-2}$. Then $\{\mathscr{s}, \ell_{\mu-1}\} = \bar{\varepsilon}_Q \mathscr{r}_{\nu-1} \in \bar{\varepsilon}_Q[Q]$, $\{\mathscr{s}, \mathscr{r}_{\nu-1}\} = \omega_Q^- \mathscr{r}_{\nu-2} \in \omega_Q^-[Q]$ and $\{\mathscr{s}, \ell_{\mu-1}, \mathscr{r}_{\nu-1}\} = \omega_Q^- \ell_{\mu-2} \in \omega_Q^-[Q]$.

2.9. Theorem. *The following assertions* (i), (ii), (iii) *are equivalent.*

(i) $P \in \mathscr{M}_S$.

(ii) $P \in \mathscr{M}$ *and* $P_2, P_2', P_3, P_3', P_4$ *cannot be embedded into* P.

(iii) $P \in \mathbf{O}\Gamma_S$.

Proof. For an arbitrary P there exist a chain I and a set $\{P_i;\ i \in I\}$ of ordinally indecomposable posets such that $P \cong \sum_{i \in I} P_i$ by 1.3. Consider the statements

(a) $P_i \in \mathscr{M}_S$ for all $i \in I$.

(b) $P_i \in \mathscr{M}$ and $P_2, P_2', P_3, P_3', P_4$ cannot be embedded into P_i for all $i \in I$.

(c) For each $i \in I$ there is $Q_i \in \Gamma_S$ with $P_i \cong Q_i$.

It follows by 2.8 that (a), (b), (c) are equivalent. Further, (a) $\Leftrightarrow$ (i) according to 1.5 (i), (ii), (b) $\Leftrightarrow$ (ii) by 1.5 (i) and the ordinal indecomposability of $P_2, P_2', P_3, P_3', P_4$, (c) $\Rightarrow$ (iii) trivially and (iii) $\Rightarrow$ (c) with regard to 2.4, 1.3.

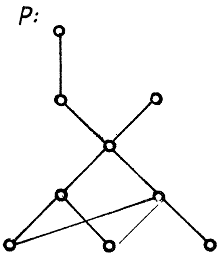

Figure 3

As $P \cong P_3(0, 0, 2) + 1 + P_2^2$, it holds that $P \in \mathbf{O}\Gamma_S$ for the poset P from Fig. 3. Then $P \in \mathscr{P}_S$ and one can easily check that $\mathrm{Gs}(P) \cong 2^4 \times 3$ using 3.18 [6].

3.1. Definition. We put

$$\Gamma_C = \{1\} \cup \{P_2^\mu;\ 0 < \mu \leqq \omega_0\} \cup \{P_3^1\}\,.$$

3.2. Lemma. *The following assertions* (i), (ii), (iii) *are equivalent.*

(i) *P is ordinally indecomposable and $P \in \mathcal{M}_C$.*

(ii) *P is ordinally indecomposable, $P \in \mathcal{M}$ and the posets $P_2, P_2', P_2(0, 0, 2)$, P_3, P_3', P_3^2, P_4 from* Fig. 1, 2 *cannot be embedded into P.*

(iii) *There is $Q \in \Gamma_C$ such that $P \cong Q$.*

Proof. (i) $\Leftrightarrow$ (iii): Assume that $P \in \mathcal{M}_C$ and P is ordinally indecomposable. Then obviously $P \in \mathcal{M}_S$ and there is $Q \in \Gamma_S$ such that $P \cong Q$ according to 2.8.

(a) Let $Q = P_2\pi$ where $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ is such that $\alpha_\mu > 1$. Then $\mu > 1$ and $A = A_\pi(1, 1) \notin \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q]$. Indeed, $A \notin \mathfrak{N}_Q$ with regard to the fact that A has only one upper bound $\ell_0 \notin A$ and $A \notin \bar{\varepsilon}_Q[Q]$ because of $Q - A = \{\ell_0, \imath_0\}$, $\imath_0 \in \bar{\varepsilon}_Q\ell_0$, $\ell_0 \in \bar{\varepsilon}_Q\imath_0$. But then $Q \notin \mathcal{M}_C$ and also $P \notin \mathcal{M}_C$ – a contradiction.

(b) Let $Q = P_3\pi$, where $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ is a finite description and let $\nu = \alpha_\mu$. Clearly, $P_2\pi$ is a final segment in Q. If $\nu > 1$ then $\mu > 1$ and $B_\pi(1, 1) \notin \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q]$ with regard to (a) and 2.1 (i), (ii) [6]. In the case $\nu = 1$, $\mu > 1$ put $A = \{s, \ell_{\mu-1}\}$. As $\ell_{\mu-2} \notin A$, $\ell_{\mu-2}$ is the least upper bound of A and $\ell_{\mu-2} \parallel \imath_0$, we have $A \in \mathfrak{D}_Q - (\mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q])$.

The conclusions of (a) and (b) imply $Q \in \Gamma_C$.

Conversely, if $P \cong Q$ for some $Q \in \Gamma_C$ then $Q \in \Gamma_S$ and P is ordinally indecomposable, $P \in \mathcal{M}$ according to 2.4. Whenever $Q = 1$ or $Q = P_3^1$ then $P \in \mathcal{P}_C$ trivially. Suppose $Q = P_2^\mu$ for an arbitrary μ, $0 < \mu \leqq \omega_0$. It follows by 2.6 and by 2.7 [6] that it is sufficient to prove $A_\pi(\gamma, 0) \in \mathfrak{N}_Q \cup \bar{\varepsilon}_Q[Q]$ for all $\gamma < \mu + 1$ where $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ is such a description that $\alpha_\mu = 1$. But, obviously, $A_\pi(0, 0) = Q \in \mathfrak{N}_Q$ and $A_\pi(\gamma, 0) = \bar{\varepsilon}_Q\ell_{\gamma-1}$ for all γ, $0 < \gamma < \mu + 1$.

(ii) $\Leftrightarrow$ (iii): Suppose that P is ordinally indecomposable, $P \in \mathcal{M}$ and P_2, P_2', $P_2(0, 0, 2)$, P_3, P_3', P_3^2, P_4 cannot be embedded into P. Then there are $Q \in \Gamma_S$ and an isomorphism $\varkappa : Q \to P$ by 2.8. If $Q = P_2\pi$, $Q = P_3\pi$ for a description or finite description $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ such that $\alpha_\mu > 1$, respectively, then we define $\iota : P_2(0, 0, 2) \to Q$ by $\iota x = x$ for $x = \ell_0, \ell_1, \imath_0, \imath_1$. If $Q = P_3\pi$ for a finite description $\pi = (\alpha_\gamma)_{\gamma<\mu+1}$ such that $\alpha_\mu = 1$, $\mu > 1$ then let $\iota : P_3^2 \to Q$ be a map assigning x to $x = \ell_0, \imath_0, s$ and $\ell_{\mu-1}$ to ℓ_1. In the first (second) case ι is an embedding of $P_2(0, 0, 2)$ (of P_3^2) into Q; but then $\varkappa\iota$ is an embedding of $P_2(0, 0, 2)$ (of P_3^2) into P which is a contradiction.

Conversely, if $P \cong Q$ for some $Q \in \Gamma_C$ then $Q \in \Gamma_S$ and P is ordinally indecomposable, $P \in \mathscr{M}$, $P_2, P_2', P_3, P_3', P_4$ cannot be embedded into P by 2.8. In the case $Q \in \{1, P_3^1\}$ it is obvious that there is no embedding of $P_2(0, 0, 2)$ and P_3^2 into Q. As $bP_3^2 = 3$ and $bP_2^\mu = 2$, P_3^2 cannot be embedded into P_2^μ for each μ, $0 < \mu \leqq \omega_0$. Finally, suppose that there is an embedding ι of $P_2(0, 0, 2)$ into P_2^μ for some μ, $0 < < \mu \leqq \omega_0$. As $P_2^\mu = L_\pi \cup R_\pi$ and $L_\pi = \{\ell_\gamma; \gamma < \mu\}$ is a chain, $R_\pi = \{r_0\}$, at least three elements of the set $\iota[P_2(0, 0, 2)]$ are members of L_π, i.e. they form a chain. But this contradicts the fact that there is no three-element chain in $P_2(0, 0, 2)$.

3.3. Theorem. *The following assertions* (i), (ii), (iii) *are equivalent.*

(i) $P \in \mathscr{M}_C$.

(ii) $P \in \mathscr{M}$ *and* $P_2, P_2', P_2(0, 0, 2), P_3, P_3', P_3^2, P_4$ *cannot be embedded into* P.

(iii) $P \in \mathbf{O}\Gamma_C$.

Proof. This statement can be proved by the same method as 2.9 using 1.5 (iii) instead of 1.5 (ii) and 3.2 instead of 2.8.

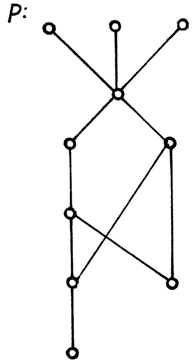

Figure 4

As $P \cong P_2^2 + P_2^2 + 1 + P_3^1$, it holds that $P \in \mathbf{O}\Gamma_C$ for the poset P from Fig. 4. Then $P \in \mathscr{P}_C$ and one can easily check that $\mathrm{Gs}(P) \cong 2^6$ by means of 4.3 [6].

4. ON THE CLASSES $\mathscr{M}_T$ AND $\mathscr{P}_T$

4.1. Lemma. *Suppose that P is ordinally indecomposable and satisfies the* ACC. *Then* $\mathfrak{D} \subseteq \mathfrak{N} \cup \{\emptyset\}$ *if and only if* $P \cong 1$ *or* $P \cong P_2^1$.

Proof. If $\mathfrak{D} \subsetneqq \mathfrak{N} \cup \{\emptyset\}$ then $\mathfrak{D} \subseteq \mathfrak{N} \cup \bar{\varepsilon}[P]$ by 2.7 [6]. This and 3.2 give $P \cong Q$ for some $Q \in \Gamma_C$.

Assume $Q \in \{P_2^\mu; 1 < \mu \leqq \omega_0\} \cup \{P_3^1\}$ and denote $A = Q - \{\ell_0\}$. As $A \notin \{\emptyset, Q\}$, we have $A \in \mathfrak{N}_Q \cup \{\emptyset\}$ iff $A = \bigcap \omega_Q[X]$ for some $X \subseteq Q$. If this is the case then there is $x \in X$ with $\ell_0 \notin \omega_Q x$. That means $A = \omega_Q x$ and x is a greatest element in A. But A has no greatest element for the following reasons. If $Q = P_2^{\mu}$ for $1 < \mu \leqq \omega_0$ then $\ell_1, \imath_0$ are two different maximal elements in A. In the case $Q = P_3^1$ we have $A = \{\mathscr{s}, \imath_0\}$ and $\mathscr{s} \parallel \imath_0$. Hence $A \notin \mathfrak{N}_Q \cup \{\emptyset\}$ so that $\mathfrak{D}_Q \subseteq \mathfrak{N}_Q \cup \{\emptyset\}$. Thus it is $P \cong 1$ or $P \cong P_2^1$. The converse implication is true obviously.

4.2. Theorem. *$P \in \mathcal{M}_T$ if and only if there exist a chain I and a set $\{Q_i; i \in I\} \subseteq \{1, P_2^1\}$ with the following properties. $P \cong \sum_{i \in I} Q_i$ and*

$$Q_i = 1 \Rightarrow \textit{there is} \quad j \in I \quad \textit{satisfying} \quad j \prec i\,, \quad Q_j = 1 \quad \textit{for all} \quad i \in I\,.$$

Proof. This statement can be proved by the same method as 2.9 using 1.5 (iv) instead of 1.5 (ii) and 4.1 instead of 2.8.

4.3. Corollary. *If $P \in \mathcal{M}_T$ then for each $a \in P$ there is at most one element $b \in P$ with the property $a \parallel b$.*

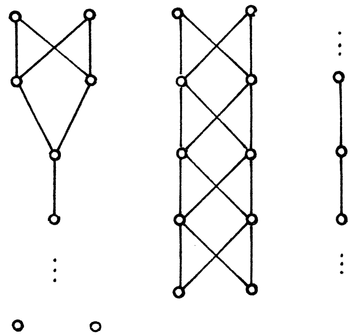

Figure 5

In Fig. 5 there are three diagrams of posets from $\mathcal{M}_T$.

4.4. Theorem. *Every poset can be embedded into a poset from the class $\mathcal{P}_T$.*

Proof. In the case $P = \emptyset$ the statement is true trivially. Otherwise we denote by $\hat{P}$ the set $P \times \omega_0$ ordered in the following way. For arbitrary $(a, i), (b, j) \in \hat{P}$ it holds that $(a, i) \leqq (b, j)$ if either (1) $i = j$ and $a \in \omega b$ or (2) $i + 1 = j$ and $a \in \bar{\varepsilon} b$ or (3) $i + 1 < j$.

We put $\bar{A} = \hat{P} - A$ and $A_+ = \{(a, i + 1); (a, i) \in A\}$ for each $A \subseteq \hat{P}$.

(a) The relation $\leqq$ is an ordering on $\hat{P}$: $\leqq$ is reflexive obviously. Antisymmetry: Suppose $(a, i) \leqq (b, j)$, $(b, j) \leqq (a, i)$ for some $(a, i), (b, j) \in \hat{P}$. Then $i \leqq j$, $j \leqq i$

and thus $i = j$. This and the supposition imply $a \in \omega b$, $b \in \omega a$ which gives $a = b$. Hence $(a, i) = (b, j)$. Transitivity: Let (a, i), (b, j), (c, k) be arbitrary elements from $\hat{P}$ satisfying $(a, i) \leqq (b, j)$, $(b, j) \leqq (c, k)$. Consider the cases

(α) $i = j,\ a \in \omega b,\ j = k,\ b \in \omega c,$

(β) $i + 1 = j,\ a \in \bar{\varepsilon} b,\ j = k,\ b \in \omega c$ and

(γ) $i = j,\ a \in \omega b,\ j + 1 = k,\ b \in \bar{\varepsilon} c.$

Then $(\alpha) \Rightarrow i = k, a \in \omega c$, $(\beta) \Rightarrow i + 1 = k, a \in \bar{\varepsilon} c$, $(\gamma) \Rightarrow i + 1 = k, a \in \bar{\varepsilon} c$ and each of the remaining six possibilities implies $i + 1 < k$ so that $(a, i) \leqq (c, k)$ in all cases.

(b) $\mathfrak{D}_{\hat{P}} \subseteq \mathfrak{N}_{\hat{P}}$: Let us take $A \in \mathfrak{D}_{\hat{P}} - \{\hat{P}\}$ arbitrarily. It is sufficient to prove that $A = \bigcap \omega[(\bar{A})_+]$, which is equivalent to $(a, i) \in A \Leftrightarrow (a, i) \leqq (b, j)$ for all $(b, j) \in (\bar{A})_+$.

The direct implication: Assume $(a, i) \in A$. If $(b, j) \in (\bar{A})_+$ then $(b, j - 1) \in \bar{A}$ and we have $(b, j - 1) \nleqq (a, i)$ according to $A \in \mathfrak{D}_{\hat{P}}$. By this we immediately obtain $i \leqq j$.

In the case $i = j$ we have $(b, i - 1) \nleqq (a, i)$. Then $b \notin \bar{\varepsilon} a$ and we get $a \in \omega b$, which means $(a, i) \leqq (b, i) = (b, j)$.

If $i + 1 = j$ then $(b, i) \nleqq (a, i)$. This consecutively implies $b \notin \omega a$, $b \nleqq a$, $a \in \bar{\varepsilon} b$. The last assertion says $(a, i) \leqq (b, i + 1) = (b, j)$.

It is obvious that $i + 1 < j \Rightarrow (a, i) \leqq (b, j)$.

The converse implication: If $(a, i) \notin A$ then $(a, i) \in \bar{A}$, $(a, i + 1) \in (\bar{A})_+$ and, clearly, $(a, i) \nleqq (a, i + 1)$.

(c) The statement of the theorem is a consequence of (a), (b) and of the fact that $\iota : P \to \hat{P}$, defined by $\iota a = (a, 0)$ for all $a \in P$, is an embedding of P into $\hat{P}$.

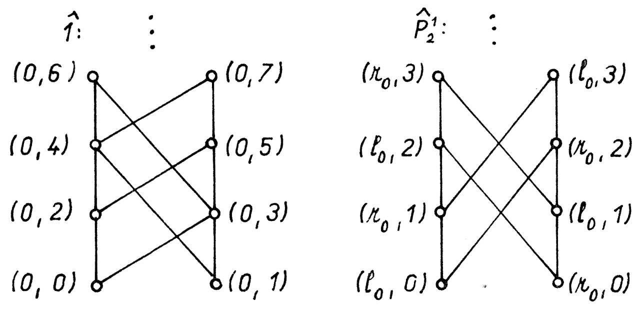

Figure 6

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